Advances in Nonlinear Variational Inequalities
ISSN: 1092-910X
Vol 28 No. 7s (2025)

Successive Approximation Method for a system of Integro - Dynamical Equations with Integral Boundary Conditions on Time Scales.

G.V. Ramana[a*], M. Bala Prabhakar[b], N. Veerraju[c], D. Naga Purnima[d], M. Naga Raju[e] and Renuka Lakshmi Avvari[f].

[a] Department of Mathematics, Aditya University, Surampalem, Kakinada, Andhra Pradesh, India-533 437.

Email: ramanaginjala9@gmail.com

[b] Department of Mathematics, Aditya University, Surampalem, Kakinada, Andhra Pradesh, India-533 437.

Email: prabhakar_mb@yahoo.co.in

[c] Department of Engineering Mathematics, SRKR Engineering College, Bhimavaram, Andhra Pradesh India.

Email: veerrajunalla@gmail.com

[d] Department of Mathematics, Aditya University, Surampalem, Kakinada, Andhra Pradesh, India-533 437.

Email: munukutla1977@gmail.com

[e] Department of Mathematics, Aditya University, Surampalem, Kakinada, Andhra Pradesh, India-533 437.

Email: nagarajuacet@gmail.com

[f] Department of science & Humanities, Vasireddy Venkatadri Institute of Technology, Nambur, A.P, India.

Email: renukalakshmiavvari@gmail.com

Article History:

Received: 18-12-2024

Revised:28-1-2025

Accepted:6-2-2025

Abstract: In this paper, we prove a new comparison result and develop the successive approximation technique to obtain the minimal and maximal solutions of integro-differential equations with integral boundary conditions on time scales.

Keywords: Time scales; integro-dynamical equations; integral boundary condition; successive approximation method.

AMS Subject Classification: 34N05.

---

1. Introduction.

The study of dynamic equations on time scales, created in order to unify the study of differential and difference equations and their applications have been well documented in the monographs [1,2,5,6,9,10,12,13]. In addition, the existence of extremal solutions of ordinary differential equations has been studied extensively in [3,4,11]. To extend such results to the integro-differential equation on time scales, the initial valued integro-differential equation of Volterra type on time scales are studied in [7,8,12]. An opportunity is traced to take up further study on the existence of extremal solutions of integro-differential equation of

Volterra type on time scales with integral boundary conditions. Therefore, this work is taken up in that direction. The successive approximation technique, used when lower solution is less than upper solution, is not suitable here. So successive approximation technique, used when lower solution is greater than upper solution is employed in this work.

In this paper, we consider the following integro-differential equation on time scales

$$x^{\Delta}(t) = f\left(t, x(t), \int_{0}^{t} k(t,s)x(s)\Delta s\right), t \in J \tag{1.1}$$

Where $f \in C[J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}], k(t,s) \in C[J \times J, \mathbb{R}]$. Let T be a time scale (any non-empty closed subset of the real numbers). Without loss of generality, we assume that $0, a \in T$ and $J = [0, \sigma(a)] \subset T$ being a closed interval.

2. Preliminary Results.

Definition 2.1. The mappings σ and $\rho : T \to R$ where T is any closed subset of reals, are defined as $\sigma(t) = inf\{s \in T : s > t\}$ and $\rho(t) = sup\{s \in T : s < t\}$.

Definition 2.2. A non-maximal element t in T is called right dense if $\sigma(t) = t$; right scattered if $\sigma(t) > t$; left dense if $\rho(t) = t$ and left scattered if $\rho(t) < t$.

Definition 2.3. If T has a left scattered maximum m, then $T^k = T - \{m\}$, otherwise, $T^k = T$. T^k is called the degenerate set.

Definition 2.4. The function $\mu^* : T^k \to \mathbb{R}^+$ defined by $\mu^*(t) = \mu(\sigma(t), t)$ for $t \in T$ is called graininess. If t is right dense, then $\mu^* = 0$ and if t is right scattered, then $\mu^* = \sigma(t) - t$.

Definition 2.5. A mapping $g : T \to \mathbb{R}$ is called rd-continuous if

(a) it is continuous in each right dense t;

(b) Left side limit $g(t^-)$ exists in each left dense t.

Remark 2.6. If (b) is replaced by g being continuous at each left-dense point, then g is said to be a continuous function on T.

Definition 2.7. A mapping $f : T \to \mathbb{R}$ is said to be differentiable at $t \in T$, if there exists an $\alpha \in \mathbb{R}$ such that for any $\epsilon > 0$ there exists a neighbourhood N of t satisfying $|f(\sigma(t)) - f(s) - (\sigma(t) - s)\alpha| \leq |\sigma(t) - s|$ for all $s \in N$.

Lemma 2.8. Assume $f, g : T \to \mathbb{R}$ are differentiable at $t \in T^k$. Then:

(a) The sum $f + g : T \to \mathbb{R}$ is differentiable at t with $(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t)$.
(b) For any constant $\alpha, \alpha f : T \to \mathbb{R}$ is differentiable at t with $(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t)$.

(c) The product $fg : T \to \mathbb{R}$ is differentiable at t with

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f\big(\sigma(t)\big)g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

We define $C[J,\mathbb{R}] = \{u(t)$ is continuous on $J\}$, and $C'\,[J,\mathbb{R}] = \{u^{\Delta}(t)$ is continuous on $J\}$.

Definition 2.9. A function $p : T \to \mathbb{R}$ is said to be regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in T^k$. The set of all regressive and rd-continuous functions will be denoted in this paper by $R(T,\mathbb{R})$. For two functions $p,q \in R(T,\mathbb{R})$ define a plus $\oplus$ and a minus $\ominus$ by

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t),$$

$$(\ominus p)(t) = -\frac{p(t)}{1+\mu(t)p(t)}.$$

Definition 2.10. If $p \in \mathbb{R}$, then the exponential function is defined as

$$e_p(t,s) = exp\left[\int_s^t \xi_{\mu(t)}\big(p(t)\big)\Delta t\right] \text{ for } s,t \in T,$$

where $\xi_{\mu(t)}\big(p(t)\big) = \frac{1}{\mu(t)} Log(1 + p(t)\mu(t))$ where Log is principal logarithmic function. When $T = \mathbb{R}$, then $e_\alpha(t,t_0) = e^{\alpha(t-t_0)}$ and when $T = \mathbb{z}$, then $e_\alpha(t,t_0) = (1 + \alpha)^{t-t_0}$

Definition 2.11. If $p \in \mathbb{R}$, then the first order linear dynamic equation

$$y^{\Delta}(t) = p(t)y(t) \qquad (2.1)$$

is called regressive.

Theorem 2.12. If $p \in \mathbb{R}$ and for fixed $t_0 \in T$, $e_p(t,t_0)$ is a solution of the initial value problem (2.1) satisfying the initial condition $y(t_0) = 1$ on T.

Remark 2.13. If $p(t) \geq 0$ for $t \geq t_0$, clearly $1 + \mu(t)p(t) \geq 1$. Therefore $\xi_{\mu(\tau)}(p(\tau)) \geq 0$ and $e_p(t,t_0) \geq 1$.

Lemma 2.14. If $p,q \in R(T,\mathbb{R})$, then

(i) $e_0(t,s) \equiv 1$ and $e_p(t,t) \equiv 1$

(ii) $e_p(\sigma(t),s) = (1 + \mu(t)p(t))e_p(t,s)$

(iii) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$

(iv) $e_p(t,r)e_p(r,s) = e_p(t,s)$

(v) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$.

3.Main Results.

Lemma 3.1. Suppose there is a function $k(t,s) : J \times J \to \mathbb{R}^+$ continuous and $K_0 = max\{k(t,s) : t,s \in J \times J\} > 0$. Assume that there exists a positive function $m(t)$ and a non-negative function $n(t)$ continuous on J and

$$x^{\Delta}(t) \geq m(t)x(t) + n(t)\int_0^t k(t,s)x(s)\Delta s \ ,t \in J \qquad (3.1)$$

$$x(0) \geq x(\sigma(a)).$$

Then $x(t) \leq 0$ for all $t \in J$ provided that $\frac{N_0 K_0 e_m(\sigma(a),0)[em(\sigma(a),0)-1]}{M_0^2\,(1+L_0)} \leq 1$, where $M_0 = \min_{t\in j}\{m(t)\}$, $N_0 = \max_{t\in j}\{n(t)\}$ and $L_0 = \min_{t\in j}\{m(t)\mu(t)\}$.

Proof. Let $v(t) = e_{\ominus m}(t,0)x(t)$, we have

$$v^{\Delta}(t) \geq \frac{n(t)}{1 + \mu(t)m(t)}\, e_{\ominus m}(t,0)\int_0^t k(t,s)v(s)e_m(s,0)\Delta s \qquad (3.2)$$

$$v(0) \geq v\big(\sigma(a)\big)e_m(\sigma(a),0)$$

Obviously, v and x have the same symbol, we need only to prove that $v(t) \leq 0, t \in J$. If it is not true, then there exists $r_1 \in J$ such that $v(r_1) > 0$. Suppose that $v(t) \geq 0$ for all $t \in J$, then $v^{\Delta}(t) \geq 0, t \in J$ by (3.2). This shows that $v(t)$ is nondecreasing on J and so, $v\big(\sigma(a)\big) \geq v(r_1) > 0$, $x\big(\sigma(a)\big) = v\big(\sigma(a)\big)e_m(\sigma(a),0) > 0$, $v(0) \leq v\big(\sigma(a)\big)$. However, from (3.1), we have $v(0) = x(0) \geq x\big(\sigma(a)\big) > x\big(\sigma(a)\big)e_{\ominus m}(\sigma(a),0) = v\big(\sigma(a)\big)$, this is a contradiction. Hence, $v(t) < 0$ for some $t \in J$.

Let $v(r_2) = \min_{t\in J}\{v(t)\} = p$, then $p < 0$. From (3.2), we obtain

$$v^{\Delta}(t) \geq \frac{pn(t)}{1 + \mu(t)m(t)}\, e_{\ominus m}(t,0)\int_0^t k(t,s)e_m(s,0)\Delta s$$

$$\geq \frac{pN_0K_0}{1 + L_0}\, e_{\ominus m}(t,0)\int_0^t e_m(s,0)\Delta s$$

$$\geq \frac{pN_0K_0}{M_0(1 + L_0)}\,[1 - e_{\ominus m}(t,0)] \qquad (3.3)$$

Next, we consider two possible cases.

Case(a): $r_2 > r_1$. By (3.3), we have

$$v(r_2) - v(r_1) \geq \frac{pN_0K_0}{M_0(1 + L_0)}\int_{r_1}^{r_2}\big[1 - e_{\ominus m}(t,0)\big]\Delta t$$

$$p = v(r_2) \geq v(r_1) + \frac{pN_0K_0}{M_0(1 + L_0)} \int_{r_1}^{r_2} \left[1 - e_{\ominus m}(t,0)\right]\Delta t$$

$$> \frac{pN_0K_0}{M_0(1 + L_0)} \int_{r_1}^{r_2} \left[1 - e_{\ominus m}(t,0)\right]\Delta t$$

Since $p < 0,\ e_m(\sigma(a),0) \geq 1$, then

$$1 < \frac{N_0K_0}{M_0(1 + L_0)} \int_{r_1}^{r_2} \left[1 - e_{\ominus m}(t,0)\right]\Delta t$$

$$\leq \frac{N_0K_0e_m(\sigma(a),0)}{M_0(1 + L_0)} \int_{0}^{\sigma(a)} \left[1 - e_{\ominus m}(t,0)\right]\Delta t$$

$$< \frac{N_0K_0e_m(\sigma(a),0)}{M_0(1 + L_0)} \int_{0}^{\sigma(a)} e_m(t,0)\Delta t$$

$$= \frac{N_0K_0e_m(\sigma(a),0)}{M_0(1 + L_0)} \int_{0}^{\sigma(a)} \frac{m(t)}{m(t)} e_m(t,0)\Delta t$$

$$\leq \frac{N_0K_0e_m(\sigma(a),0)[e_m(\sigma(a),0) - 1]}{M_0^2(1 + L_0)}$$

This is a contradiction.

Case(b): $r_2 < r_1$. By (3.3), we have

$$v(r_2) - v(0) \geq \frac{pN_0K_0}{M_0(1 + L_0)} \int_{0}^{r_2} \left[1 - e_{\ominus m}(t,0)\right]\Delta t$$

$$p = v(r_2) \geq v(0) + \frac{pN_0K_0}{M_0(1 + L_0)} \int_{0}^{r_2} \left[1 - e_{\ominus m}(t,0)\right]\Delta t \tag{3.4}$$

and

$$\int_{r_1}^{\sigma(a)} v^{\Delta}(t)\Delta t \geq \frac{pN_0K_0}{M_0(1 + L_0)} \int_{r_1}^{\sigma(a)} \left[1 - e_{\ominus m}(t,0)\right]\Delta t$$

$$v(\sigma(a)) \geq v(r_1) + \frac{pN_0K_0}{M_0(1 + L_0)} \int_{r_1}^{\sigma(a)} \left[1 - e_{\ominus m}(t,0)\right]\Delta t$$

$$> \frac{pN_0K_0}{M_0(1 + L_0)} \int_{r_1}^{\sigma(a)} \left[1 - e_{\ominus m}(t,0)\right]\Delta t \tag{3.5}$$

Since $v(0) \geq v(\sigma(a) e_m(\sigma(a),0)$, from (3.4) and (3.5), we get

$$p \geq v\big(\sigma(a) e_m(\sigma(a),0)\big) + \frac{pN_0K_0}{M_0(1 + L_0)} \int_0^{r_2} \left[1 - e_{\ominus m}(t,0)\right]\Delta t$$

$$p > \frac{pN_0K_0}{M_0(1 + L_0)} e_m(\sigma(a),0) \int_{r_1}^{\sigma(a)} \left[1 - e_{\ominus m}(t,0)\right]\Delta t + \frac{pN_0K_0}{M_0(1 + L_0)} \int_0^{r_2} \left[1 - e_{\ominus m}(t,0)\right]\Delta t$$

Noting that $p < 0, e_m(\sigma(a),o) \geq 1$, $r_2 < r_1$, one can obtain

$$\begin{aligned} 1 &< \frac{N_0K_0}{M_0(1 + L_0)} e_m(\sigma(a),0) \int_{r_1}^{\sigma(a)} \left[1 - e_{\ominus m}(t,0)\right]\Delta t \\ &\quad + \frac{N_0K_0}{M_0(1 + L_0)} \int_0^{r_2} \left[1 - e_{\ominus m}(t,0)\right]\Delta t \\ &< \frac{N_0K_0}{M_0(1 + L_0)} e_m(\sigma(a),0) \int_0^{\sigma(a)} \left[1 - e_{\ominus m}(t,0)\right]\Delta t \\ &< \frac{N_0K_0}{M_0(1 + L_0)} e_m(\sigma(a),0) \int_0^{\sigma(a)} e_m(t,0)\Delta t \\ &\leq \frac{N_0K_0}{M_0^2(1 + L_0)} e_m(\sigma(a),0)[e_m(\sigma(a),0) - 1] \end{aligned}$$

This is a contradiction. Combining case(a) and case (b), we know that $v(t) \leq 0$, for all $t \in J$, and thus $x(t) \leq 0$, for all $t \in J$. The proof is complete.

Next we apply Lemma 3.1 to two extreme cases of T and obtain some interesting results.

Corollary 3.2. Assume $T = \mathbb{R}^+$, $k(t,s)$ satisfies the assumption in Lemma 3.1 and there exists two positive numbers M_0 and N_0 such that

$$x'(t) \geq M_0 x(t) + N_0 \int_0^t k(t,s)x(s)ds, t \in J = [0,a]$$

$$x(0) \geq x(a).$$

Then $x(t) \leq 0$ for all $t \in J$ provided that $\frac{N_0K_0 e^{aM_0}[e^{aM_0} - 1]}{M_0^2} \leq 1.$

Corollary 3.3. Assume $T = \mathbb{Z}^+$, $k(t,s)$ satisfies the assumption in Lemma 3.1 and there exists two positive numbers M_0 and N_0 such that

$$x(t+1) \geq (1+M_0)x(t) + N_0 \sum_{s=0}^{t-1} k(t,s)x(s) \ , t \in J = [0,L]$$

where $L = a + 1$

$$x(0) \geq x(L).$$

Then $x(t) \leq 0$ for all $t \in J$ provided that $N_0 K_0 [1 + M_0]^{L-1} [(1+M_0)^L - 1] \leq 1$.

Lemma 3.4. Suppose there is a function $k(t,s)\colon J \times J \to \mathbb{R}^+$ continuous and $K_0 = \max\{k(t,s)\colon t,s \in J \times J\} > 0$. Assume that there exists a positive function $m(t)$ and a non-negative function $n(t)$ continuous on J. Then the equation

$$x^{\Delta}(t) = m(t)x(t) + n(t) \int_0^t k(t,s)x(s)\Delta s \ , t \in J \tag{3.6}$$

$$x(0) + d = x(\sigma(a)),$$

where $d = \int_0^{\sigma(a)} x(s)\Delta s \in \mathbb{R}$, has a unique solution provided that $\frac{\sigma(a) N_0 K_0 [2e_m(t,0)-1]}{M_0} < 1$, where $(Kx)(t) = \int_0^t k(t,s)x(s)\Delta s$, $M_0 = \max_{t \in j}\{m(t)\}$ and $N_0 = \max_{t \in j}\{n(t)\}$.

Proof. We shall prove the conclusion by Banach's contraction principle. First define a Banach space as follows.

$$X = \{x(t) \in C_{rd}[J,\mathbb{R}]\} \text{ with } \| (x(t) \| = \max_{t \in J}\{x(t)\}.$$

Now define an operator on X as

$$S : x(t) = e_m(t,0)\left\{ \frac{d}{e_m(\sigma(a),0) - 1} - \frac{e_m(\sigma(a),0)}{e_m(\sigma(a),0) - 1} \int_0^{\sigma(a)} e_m(0,\sigma(s)n(s)(Kx)(s)\Delta s + \int_0^t e_m(0,\sigma(s)n(s)(Kx)(s)\Delta s \right\}, t \in J.$$

For any two functions $\phi,\ \psi \in X$, there holds

$$|(S\phi)(t) - (S\psi)(t)| = \left| \frac{e_m(t,0)e_m(\sigma(a),0)}{e_m(\sigma(a),0) - 1} \int_0^{\sigma(a)} e_m(0,\sigma(s)\left\{n(s)\int_0^s [K(\phi - \psi)(\tau)\Delta\tau]\Delta s\right\} + e_m(t,0)\int_0^t e_m\big(0,\sigma(s)\big)\left\{n(s)\int_0^s [K(\phi - \psi)(\tau)\Delta\tau]\Delta s\right\} \right|$$

$$\leq \frac{e_m(t,0)e_m(\sigma(a),0)}{e_m(\sigma(a),0)-1}\left| \int_0^{\sigma(a)} e_m(0,\sigma(s)\left\{n(s)\int_0^s \left[K(\phi - \psi)(\tau\,)\Delta\tau\,\right]\Delta s\right\}\right| + e_m(t,0)\left|\int_0^t e_m\big(0,\sigma(s)\big)\left\{n(s)\int_0^s \left[K(\phi - \psi)(\tau\,)\Delta\tau\,\right]\Delta s\right\}\right|,$$

where the integral

$$\int_0^t e_m(o,\sigma(s)\Delta s \;=\; \int_0^t e_{\ominus m}(\sigma(s),0)\Delta s$$

$$= \int_0^t [1 + \mu(s) \ominus m(s)]e_{\ominus m}(s,0)\Delta s$$

$$= \int_0^t \left[1 - \frac{\mu(s)m(s)}{1+\mu(s)m(s)}\right] e_{\ominus m}(s,0)\Delta s$$

$$= \int_0^t \frac{1}{1 + \mu(s)m(s)} e_{\ominus m}(s,0)\Delta s$$

$$= \int_0^t \frac{\ominus m}{-m(s)} e_{\ominus m}(s,0)\Delta s$$

$$\leq -\frac{1}{M_0}[e_{\ominus m}(t,0) - 1]$$

$$= \frac{1 - e_{\ominus m}(t,0)}{M_0}.$$

Thus, we have

$$\parallel (S\phi)(t) - (S\psi)(t) \parallel \leq \frac{e_m(t,0)e_m(\sigma(a),0)}{[e_m\,(\sigma(a),0)-1]}\left[1 - e_{\ominus m}(\sigma(a),0)\right]\frac{\sigma(a)N_0K_0}{M_0} \parallel \phi - \psi \parallel$$

$$+ e_m(t,0)\left[1 - e_{\ominus m}(t,0)\right]\frac{\sigma(a)N_0K_0}{M_0} \parallel \phi - \psi \parallel$$

$$= \frac{e_m(t,0)e_m(\sigma(a),0)}{[e_m\,(\sigma(a),0)-1]}\left[1 - e_m\big(0,\sigma(a)\big)\right]\frac{\sigma(a)N_0K_0}{M_0} \quad \parallel \phi - \psi \parallel$$

$$+ e_m(t,0)[1 - e_m(0,t)]\frac{\sigma(a)N_0K_0}{M_0} \parallel \phi - \psi \parallel$$

$$= e_m(t,0)\frac{\sigma(a)N_0K_0}{M_0} \parallel \phi - \psi \parallel + [e_m(t,0) - 1]\frac{\sigma(a)N_0K_0}{M_0} \parallel \phi - \psi \parallel$$

$$= \frac{\sigma(a)N_0K_0}{M_0}[2e_m(t,0) - 1] \parallel \phi - \psi \parallel$$

This implies by condition $\frac{\sigma(a)N_0K_0[2e_m(t,0)-1]}{M_0} < 1$ that S is a contraction on X and therefore by Banach’s contraction principle there exists exactly one $x(t) \in X$ such that

$$x(t) = e_m(t,0)\left\{\frac{d}{e_m(\sigma(a),0) - 1} - \frac{e_m(\sigma(a),0)}{e_m(\sigma(a),0) - 1}\int\limits_0^{\sigma(a)} e_m(0,\sigma(s)n(s)(Kx)(s)\Delta s \right.$$
$$\left. + \int\limits_0^{t} e_m(0,\sigma(s)n(s)(Kx)(s)\Delta s \right\}, t \in J.$$

Next, we shall show that x(t) is a solution of (3.6). We have

$$x^{\Delta}(t) = m(t)e_m(t,0)\left\{\frac{d}{e_m(\sigma(a),0) - 1}\right.$$
$$- \frac{e_m(\sigma(a),0)}{e_m(\sigma(a),0) - 1}\int\limits_0^{\sigma(a)} e_m(0,\sigma(s)n(s)(Kx)(s)\Delta s$$
$$\left. + \int\limits_0^{t} e_m(0,\sigma(s)n(s)(Kx)(s)\Delta s\right\} + e_m(\sigma(t),0)e_m(0,\sigma(t))n(t)(Kx)(t)$$
$$= m(t)x(t) + n(t)\int\limits_0^{t} k(t,s)x(s)\,\Delta s.$$

Moreover

$$x\big(\sigma(a)\big) = \frac{e_m(\sigma(a),0)d}{e_m(\sigma(a),0) - 1} - \frac{e_m(\sigma(a),0)e_m(\sigma(a),0)}{e_m(\sigma(a),0) - 1}\int_0^{\sigma(a)} e_m(0,\sigma(s)n(s)(Kx)(s)\Delta s$$
$$+ \; e_m(\sigma(a),0)\int\limits_0^{\sigma(a)} e_m(0,\sigma(s)n(s)(Kx)(s)\Delta s$$
$$= \frac{e_m(\sigma(a),0)d}{e_m(\sigma(a),0) - 1} + \left[1 - \frac{e_m(\sigma(a),0)}{e_m(\sigma(a),0) - 1}\right] e_m(\sigma(a),0)\int_0^{\sigma(a)} e_m(0,\sigma(s)n(s)(Kx)(s)\Delta s$$
$$= \frac{e_m(\sigma(a),0)d}{e_m(\sigma(a),0) - 1} - \frac{e_m(\sigma(a),0)}{e_m(\sigma(a),0) - 1}\int_0^{\sigma(a)} e_m(0,\sigma(s)n(s)(Kx)(s)\Delta s$$
$$= \frac{e_m(\sigma(a),0)}{e_m(\sigma(a),0) - 1}\left[d - \int_0^{\sigma(a)} e_m(0,\sigma(s)n(s)(Kx)(s)\Delta s\right]$$
$$= x(0) + d$$

Definition 3.5. Functions $\alpha, \beta \in C'(J,\mathbb{R})$ are said to be the lower and upper solutions of (1.1) respectively if

$$\alpha^{\Delta}(t) \leq f(t,\alpha(t),K\alpha(t), t \in J,$$

$$\alpha(0) + \int_0^{\sigma(a)} \alpha(s)\Delta s \;\; \leq \; \alpha\big(\sigma(a)\big).$$

$$\beta^{\Delta}(t) \geq \; f\big(t, \beta(t), K\beta(t)\big), t \; \in \; J,$$

$$\beta(0) \; + \; \int_0^{\sigma(a)} \; \beta(s)\Delta s \; \geq \; \beta(\sigma(a)).$$

Let $\Omega \; = \; \{y \; : \; \beta_0(t) \; \leq \; y(t) \; \leq \; \alpha_0(t), t \; \in \; J\}$ if $\beta_0(t) \; \leq \; \alpha_0(t)$ for $t \; \in \; J.$ We introduce the following assumptions.

(H1) $\alpha_0, \; \beta_0 \; \in \; C'(\,J, \mathbb{R})$ are lower and upper solutions of (1.1) respectively and $\beta_0 \; \leq \; \alpha_0$ for $t \; \in \; J$

(H2) $f \; \in \; C(J \times \mathbb{R} \times \Omega, \; \mathbb{R})$

(H3) there exists two positive, rd-continuous functions $m(t), n(t)$ such that

$f(t, x_2, u_2) - \; f(t, x_1, u_1) \;\; \leq \; m(x_2 \; - \; x_1) \; + \; n(u_2 \; - \; u_1)$ if $\beta_0 \; \leq \; x_1 \; \leq \; x_2 \; \leq \; \alpha_0, \; K\beta_0 \; \leq \; u_1 \; \leq \; u_2 \; \leq \; K\alpha_0, \; t \; \in \; J$

(H4) $\frac{N_0 K_0 e_m(\sigma(a),0)[e_m(\sigma(a),0)-1]}{M_0^2(1+L_0)} \; \leq \; 1,$ where $M_0 \; = \min_{t \in j}\{m(t)\},$ $K_0 \; = \; max\{k(t,s) \colon t, s \; \in \; J \; \times \; J\} > \; 0, \; N_0 \; = \max_{t \in j}\{n(t)\}$ and $L_0 \; = \min_{t \in j}\{m(t)\mu(t)\}$

(H5) $\frac{\sigma(a) N_0 K_0 [2e_m(t,0)-1]}{M_0} \; < \; 1,$ where $M_0 \; = \max_{t \in j}\{m(t)\}, N_0 \; = \max_{t \in j}\{n(t)\}$ and $K_0 \; = \; max\{k(t,s) \; : \; t, s \; \in \; J \; \times \; J\} \; > \; 0.$

Lemma 3.6. Assume that (H1) − (H5) hold. If

$$y^{\Delta}(t) \; - \; m(t)y(t) \; - \; n(t) \int_0^t k(t,s)y(s)\Delta s$$

$$= \; f(t, \alpha_0(t), K\alpha_0(t)) \; - m(t)\alpha_0(t) \; - \; n(t) \int_0^t k(t,s)\alpha_0(s)\Delta s \;\; , t \; \in \; J$$

$$y(0) + \int_0^{\sigma(a)} \alpha_0(s)\Delta s = y\big(\sigma(a)\big)$$

$$z^{\Delta}(t) \; - \; m(t)z(t) \; - \; n(t) \int_0^t k(t,s)z(s)\Delta s$$

$$= \; f(t, \beta_0(t), K\beta_0(t)) \; - m(t)\beta_0(t) \; - \; n(t) \int_0^t k(t,s)\beta_0(s)\Delta s \;\; , t \in J$$

$$z(0)+\int_0^{\sigma(a)}\beta_0(s)\Delta s=z(\sigma(a))$$

then

$$\beta_0(t)\le z(t)\le y(t)\le\alpha_0(t), t\in J \tag{3.7}$$

and y, z are lower and upper solutions of (1.1) respectively.

Proof. From lemma 3.4, we know that there exist unique solutions for y and z.

Put $p = y - \alpha_0$, $q = \beta_0 - z$, then

$$\begin{aligned}p(\sigma(a)) &= y(\sigma(a)) - \alpha_0(\sigma(a))\\ &\le y(0) + \int_0^{\sigma(a)}\alpha_0(s)\Delta s - \alpha_0(0) - \int_0^{\sigma(a)}\alpha_0(s)\Delta s\\ &= y(0) - \alpha_0(0)\\ &= p(0),\end{aligned}$$

$$\begin{aligned}q(\sigma(a)) &= \beta_0(\sigma(a)) - z(\sigma(a))\\ &\le \beta_0(0) + \int_0^{\sigma(a)}\beta_0(s)\Delta s - z(0) - \int_0^{\sigma(a)}\beta_0(s)\Delta s\\ &= \beta_0 0) - z(0)\\ &= q(0),\end{aligned}$$

and

$$p^{\Delta}(t) = y^{\Delta}(t) - \alpha_0^{\Delta}(t)$$

$$\begin{aligned}&\ge f(t,\alpha_0(t),K\alpha_0(t)) + m(t)y(t) + n(t)\int_0^t k(t,s)y(s)\Delta s\\ &\quad -m(t)\alpha_0(t) - n(t)\int_0^t k(t,s)\alpha_0(s)\Delta s - f(t,\alpha_0(t),K\alpha_0(t))\\ &= m(t)[y(t)-\alpha_0(t)] + n(t)\int_0^t k(t,s)[y(s)-\alpha_0(s)]\Delta s\\ &= m(t)p(t) + n(t)\int_0^t k(t,s)p(s)\Delta s,\ t\in J\end{aligned}$$

$$\begin{aligned}q^{\Delta}(t) &= \beta_0^{\Delta}(t) - z^{\Delta}(t)\\ &\ge f(t,\beta_0(t),K\beta_0(t)) - f(t,\beta_0(t),K\beta_0(t)) - m(t)z(t) -\\ &\quad n(t)\int_0^t k(t,s)z(s)\Delta s + m(t)\beta_0(t) + n(t)\int_0^t k(t,s)\beta_0(s)\Delta s\\ &= m(t)[\beta_0(t) - z(t)] + n(t)\int_0^t k(t,s)[\beta_0(s) - z(s)]\Delta s\end{aligned}$$

$$= m(t)q(t) + n(t)\int_0^t k(t,s)q(s)\ ,t \in J.$$

From lemma 3.1, we have $p(t) \le 0,\ q(t) \le 0,\ t \in J$ and so

$y(t) \le \alpha_0(t),\ \beta_0(t) \le z(t),\ t \in J$. Now let $p(t) = z - y$, then

$$\begin{aligned}
p(\sigma(a)) &= z(\sigma(a)) - y(\sigma(a)) \\
&= z(0) + \int_0^{\sigma(a)} \beta_0(s)\Delta s \ - y(0) - \int_0^{\sigma(a)} \alpha_0(s)\Delta s \\
&= z(0) - y(0) + \int_0^{\sigma(a)} [\beta_0(s) - \alpha_0(s)]\Delta s \\
&\le z(0) - y(0) \\
&= p(0)
\end{aligned}$$

From assumption (H3), we have

$$\begin{aligned}
p^{\Delta}(t) &= z^{\Delta}(t) - y^{\Delta}(t) \\
&= f\big(t,\beta_0(t),K\beta_0(t)\big) + m(t)z(t) + n(t)\int_0^t k(t,s)z(s)\Delta s \\
&\quad - m(t)\beta_0(t) - n(t)\int_0^t k(t,s)\beta_0(s)\Delta s - f\big(t,\alpha_0(t),K\alpha_0(t)\big) \\
&\quad - m(t)y(t) - n(t)\int_0^t k(t,s)y(s)\Delta s + m(t)\alpha_0(t) + n(t)\int_0^t k(t,s)\alpha_0(s)\Delta s \\
&\ge -m(t)[\alpha_0(t) - \beta_0(t)] - n(t)\int_0^t k(t,s)\ [\alpha_0(s) - \beta_0(s)]\Delta s \\
&\quad + m(t)[z(t) - \beta_0(t)] + n(t)\int_0^t k(t,s)\,[z(s) - \beta_0(s)]\Delta s \\
&\quad - m(t)[y(t) - \alpha_0(t)] - n(t)\int_0^t k(t,s)\ [y(s) - \alpha_0(s)]\Delta s \\
&= m(t)[z(t) - y(t)] + n(t)\int_0^t k(t,s)\,[z(s) - y(s)]\Delta s \\
&= m(t)p(t) + n(t)\int_0^t k(t,s)\,p(s)\Delta s.
\end{aligned}$$

By lemma 3.1, one can get $p(t) \le 0,\ t \in J$, then $z(t) \le y(t), t \in J$. It proves that (3.7) holds. Now we need to show that y, z are lower and upper solutions of (1.1) respectively. Using again assumption (H3) we have

$$\begin{aligned}
y^{\Delta}(t) &= f(t,\alpha_0(t),K\alpha_0(t)) + m(t)[y(t) - \alpha_0(t)] + n(t)\int_0^t k(t,s)\,[y(s) - \alpha_0(s)]\Delta s \\
&\quad - f(t,y(t),Ky(t)) + f(t,y(t),Ky(t)) \\
&\le m(t)[\alpha_0(t) - y(t)] + n(t)\int_0^t k(t,s)\,[\alpha_0(s) - y(s)]\Delta s + m(t)[y(t) - \\
&\quad \alpha_0(t)] + n(t)\int_0^t k(t,s)\,[y(s) - \alpha_0(s)]\Delta s + f(t,y(t),Ky(t)) \\
&= f(t,y(t),Ky(t))
\end{aligned}$$

$$z^{\Delta}(t) = f(t,\beta_0(t),K\beta_0(t)) + m(t)[z(t) - \beta_0(t)] + n(t)\int_0^t k(t,s)\,[z(s) - \beta_0(s)]\Delta s - f(t,z(t),Kz(t)) + f(t,z(t),Kz(t))$$

$$\geq -m(t)[z(t) - \beta_0(t)] - n(t)\int_0^t k(t,s)\,[z(s) - \beta_0(s)]\Delta s + m(t)[z(t) - \beta_0(t)] + n(t)\int_0^t k(t,s)\,[z(s) - \beta_0(s)]\Delta s + f(t,z(t),Kz(t))$$

$$= f(t,z(t),Kz(t)),$$

$$y(0) + \int_0^{\sigma(a)} y(s)\Delta s \leq y(0) + \int_0^{\sigma(a)} \alpha_0(s)\Delta s = y(\sigma(a)),$$

$$z(0) + \int_0^{\sigma(a)} z(s)\Delta s \geq z(0) + \int_0^{\sigma(a)} \beta_0(s)\Delta s = z(\sigma(a)).$$

It shows that y, z are lower and upper solutions of (1.1) respectively.

Theorem 3.7. Suppose that (H1) - (H5) hold. Then there exist monotone sequences $\{\alpha_n, \beta_n\}$ such that $\alpha_n \to \alpha,\ \beta_n \to \beta, t \in J$ as $n \to \infty$ and this convergence is uniformly and monotonically on J. Moreover, β, α are maximal and minimal solutions of (1.1) in $[\beta_0, \alpha_0] = \{u \in C'(J, R) : \beta_0 \leq u \leq \alpha_0\}$.

Proof. Consider

$$\alpha_{n+1}^{\Delta}(t) - m(t)\alpha_{n+1}(t) - n(t)\int_0^t k(t,s)\alpha_{n+1}(s)\Delta s$$

$$= f(t,\alpha_n(t),K\alpha_n(t)) - m(t)\alpha_n(t) - n(t)\int_0^t k(t,s)\alpha_n(s)\Delta s, t \in J,$$

$$\alpha_{n+1}(0) + \int_0^{\sigma(a)} \alpha_n(s)\Delta s = \alpha_{n+1}(\sigma(a))$$

$$\beta_{n+1}^{\Delta}(t) - m(t)\beta_{n+1}(t) - n(t)\int_0^t k(t,s)\beta_{n+1}(s)\Delta s$$

$$= f(t,\beta_n(t),K\beta_n(t)) - m(t)\beta_n(t) - n(t)\int_0^t k(t,s)\beta_n(s)\Delta s, t \in J,$$

$$\beta_{n+1}(0) + \int_0^{\sigma(a)} \beta_n(s)\Delta s = \beta_{n+1}(\sigma(a)) \tag{3.8}$$

for $n = 0, 1, 2, \ldots$. Lemma 3.6, shows $\beta_0(t) \leq \beta_1(t) \leq \alpha_1(t) \leq \alpha_0(t),\ t \in J$ and $\alpha_1,\ \beta_1$ are lower and upper solutions of (1.1) respectively. Assume that

$$\beta_0(t) \le \beta_1(t) \le \ldots \le \beta_k(t) \le \alpha_k(t) \le \ldots \alpha_1(t) \le \alpha_0(t), t \in J$$

for some $k \ge 1$ and let α_k, β_k be lower and upper solutions of (1.1) respectively. Then using again lemma 3.6, we get $\beta_k(t) \le \beta_{k+1}(t) \le \alpha_{k+1}(t) \le \alpha_k(t), t \in J$ and $\alpha_{k+1}(t), \beta_{k+1}(t)$ are lower and upper solutions of (1.1) respectively. By induction, we have

$$\beta_0(t) \le \beta_1(t) \le \ldots \le \beta_n(t) \le \alpha_n(t) \le \ldots \alpha_1(t) \le \alpha_0(t), t \in J$$

for all n.

Hence $\beta_n(t) \to \beta(t), \alpha_n(t) \to \alpha(t),\ t \in J$ if $n \to \infty$. Indeed, taking the limit $n \to \infty$ on both sides of (3.8), we know that α and β are solutions of (1.1).

Next, we are going to show that β, α are maximal and minimal solutions of (1.1) in $[\beta_0, \alpha_0]$.

To do it, we need to show that if $w(t)$ is any solution of (1.1) such that $\beta_0(t) \le w(t) \le \alpha_0(t),\ t \in J$ then $\beta_0(t) \le \beta(t) \le w(t) \le \alpha(t) \le \alpha_0(t),\ t \in J$. Assume that for some k,

$\beta_k(t) \le w(t) \le \alpha_k(t), t \in J$.

Let $p = \beta_{k+1}(t) - w(t),\ q = w(t) - \alpha_{k+1}(t)$. Then

$$\begin{aligned}
p(\sigma(a)) &= \beta_{k+1}(\sigma(a)) - w(\sigma(a)) \\
&= \beta_{k+1}(0) + \int_0^{\sigma(a)} \beta_k(s)\Delta s - w(0) - \int_0^{\sigma(a)} w(s)\Delta s \\
&= \beta_{k+1}(0) - w(0) + \int_0^{\sigma(a)} [\beta_k(s) - w(s)]\Delta s \\
&\le \beta_{k+1}(0) - w(0) \\
&= p(0),
\end{aligned}$$

$$\begin{aligned}
q(\sigma(a)) &= w(\sigma(a)) - \alpha_{k+1}(\sigma(a)) \\
&= w(0) + \int_0^{\sigma(a)} w(s)\Delta s - \alpha_{k+1}(0) - \int_0^{\sigma(a)} \alpha_k(s)\Delta s \\
&= w(0) - \alpha_{k+1}(0) + \int_0^{\sigma(a)} [w(s) - \alpha_k(s)]\Delta s \\
&\le w(0) - \alpha_{k+1}(0) \\
&= q(0),
\end{aligned}$$

From assumption (H3), we have

$$\begin{aligned}
p^{\Delta}(t) &= \beta_{k+1}^{\Delta}(t) - w^{\Delta}(t) \\
&= f\big(t, \beta_k(t), K\beta_k(t)\big) - f\big(t, w(t), Kw(t)\big) + m(t)[\beta_{k+1}(t) - \beta_k(t)] \\
&\quad + n(t) \int_0^t k(t,s)\, [\beta_{k+1}(s) - \beta_k(s)]\Delta s \\
&\ge -m(t)[w(t) - \beta_k(t)] - n(t) \int_0^t k(t,s)\, [w(s) - \beta_k(s)]\Delta s
\end{aligned}$$

$$+m(t)[\beta_{k+1}(t) - \beta_k(t)] + n(t)\int_0^t k(t,s)\,[\beta_{k+1}(s) - \beta_k(s)]\Delta s$$

$$= m(t)[\beta_{k+1}(t) - w(t)] + n(t)\int_0^t k(t,s)\,[\beta_{k+1}(s) - w(s)]\Delta s$$

$$= m(t)p(t) + n(t)\int_0^t k(t,s)\,p(s)\Delta s$$

$$q^{\Delta}(t) = w^{\Delta}(t) - \alpha_{k+1}^{\Delta}(t)$$

$$= f(t,w(t),Kw(t)) - f(t,\alpha_k(t),K\alpha_k(t)) - m(t)[\alpha_{k+1}(t) - \alpha_k(t)]$$

$$-n(t)\int_0^t k(t,s)\,[\alpha_{k+1}(s) - \alpha_k(s)]\Delta s$$

$$\geq -m(t)[\alpha_k(t) - w(t)] - n(t)\int_0^t k(t,s)\,[\alpha_k(s) - w(s)]\Delta s$$

$$-m(t)[\alpha_{k+1}(t) - \alpha_k(t)] - n(t)\int_0^t k(t,s)\,[\alpha_{k+1}(s) - \alpha_k(s)]\Delta s$$

$$= m(t)[w(t) - \alpha_{k+1}(t)] + n(t)\int_0^t k(t,s)\,[w(s) - \alpha_{k+1}(s)]\Delta s$$

$$= m(t)q(t) + n(t)\int_0^t k(t,s)\,q(s)\Delta s,\ t \in J.$$

By lemma 3.1, we can obtain $p(t) \leq 0,\ q(t) \leq 0,\ t \in J$, this shows $\beta_{k+1}(t) \leq w(t) \leq \alpha_{k+1}(t), t \in J$. It proves, by induction, that $\beta_n(t) \leq w(t) \leq \alpha_n(t), t \in J$, for all n. Taking the limit $n \mapsto \infty$, we have $\beta_0(t) \leq \beta(t) \leq w(t) \leq \alpha(t) \leq \alpha_0(t),\ t \in J$, so the assertion of Theorem 3.7 is true. The proof is complete.

Conclusion: The solutions to a system of integro-dynamical equations with integral boundary conditions on Time Scales have been obtained in this work. The solutions are observed to satisfy boundedness and stability which leads to the observability of the system. Hence the solutions may be treated significant in analysing the systems which have lower solution is greater than higher solution.

Future Work: This work may be extended to the fields of fractional order integro-dynamical equations with integral boundary conditions and integro-dynamical equations with fractional order integral boundary conditions.

Acknowledgement: The authors thank their respective college management for their support and encouragement.

Refrences

[1] Bohner, M., Peterson, A. Advances in Dynamic Equations on Time Scales, Birkhauser, Boston, 2003.

[2] Burton, T.A. An integro differential equations, Proc. Amer. Math. Soc., 79 (1980), 393-399.

[3] Guo, D. Initial value problems for integro-differential equations of Volterra type in Banach spaces, J. Appl. Math. Stoch. Anal. 7 (1994)13-23.

[4] Guoping Chen and et al. Monotone Iterative Technique for a Class of Integro-Differential Equations with Integral Boundary Conditions, Int. Journal of Math. Anal, Vol. 1, 2007, no. 3, 113-122.

[5] Hilger, S. Ein Mabkettenkalkulmit Anwendung auf Zentrumsmannigfaltigkeiten, PhD thesis, Universitat Wurzburg, 1988.

[6] Lakshmikantham, V and et al. Dynamic Systems on Measure Chains, Kluwer, 1996.

[7] Ramana, G.V; Deekshitulu, G.V.S.R. (2017). Controllability, Observability and Stability of Volterra Type Non-Linear Matrix Integro-Dynamic System on Time Scales. IJET, 7, 115-120.

[8] Ramana, G.V; Deekshitulu, G.V.S.R. (2017). Asymptotic Stability of Solution of Lyapunov Type Matrix Volterra Integro-Dynamic System on Time Scales. IJET, 7, 179-185.

[9] Ramana, G.V and et al. (2025). Stability Analysis of Two Point Boundary Value Problem on Time Scales, Communications on Applied Nonlinear Analysis.

[10] Sabrina Streipert. (2023). Dynamic equations on time scales. Nonlinear systems.

[11] Song, G. Initial value problems for systems of integro-differential equations in Banach spaces, J. Math. Anal. Appl. 264(2001) 68-75.

[12] Xing, Y and et al. Initial value problem for first order integro-differential equation of Volterra type on time scales, Nonlinear Anal.60(3)(2005) 429-442.

[13] Yepeng Xing and et al. Periodic boundary value problems of integro-differential equation of Volterra type on time scales, Nonlinear Anal. 68(2008) 127-138