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A New Family of Power Function- Lindley Distribution

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Abstract:

As data types evolved many problems related to the representation and modeling of this data began to emerge creating a need to find solutions. Over the past few years many methods have been proposed to generate new families of continuous distributions and study them in various applications including medicine engineering physics economics finance biology and environmental studies and the solution is to extend these distributions.

In recent years many methods have been proposed to create new distributions. We generate a new of family distributions named PF- LD . Some properties of our distribution are introduced. Such as rth moment function, reliability function, hazard rate function, Shannon entropy function and stress-strength models. A simulation study was carried out for this compare the performance of different estimates from PF-LD. We create random variable from PF-LD for different sample sizes and different parameters values. The simulation study was repeated every (1000N) with a sample sizes (n=30,60,160). And numerical simulations can be performed for different parameter values by solving nonlinear equations and calculating MSE for parameter estimation. The results we obtained using the Matlab code observed a lower MSE when the default parameter values were small. (β=0.5, λ=0.5, α =0.5). for all sample size . Based on the knowledge and information obtained from this research, we recommend, applying the proposed distribution to real data.

Keywords: distribution Power function, , Lindley distribution, reliability function, hazard rate function, Shannon entropy, stress-strength.

1. Interdiction

Problems related to life data analysis and model failure involve multiple deterministic distributions. The power-law distribution and the Lindley distribution are the most commonly used distributions in life data analysis. The development of data types and the appearance of many problems in modeling these data have made it necessary to find more flexible distributions to deal with the data and represent them more accurately. One of the most commonly used distributions is the power function distribution, and this is especially true with Pareto distribution as it is used to solve the lack of time for equipment and engineering data. Researchers Meniconi and Barge (1995) have demonstrated that the power function has a good distribution. Najm, A (2022), The Lindley-Rayleigh (LR) lifetime distribution was proposed by him to make the Lindley distribution more flexible by adding another parameter. The main objective of this research is to propose a new probability distribution called power function – Lindley distribution (P-LD).

Suppose that X has a Power function distribution, the PDF and CDF of the distribution can be written as; Salman, M. (2020) and (Forbes, C et al 2015), .

$$f(\,x;\theta,\alpha\,)=\frac{\theta x^{\theta-1}}{\alpha^{\theta}} \qquad 0<x<\alpha\,, \qquad \alpha\,,\theta>0 \tag{1}$$

The corresponding CDF, is given by;

$$F(x;\theta,\alpha)=\left(\frac{x}{\alpha}\right)^{\theta} \qquad 0<x<\alpha\,, \qquad \alpha\,,\theta>0 \tag{2}$$

2. H-Q Distribution

Assume that $Q(x) and\, q(x)$ are any continuous distribution, represent the cdf and pdf respectively for any continuous distribution in the interval [0 ∞). The general formula of the reliability function of this class is given by Boshi M.A.(2019).

$$R(x)_{H-Q}=\int_{0}^{-\ln Q(x)} h(x)\,dx=H(-\,ln\,Q(x)) \tag{3}$$

Depending on (3), A general CDF formula for this family is presented;

$$F(x)_{H-Q}=1-R(x)_{H-Q}=1-P(-\ln L(x)) \tag{4}$$

And the pdf given as;

$f(x)_{H-Q}=\frac{d}{dx}\left[F(x)_{H-Q}\right]=\frac{d}{dx}\left[R(x)_{H-Q}\right]$, will be,

$$f(x)_{H-Q}=\frac{q(x)}{Q(x)}h(-\,ln\,Q(x)) \tag{5}$$

3- Power function distribution- Q Distribution

Here we introduce a new generation of time series distributions based on the power function (PF) distribution in the interval [0,∞].

Let $H(.)and\ h(.)$ That mentioned in (3), (4) and (5) Hence the cdf and pdf of the distribution will be, Nadarajah and Rocha (2016).

Remember (1) and (2), with tow positive parameters $\theta and\ \alpha$ as,

$$H(-\ln Q(x);\ \theta,\alpha)=1-\left(\frac{-lnQ(x)}{\alpha}\right)^{\theta} \tag{6}$$

$$h(-\ln Q(x);\ \theta,\alpha)=\frac{\theta[-lnQ(x)]^{\theta-1}}{\alpha^{\theta}} \tag{7}$$

By substituting (6) and (5) into (4) and (5) we obtain the cdf and pdf of a new continuous distribution called the q-power function (denoted by PF-Q).

$$F(x)_{PF-Q}=1-\left[\frac{-lnQ(x)}{\alpha}\right]^{\theta} \tag{8}$$

$$f(x)_{PF-Q}=\frac{q(x)}{Q(x)}\frac{\theta[-lnQ(x)]^{\theta-1}}{\alpha^{\theta}} \tag{9}$$

4- Power function- Lindley distribution

As a first case, assume that Q(x) and q(x) in (8) and (9) represents the cdf and pdf of Lindley distribution (Abdullah, N. 2018), recalls (6) and (7) with one positive parameter β, given respectively by,

$$Q(x;\ \beta) = 1 - \left[1 + \frac{\beta}{1+\beta}x\right]e^{-\beta x} \quad (10)$$

$$q(x;\ \beta) = \frac{\beta^2}{(1+\beta)}(1+x)e^{-\beta x} \quad (11)$$

According to (8), the cdf of new distribution called Power function- Lindley distribution (symbolized by PF-LD) will be,

$$F(x)_{PF-LD} = 1 - \left[\frac{-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)}{\alpha}\right]^{\theta} \quad (12)$$

And according to (9) the pdf as,

$$f(x)_{PF-LD} = \frac{\frac{\beta^2}{(1+\beta)}(1+x)e^{-\beta x}}{1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}} \frac{\theta\left[-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right]^{\theta-1}}{\alpha^{\theta}} \quad (13)$$

The reliability function of PF-LD can be obtained as,

$$\mathrm{R(x)_{PF-LD}} = 1 - F(x)_{PF-LD}$$

$$\mathrm{R(x)_{PF-LD}} = 1 - \left\{1 - \left[\frac{-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)}{\alpha}\right]^{\theta}\right\}$$

The hazard rate function of PF-LD as,

$$\mathrm{H(x)_{PF-LD}} = \frac{\mathrm{f(x)_{PF-LD}}}{\mathrm{R(x)_{PF-LD}}}$$

$$H(x)_{PF-LD} = \frac{\frac{\frac{\beta^2}{(1+\beta)}(1+x)e^{-\beta x}}{1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}} \frac{\theta\left[-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right]^{\theta-1}}{\alpha^{\theta}}}{1-\left\{1-\left[\frac{-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)}{\alpha}\right]^{\theta}\right\}}$$

5- Properties of the $PF - LD$

First property that is $f(x)_{PF-LD}$ is pdf,

lim $x \to 0$

$$\lim_{x\to 0} F(x)_{PF-LD} = \lim_{x\to 0}\left\{1 - \left[\frac{-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)}{\alpha}\right]^{\theta}\right\} = 0$$

Now , lim $x \to \infty$

$$\lim_{x\to\infty} F(x)_{PF-LD} = \lim_{x\to\infty}\left\{1 - \left[\frac{-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)}{\alpha}\right]^{\theta}\right\} = 1$$

i- r-th Moment:

The function of r^{th} moment of $PF-LD$, can be obtained from $\int_0^\infty x^r f(x)_{PF-LD} dx$. According to (13). The r^{-th} moment of $PF-LD$ is given by,

$$E(X^r)_{PF-LD} = \int_0^\infty x^r \left\{ \frac{\frac{\beta^2}{(1+\beta)}(1+x)e^{-\beta x}}{1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}} \frac{\theta\left[-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right]^{\theta-1}}{\alpha^\theta} \right\} dx \quad (14)$$

Let, $I = -ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right), \quad II = (1+x)e^{-\beta x}$

$III = \left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)$

By using the formulas [5],

$$(1-u)^b = \sum_{i=0}^\infty (-1)^i C_i^b u^i; \quad |u| < 1, b > 0 \ , \quad (15)$$

And $e^{-u} = \sum_{k=0}^\infty \frac{(-1)^k}{k!} u^k$, (16)

$$[-\ln(1-z)]^p = \sum_{m=0}^\infty \sum_{s=0}^\infty C_m^p b_{s,m}\, z^{p+m+s} \quad (17)$$

Where $b_{s,m} = (sa_o)^{-1} \sum_l^s [m(l+1)-s]al\, b_{s-l,m,} b_{o,m} = {a_o}^m \ and \ a_s = (s+2)^{-1}$

$$(a+b)^n = \sum_{k=0}^n C_k^n a^{n-k} b^k = \sum_{k=0}^n C_k^n a^k b^{n-k} \quad (18)$$

$;n \geq 0 \quad where \quad C_k^n = \frac{n!}{k!(n-k)!}$ Called Binomial Coefficients

$$\ln(x) = 2\sum_{k=0}^\infty \frac{(x-1)^{2k+1}}{(x+1)^{2k+1}} \quad ,x > 0 \quad (19)$$

$$(a+u)^{-n} = \sum_{i=0}^\infty C_i^{-n}\, a^{-n-i} u^i \quad (20)$$

$where, \quad C_i^{-n} = (-1)^i \binom{n+i-1}{i}$, called invers binomial

$$ln(u) = \sum_{i=0}^\infty \frac{(-1)^i (u-1)^{i-1}}{i+1} \ ; \quad 0 < u \leq 2 \quad (21)$$

According to (17) I will be,

$$I = -ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)$$

$$I = \sum_{m=0}^\infty \sum_{s=0}^\infty C_m^1 b_{s,m} \left(\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)^{m+s+1}$$

$$= \sum_{m=0}^\infty \sum_{s=0}^\infty C_m^1 b_{s,m} \left(\left[1+\frac{\beta}{1+\beta}x\right]^{m+s+1} \left(e^{-\beta x}\right)^{m+s+1}\right)$$

According to and (18) I will be,

$$I = \sum_{m=0}^\infty \sum_{s=0}^\infty C_m^1 b_{s,m} \sum_{k=0}^{m+s+1} C_k^{m+s+1} \left(\frac{\beta}{1+\beta}x\right)^k e^{-\beta(m+s+1)x}$$

Now,

$II = (1+x)e^{-\beta x}$

According to (18) II will be,

$$II = \sum_{i=0}^{1} C_i^1 x^i\, e^{-\beta x}$$

According to (15) III will be,

$$III = \sum_{n=0}^{\infty} (-1)^n\, C_n^1 \left[1+\frac{\beta}{1+\beta}x\right]^n \left(e^{-\beta x}\right)^n$$

$$III = \sum_{n=0}^{\infty} (-1)^n\, C_n^1 \left[1+\frac{\beta}{1+\beta}x\right]^n \left(e^{-\beta x}\right)^n$$

Also according to (18) we get,

$III = \sum_{n=0}^{\infty} (-1)^n\, C_n^1 \sum_{u=0}^{n} C_u^n \left(\frac{\beta}{1+\beta}\right)^u x^u\, e^{-\beta n x}$

Substitutes I, II and III in (14) we get,

$$E(X^r)_{PF-LD} = \int_0^\infty x^r \left\{\frac{\theta \sum_{i=0}^{1} C_i^1 x^i e^{-\beta x} \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} C_m^1 b_{s,m} \sum_{k=0}^{m+s+1} C_k^{m+s+1}\left(\frac{\beta}{1+\beta}x\right)^k e^{-\beta(m+s+1)x}}{\alpha^\theta \sum_{n=0}^{\infty}(-1)^n C_n^1 \sum_{u=0}^{n} C_u^n \left(\frac{\beta}{1+\beta}\right)^u x^u e^{-\beta n x}}\right\} dx$$

$$= \int_0^\infty x^r \left\{\frac{\theta \sum_{i=0}^{1} C_i^1 x^i \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} C_m^1 b_{s,m} \sum_{k=0}^{m+s+1} C_k^{m+s+1}\left(\frac{\beta}{1+\beta}\right)^k x^k e^{-\beta(m+s+2-n)x}}{\alpha^\theta \sum_{n=0}^{\infty}(-1)^n C_n^1 \sum_{u=0}^{n} C_u^n \left(\frac{\beta}{1+\beta}\right)^u x^u}\right\} dx$$

$$= \int_0^\infty x^r \left\{\frac{\theta \sum_{i=0}^{1} C_i^1 \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} C_m^1 b_{s,m} \sum_{k=0}^{m+s+1} C_k^{m+s+1}\left(\frac{\beta}{1+\beta}\right)^k x^{k+i-u} e^{-\beta(m+s+2-n)x}}{\alpha^\theta \sum_{n=0}^{\infty}(-1)^n C_n^1 \sum_{u=0}^{n} C_u^n \left(\frac{\beta}{1+\beta}\right)^u}\right\} dx$$

$$= \int_0^\infty x^r \left\{\frac{\theta \sum_{i=0}^{1} C_i^1 \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} C_m^1 b_{s,m} \sum_{k=0}^{m+s+1} C_k^{m+s+1}\left(\frac{\beta}{1+\beta}\right)^k x^{(k+i-u+1)-1} e^{-\beta(m+s+2-n)x}}{\alpha^\theta \sum_{n=0}^{\infty}(-1)^n C_n^1 \sum_{u=0}^{n} C_u^n \left(\frac{\beta}{1+\beta}\right)^u}\right\} dx$$

$$= \left\{\frac{\theta \sum_{i=0}^{1} C_i^1 \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} C_m^1 b_{s,m} \sum_{k=0}^{m+s+1} C_k^{m+s+1}\left(\frac{\beta}{1+\beta}\right)^k \int_0^\infty x^{(r+k+i-u+1)-1} e^{-\beta(m+s+2-n)x} dx}{\alpha^\theta \sum_{n=0}^{\infty}(-1)^n C_n^1 \sum_{u=0}^{n} C_u^n \left(\frac{\beta}{1+\beta}\right)^u}\right\}$$

By using the formula, $\int_0^\infty x^{\alpha-1}\, e^{-\beta x} dx = \frac{\Gamma(\alpha)}{\beta^\alpha}$

Thus the function of r^th moment of the $PF-LD$, is given by,

$$E(X^r)_{PF-LD} = \left\{\frac{\theta \sum_{i=0}^{1} C_i^1 \sum_{m=0}^{\infty}\sum_{s=0}^{\infty} C_m^1 b_{s,m} \sum_{k=0}^{m+s+1} C_k^{m+s+1}\left(\frac{\beta}{1+\beta}\right)^k \Gamma(r+k+i-u+1)}{\alpha^\theta \sum_{n=0}^{\infty}(-1)^n C_n^1 \sum_{u=0}^{n} C_u^n \left(\frac{\beta}{1+\beta}\right)^u \beta^{r+k+i-u+1}}\right\} \qquad (22)$$

Depending on the particular $E(x^r)_{TR\_R}$; ($r = 1,2,3,4$), other properties of this distribution such as, $\left(mean\ \mu = \text{E}(x)\right)$, $variance$ $\left(var(x) = \sigma^2 = E(x^2) - \left(E(x)\right)^2\right)$ the skewness coefficient $\left(SK = \frac{E[X-\mu]^3}{\sigma^3} = \frac{E(x^2)-3\mu E(x^2)+2\mu^2}{(\sigma^2)^{\frac{3}{2}}}\right)$. Coefficient of kurtosis $\left(Kr = \frac{E(X-\mu)^4}{\sigma^4} = \frac{E(x^4)-4\mu E(x^3)+6\mu^2 E(x^2)-3\mu^3}{\sigma^4}\right)$

can be found where,

$$E(X)_{PF-LD} = \left\{\frac{\theta \sum_{i=0}^{1} C_i^1 \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} C_m^1 b_{s,m} \sum_{k=0}^{m+s+1} C_k^{m+s+1} \left(\frac{\beta}{1+\beta}\right)^k \Gamma(k+i-u+2)}{\alpha^\theta \sum_{n=0}^{\infty} (-1)^n C_n^1 \sum_{u=0}^{n} C_u^n \left(\frac{\beta}{1+\beta}\right)^u \beta^{k+i-u+2}}\right\}$$

$$E(X^2)_{PF-LD} = \left\{\frac{\theta \sum_{i=0}^{1} C_i^1 \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} C_m^1 b_{s,m} \sum_{k=0}^{m+s+1} C_k^{m+s+1} \left(\frac{\beta}{1+\beta}\right)^k \Gamma(k+i-u+3)}{\alpha^\theta \sum_{n=0}^{\infty} (-1)^n C_n^1 \sum_{u=0}^{n} C_u^n \left(\frac{\beta}{1+\beta}\right)^u \beta^{k+i-u+3}}\right\}$$

$$E(X^3)_{PF-LD} = \left\{\frac{\theta \sum_{i=0}^{1} C_i^1 \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} C_m^1 b_{s,m} \sum_{k=0}^{m+s+1} C_k^{m+s+1} \left(\frac{\beta}{1+\beta}\right)^k \Gamma(k+i-u+4)}{\alpha^\theta \sum_{n=0}^{\infty} (-1)^n C_n^1 \sum_{u=0}^{n} C_u^n \left(\frac{\beta}{1+\beta}\right)^u \beta^{k+i-u+4}}\right\}$$

$$E(X^4)_{PF-LD} = \left\{\frac{\theta \sum_{i=0}^{1} C_i^1 \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} C_m^1 b_{s,m} \sum_{k=0}^{m+s+1} C_k^{m+s+1} \left(\frac{\beta}{1+\beta}\right)^k \Gamma(k+i-u+5)}{\alpha^\theta \sum_{n=0}^{\infty} (-1)^n C_n^1 \sum_{u=0}^{n} C_u^n \left(\frac{\beta}{1+\beta}\right)^u \beta^{k+i-u+5}}\right\}$$

ii- Characteristic Function $\emptyset_X\ (t)_{PF-LD}$:

The characteristic function of PF-LD is given by,

$$E(\text{e}^{\text{itx}}) = \sum_{\text{r}=0}^{\infty} \frac{(\text{it})^{\text{r}}}{\text{r}!} \text{E}(\text{X}^{\text{r}})_{\text{PF-LD}}$$

Therefore, the characteristic function of PF-LD is given by follows,

$$\emptyset_X\ (t)_{PF-LD} = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left\{\frac{\theta \sum_{i=0}^{1} C_i^1 \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} C_m^1 b_{s,m} \sum_{k=0}^{m+s+1} C_k^{m+s+1} \left(\frac{\beta}{1+\beta}\right)^k \Gamma(r+k+i-u+1)}{\alpha^\theta \sum_{n=0}^{\infty} (-1)^n C_n^1 \sum_{u=0}^{n} C_u^n \left(\frac{\beta}{1+\beta}\right)^u \beta^{r+k+i-u+1}}\right\} \quad (23)$$

iii- Shannon Entropy

The function of (SE) of the PF-LD will be given by

$$-\int_0^\infty \ln\ (f(x)_{PF-LD}) f(x)_{PF-LD}\, dx$$

By applying natural logarithm of the PDF in (13), we will get.

$$ln(f(x)_{PF-LD}) = ln\left\{\frac{\frac{\beta^2}{(1+\beta)}(1+x)e^{-\beta x}}{1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}} \frac{\theta\left[-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right]^{\theta-1}}{\alpha^\theta}\right\}$$

$$= \begin{cases} \ln\left(\frac{\beta^2}{(1+\beta)}(1+x)e^{-\beta x}\theta\left[-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right]^{\theta-1}\right) \\ -ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)+ln(\alpha^{\theta}) \end{cases}$$

$$= \begin{cases} ln\left(\frac{\beta^2}{(1+\beta)}\right)+ln(1+x)-\beta x+\ln(\theta) \\ +(\theta-1)ln\left[-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right] \\ -ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)+ln(\alpha^{\theta}) \end{cases} \quad (24)$$

Let $V = ln(1+x),\ VI = -ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)$

and $VII = ln\left[ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right]$

According to (19) V will be,

$$\mathrm{V} = 2\sum_{q=0}^{\infty}\frac{x^{2q+1}}{(x+2)^{2q+1}} = x^{2q+1}(x+2)^{-(2q+1)}$$

According to (20) V will be,

$$V = x^{2q+1}\sum_{i=0}^{\infty}\mathrm{C}_i^{-(2q+1)}\,2^{-(2q+1)-i}x^{i}$$

$$= \sum_{i=0}^{\infty}\mathrm{C}_i^{-(2q+1)}\,2^{-(2q+1)-i}x^{i+2q+1}$$

Now according to (17) VI will be,

$$VI = -ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)$$

$$VI = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}C_m^1 b_{s,m}\left(\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)^{m+s+1}$$

$$= \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}C_m^1 b_{s,m}\left(\left[1+\frac{\beta}{1+\beta}x\right]^{m+s+1}\left(e^{-\beta x}\right)^{m+s+1}\right)$$

According to and (18) VI will be,

$$VI = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}C_m^1 b_{s,m}\sum_{k=0}^{m+s+1}\mathrm{C}_k^{m+s+1}\left(\frac{\beta}{1+\beta}x\right)^{k}e^{-\beta(m+s+1)x}$$

According to and (16) we get,

$$VI = \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}C_m^1 b_{s,m}\sum_{k=0}^{m+s+1}\mathrm{C}_k^{m+s+1}\left(\frac{\beta}{1+\beta}x\right)^{k}\sum_{t=0}^{\infty}\frac{(-1)^t}{t!}(\beta(m+s+1)x)^{t}$$

$$= \sum_{m=0}^{\infty}\sum_{s=0}^{\infty}C_m^1 b_{s,m}\sum_{k=0}^{m+s+1}\mathrm{C}_k^{m+s+1}\left(\frac{\beta}{(1+\beta)}\right)^{k}\sum_{t=0}^{\infty}\frac{(-1)^t}{t!}\beta^{t}(m+s+1)^{t}x^{k+t}$$

Now,

$$VII = ln\left[-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right]$$

According to and (17) VII will be,

$$VII = ln\left[\sum_{n=0}^{\infty}\sum_{v=0}^{\infty} C_n^1 b_{v,n}\left(\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)^{1+n+v}\right]$$

$$VII = ln\left[\sum_{n=0}^{\infty}\sum_{v=0}^{\infty} C_n^1 b_{v,n}\left(1+\frac{\beta}{1+\beta}x\right)^{1+n+v} e^{-\beta(1+n+v)x}\right]$$

According to and (21) VII will be,

$$ln(u) = \sum_{j=0}^{\infty}\frac{(-1)^j}{j+1}\left(\sum_{n=0}^{\infty}\sum_{v=0}^{\infty} C_n^1 b_{v,n}\left(1+\frac{\beta}{1+\beta}x\right)^{1+n+v} e^{-\beta(1+n+v)x}-1\right)^{j-1}$$

According to (18) we get,

$$VII = \sum_{p=0}^{j-1}\sum_{j=0}^{\infty}\frac{(-1)^{2j-p-1}}{j+1}C_p^{j-1}\left(\sum_{n=0}^{\infty}\sum_{v=0}^{\infty} C_n^1 b_{v,n}\left(1+\frac{\beta}{1+\beta}x\right)^{1+n+v} e^{-\beta(1+n+v)x}\right)^p$$

$$= \left\{\begin{matrix}\sum_{p=0}^{j-1}\sum_{j=0}^{\infty}\frac{(-1)^{2j-p-1}}{j+1}C_p^{j-1} \\ \left(\sum_{n=0}^{\infty}\sum_{v=0}^{\infty} C_n^1 b_{v,n}\right)^p\left(1+\frac{\beta}{1+\beta}x\right)^{p(1+n+v)} e^{-\beta p(1+n+v)x}\end{matrix}\right\}$$

According to (16) and(18) we get,

$$VII = \left\{\begin{matrix}\sum_{p=0}^{j-1}\sum_{j=0}^{\infty}\frac{(-1)^{2j-p-1}}{j+1}C_p^{j-1} \\ \left(\sum_{n=0}^{\infty}\sum_{v=0}^{\infty} C_n^1 b_{v,n}\right)^p \\ \sum_{z=0}^{p(1+n+v)} C_z^{p(1+n+v)}\left(\frac{\beta}{1+\beta}\right)^z \\ \sum_{w=0}^{\infty}\frac{(-1)^w}{w!}\left(\beta p(1+n+v)\right)^w x^{z+w}\end{matrix}\right\}$$

substitutes V , VI and VII in (24) we get,

$$ln(f(x)_{PF-LD}) = \left\{\begin{matrix} ln\left(\frac{\beta^2}{(1+\beta)}\right)+\sum_{i=0}^{\infty} C_i^{-(2q+1)}\,2^{-(2q+1)-i}x^{i+2q+1}-\beta x+ln\,(\theta) \\ +(\theta-1)\left[\begin{matrix}\sum_{p=0}^{j-1}\sum_{j=0}^{\infty}\frac{(-1)^{2j-p-1}}{j+1}C_p^{j-1} \\ \left(\sum_{n=0}^{\infty}\sum_{v=0}^{\infty} C_n^1 b_{v,n}\right)^p \\ \sum_{z=0}^{p(1+n+v)} C_z^{p(1+n+v)}\left(\frac{\beta}{1+\beta}\right)^z \\ \sum_{w=0}^{\infty}\frac{(-1)^w}{w!}\left(\beta p(1+n+v)\right)^w x^{z+w}\end{matrix}\right] \\ +\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} C_m^1 b_{s,m}\sum_{k=0}^{m+s+1} C_k^{m+s+1}\left(\frac{\beta}{1+\beta}\right)^k\sum_{t=0}^{\infty}\frac{(-1)^t}{t!}\beta^t(m+s+1)^t x^{k+t}\end{matrix}\right\}$$

Now, the function of Shannon entropy of PF-LD can be given:

$$SH_{PF-LD} = -\int_0^\infty \left\{ \begin{array}{c} ln\left(\frac{\beta^2}{(1+\beta)}\right) + \sum_{i=0}^{\infty} C_i^{-(2q+1)}\, 2^{-(2q+1)-i} x^{i+2q+1} - \beta x + ln\,(\theta) \\ +(\theta-1)\left[\begin{array}{c} \sum_{p=0}^{j-1}\sum_{j=0}^{\infty} \frac{(-1)^{2j-p-1}}{j+1} C_p^{j-1} \\ \left(\sum_{n=0}^{\infty}\sum_{v=0}^{\infty} C_n^1 b_{v,n}\right)^p \\ \sum_{z=0}^{p(1+n+v)} C_z^{p(1+n+v)} \left(\frac{\beta}{1+\beta}\right)^z \\ \sum_{w=0}^{\infty} \frac{(-1)^w}{w!}\left(\beta p(1+n+v)\right)^w x^{z+w} \end{array} \right] \\ +\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} C_m^1 b_{s,m} \sum_{k=0}^{m+s+1} C_k^{m+s+1}\left(\frac{\beta}{1+\beta}\right)^k \sum_{t=0}^{\infty}\frac{(-1)^t}{t!}\beta^t (m+s+1)^t x^{k+t} \end{array} \right\} f(x)_{PF-LD}\, dx$$

Therefore, the function of Shannon entropy of PF-LD can be obtained as,

$$SH_{PF-LD} = \left\{ \begin{array}{c} -ln\left(\frac{\beta^2}{(1+\beta)}\right) + \sum_{i=0}^{\infty} C_i^{-(2q+1)}\, 2^{-(2q+1)-i} \int_0^\infty x^{i+2q+1} f(x)_{PF-LD} dx \\ +\beta\int_0^\infty x\, dx - ln\,(\theta) \\ -(\theta-1)\left[\begin{array}{c} \sum_{p=0}^{j-1}\sum_{j=0}^{\infty} \frac{(-1)^{2j-p-1}}{j+1} C_p^{j-1} \\ \left(\sum_{n=0}^{\infty}\sum_{v=0}^{\infty} C_n^1 b_{v,n}\right)^p \\ \sum_{z=0}^{p(1+n+v)} C_z^{p(1+n+v)} \left(\frac{\beta}{1+\beta}\right)^z \\ \sum_{w=0}^{\infty} \frac{(-1)^w}{w!}\left(\beta p(1+n+v)\right)^w \int_0^\infty x^{z+w} f(x) dx \end{array} \right] \\ -\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} C_m^1 b_{s,m} \sum_{k=0}^{m+s+1} C_k^{m+s+1}\left(\frac{\beta}{1+\beta}\right)^k \\ \sum_{t=0}^{\infty}\frac{(-1)^t}{t!}\beta^t (m+s+1)^t \int_0^\infty x^{k+t} f(x) dx \end{array} \right\}$$

$$SH_{PF-LD} = \left\{ \begin{array}{c} -ln\left(\frac{\beta^2}{(1+\beta)}\right) + \sum_{i=0}^{\infty} C_i^{-(2q+1)}\, 2^{-(2q+1)-i} E(x^{i+2q+1}) \\ +\beta E(x) - ln\,(\theta) \\ -(\theta-1)\left[\begin{array}{c} \sum_{p=0}^{j-1}\sum_{j=0}^{\infty} \frac{(-1)^{2j-p-1}}{j+1} C_p^{j-1} \\ \left(\sum_{n=0}^{\infty}\sum_{v=0}^{\infty} C_n^1 b_{v,n}\right)^p \\ \sum_{z=0}^{p(1+n+v)} C_z^{p(1+n+v)} \left(\frac{\beta}{1+\beta}\right)^z \\ \sum_{w=0}^{\infty} \frac{(-1)^w}{w!}\left(\beta p(1+n+v)\right)^w E(x^{z+w}) \end{array} \right] \\ -\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} C_m^1 b_{s,m} \sum_{k=0}^{m+s+1} C_k^{m+s+1}\left(\frac{\beta}{1+\beta}\right)^k \\ \sum_{t=0}^{\infty}\frac{(-1)^t}{t!}\beta^t (m+s+1)^t E(x^{k+t}) \end{array} \right\}$$

Where,

$E(x^{i+2q+1})$, $E(x)$, $E(x^{z+w})$, and $E(x^{k+t})$ as in (22) with ($r = i+2q+1, 1, z+w, k+t$)

vi- Stress -Strength

Spouse X and Y are two independent Stress & Strength $r.v's$ respectively, and they have PF-LD. With different parameters ($\alpha_1, \theta_1, \beta_1$). The function of Stress and Strength as obtained by,

$SS_{PF-LD} = P_r(Y < X) = \int_0^\infty f_X(x)_{PF-LD}\; F_y(x)\, dx,$

where,

$$F_y(x) = 1 - \left[\frac{-ln\left(1-\left[1+\frac{\beta_1}{1+\beta_1}x\right]e^{-\beta_1 x}\right)}{\alpha_1}\right]^{\theta_1}$$

According to (17) $F_y(x)$ will be,

$$F_y(x) = \frac{1}{\alpha_1{}^{\theta_1}}\left(\sum_{n=0}^{\infty}\sum_{v=0}^{\infty} C_n^1 b_{v,n}\right)^{\theta_1}\left(\left[1+\frac{\beta_1}{1+\beta_1}x\right]e^{-\beta_1 x}\right)^{\theta_1(1+n+v)}$$

According to (16)and (18) we get,

$$F_y(x) = \left\{\begin{matrix}\frac{1}{\alpha_1{}^{\theta_1}}\left(\sum_{n=0}^{\infty}\sum_{v=0}^{\infty} C_n^1 b_{v,n}\right)^{\theta_1} \\ \left(\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} C_m^1 b_{s,m}\sum_{k=0}^{m+s+1} C_k^{m+s+1}\left(\frac{\beta}{1+\beta}\right)^k \sum_{t=0}^{\infty}\frac{(-1)^t}{t!}\beta^t (m+s+1)^t x^{k+t}\right)^{\theta_1(1+n+v)}\end{matrix}\right\}$$

$$= \left\{\begin{matrix}\frac{1}{\alpha_1{}^{\theta_1}}\left(\sum_{n=0}^{\infty}\sum_{v=0}^{\infty} C_n^1 b_{v,n}\right)^{\theta_1} \\ \left(\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} C_m^1 b_{s,m}\sum_{k=0}^{m+s+1} C_k^{m+s+1}\left(\frac{\beta}{1+\beta}\right)^k \sum_{t=0}^{\infty}\frac{(-1)^t}{t!}\beta^t (m+s+1)^t\right)^{\theta_1(1+n+v)} \\ x^{\theta_1(1+n+v)(k+t)}\end{matrix}\right\} \quad (25)$$

Therefore, by using (25), the stress -strength of the PF-LD is given by,

$$SS_{PF-LD} = \left\{\begin{matrix}\frac{1}{\alpha_1{}^{\theta_1}}\left(\sum_{n=0}^{\infty}\sum_{v=0}^{\infty} C_n^1 b_{v,n}\right)^{\theta_1} \\ \left(\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} C_m^1 b_{s,m}\sum_{k=0}^{m+s+1} C_k^{m+s+1}\left(\frac{\beta}{1+\beta}\right)^k \sum_{t=0}^{\infty}\frac{(-1)^t}{t!}\beta^t (m+s+1)^t\right)^{\theta_1(1+n+v)} \\ \int_0^{\infty} x^{\theta_1(1+n+v)(k+t)} d\,x\end{matrix}\right\}$$

$$SS_{PF-LD} = \left\{\begin{matrix}\frac{1}{\alpha_1{}^{\theta_1}}\left(\sum_{n=0}^{\infty}\sum_{v=0}^{\infty} C_n^1 b_{v,n}\right)^{\theta_1} \\ \left(\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} C_m^1 b_{s,m}\sum_{k=0}^{m+s+1} C_k^{m+s+1}\left(\frac{\beta}{1+\beta}\right)^k \sum_{t=0}^{\infty}\frac{(-1)^t}{t!}\beta^t (m+s+1)^t\right)^{\theta_1(1+n+v)} \\ E\left(x^{\theta_1(1+n+v)(k+t)}\right)\end{matrix}\right\}$$

Where, $E\left(X^{\theta_1(1+n+v)(k+t)}\right)$ as in (22) with $\left(r = \theta_1(1+n+v)(k+t)\right)$.

6- Estimation of the $PF - LD$ Parameters

we obtain the maximum likelihood estimate (MLE) of the $PF - LD$ parameters. Let $x_1, x_2, \ldots, x_n$ be a random sample size from the probability function X~$PF - LD(\ \beta, \theta, \alpha)$, likelihood function is,

$$L(\beta, \theta, \alpha \setminus X) = \prod_{i=0}^{n}[f(x_i \setminus \beta, \theta, \alpha)]$$

$$= \prod_{i=0}^{n}\left(\frac{\frac{\beta^2}{(1+\beta)}(1+x)e^{-\beta x}}{1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}}\frac{\theta\left[-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right]^{\theta-1}}{\alpha^{\theta}}\right)$$

$$= \left(\frac{\beta^2\theta}{(1+\beta)\alpha^{\theta}}\right)^n \prod_{i=0}^{n}\left(\frac{(1+x)e^{-\beta x}\left[-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right]^{\theta-1}}{1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}}\right)$$

So, log-likelihood function is,

$$\ln L(\beta,\theta,\alpha\backslash X) = nln\left(\frac{\beta^2\theta}{(1+\beta)\alpha^\theta}\right)\sum_{i=0}^{n} ln\left(\frac{(1+x)e^{-\beta x}\left[-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right]^{\theta-1}}{1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}}\right) \quad (25)$$

Let $I = ln\left(\frac{\beta^2\theta}{(1+\beta)\alpha^\theta}\right)$ and $II = ln\left(\frac{(1+x)e^{-\beta x}\left[-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right]^{\theta-1}}{1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}}\right)$

Now

$$I = ln\left(\frac{\beta^2\theta}{(1+\beta)\alpha^\theta}\right)$$

$$= 2ln(\beta)\ ln\ (\theta) - ln(1+\beta) + \theta ln(\alpha)$$

$$II = ln\left(\frac{(1+x)e^{-\beta x}\left[-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right]^{\theta-1}}{1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}}\right)$$

$$= \left\{\begin{matrix} \ln(1+x) - \beta x + (\theta-1)ln\left[-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right] \\ -ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right) \end{matrix}\right\}$$

Substitute I and II in (25) we get,

$$\ln L(\beta,\theta,\alpha\ \backslash X) = \left\{\begin{matrix} 2nln(\beta)\ ln\ (\theta) - ln(1+\beta) + \theta ln(\alpha) \\ +\sum_{i=0}^{n}\left[\begin{matrix} ln(1+x) - \beta x + (\theta-1)ln\left[-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right] \\ -ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right) \end{matrix}\right] \end{matrix}\right\}$$

The MLEs $\widehat{\beta}$, $\widehat{\theta}$, and $\widehat{\alpha}$ are obtained respectively by solving the four nonlinear equation,

$$\frac{\partial \ln L(\beta,\theta,\alpha\ \backslash X)}{\partial\beta} = \left\{\begin{matrix} \frac{2n\ ln\ (\theta)}{\beta} - \frac{1}{(1+\beta)} + \theta ln(\alpha) \\ +\sum_{i=0}^{n}\left[\begin{matrix} \frac{ln(1+x) - x + (\theta-1)\left[x\left(1+\frac{\beta}{1+\beta}x\right)\left(e^{-\beta x}\right) + e^{-\beta x}\left(-x\beta(1+\beta)^{-2} + x(1+\beta)^{-1}\right)\right]}{\left[-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right]\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)} \\ +\frac{e^{-\beta x}\left(x\beta(1+\beta)^{-2} + x(1+\beta)^{-1}\right)}{ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)} \end{matrix}\right] \end{matrix}\right\} = 0$$

$$\frac{\partial \ln L(\beta,\theta,\alpha\ \backslash X)}{\partial\theta} = \left\{\begin{matrix} \frac{2nln(\beta)}{\theta} - ln(1+\beta) + ln(\alpha) \\ +\sum_{i=0}^{n}\left[\begin{matrix} ln(1+x) - \beta x + ln\left[-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right] \\ -ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right) \end{matrix}\right] \end{matrix}\right\} = 0$$

$$\frac{\partial \ln L(\beta,\theta,\alpha\ \backslash X)}{\partial\alpha} = \left\{\begin{matrix} 2nln(\beta)\ ln\ (\theta) - ln(1+\beta) + \frac{\theta}{\alpha} \\ +\sum_{i=0}^{n}\left[\begin{matrix} ln(1+x) - \beta x + (\theta-1)ln\left[-ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right)\right] \\ -ln\left(1-\left[1+\frac{\beta}{1+\beta}x\right]e^{-\beta x}\right) \end{matrix}\right] \end{matrix}\right\} = 0$$

7- Empirical study

We simulate a data of random variable from PF-LD for different sample sizes (30, 60, 160) and different parameter values, can be simulated by solving numerically the above nonlinear equations and the MSE,s are calculated a for parameters estimations .By using Matlab codes, we obtained the results.

Empirical MSE for the parameters estimation of **PF-LD**.

Table (1): The empirical MSE values of the parameter estimates of the PF-LD

Default parameter value			**Sample size**	**Empirical MSE**		
β	λ	α	n	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$
0.5	0.5	0.5	30	0.001325	0.000957	0.001507
			60	0.001324	0.000656	0.001582
			160	0.001130	0.000287	0.001214
		1.2	30	0.001454	0.001131	0.01513
			60	0.001303	0.000604	0.001465
			160	0.001300	0.000521	0.001311
	1.2	0.5	30	0.001423	0.001230	0.02418
			60	0.001419	0.000905	0.001852
			160	0.001346	0.000654	0.001725
		1.2	30	0.001652	0.001062	0.002542
			60	0.001374	0.000852	0.001627
			160	0.001148	0.000538	0.001745
1.2	0.5	0.5	30	0.001753	0.001227	0.002945
			60	0.001627	0.001263	0.002871
			160	0.001542	0.000942	0.001961
		1.2	30	0.001762	0.001382	0.002653
			60	0.001663	0.001275	0.002463
			160	0.001578	0.000973	0.002178
	1.2	0.5	30	0.001862	0.001582	0.002378
			60	0.001801	0.001454	0.002216
			160	0.001728	0.001033	0.001934
		1.2	30	0.001832	0.001462	0.001763
			60	0.001625	0.001201	0.001639
			160	0.001523	0.000435	0.001582

In this table you can see:

i- At a sample size of 30 when the estimated parameters were small (β = 0.5, λ = 0.5, α = 0.5) the MSE values are ($\beta = 0.001325, \lambda = 0.000957, \alpha = 0.001507$).

ii- When the sample size is 60 for the estimated parameters (β = 0.5, λ = 0.5, α = 1.2) where the MSE values are (β = 0.001303, λ = 0.000604, α = 0.001465).

i- We notice that the lowest MSE when the sample size is 160 for the estimated parameters (β=0.5, λ=0.5, α =0.5,), the MSE values are ($\beta = 0.001130,\ \lambda = 0.000287, \alpha = 0.001214$.

iii- MSE decreases with increasing sample size for all cases. This result is consistent with statistical theory.

The plot of the pdf of PF-LD with β,θ,α=0.5,1.2 are presented in Figure (1).

Figure (1): Graph of pdf of the power function **-** Lindley distribution

8- Conclusion

here in our research, we introduce a new family of continuous distributions called Power function- Lindley distribution. we derive some of important properties of PF-LD, such as the characteristic function, the r^{th} moment function, variance, mean , skewness, kurtosis, function of reliability , hazard rate function , Shannon entropy function and Stress –Strength model. We obtained the results, that the lowest MSE when the sample size is 160 for the estimated parameters (β=0.5, λ=0.5, α =0.5). Based on the knowledge and information obtained from this research, we recommend, applying the proposed distribution to real data pertaining to the Iraqi society.

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