Advances in Nonlinear Variational Inequalities
ISSN: 1092-910X
Vol 28 No. 7s (2025)

On Congruences Modulo Powers of 2 for (2, 4)-Regular Overpartitions

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Article History:

Received: 05-01-2025

Revised: 19-02-2025

Accepted: 27-02-2025

Abstract:

Recently, Naika et al. (2021) proved several congruence properties modulo powers of 2 for (*j*,*k*)-regular overpartition $\overline{p}_{j,k}(n)$ of *n* for $\overline{p}_{3,6}(n)$, $\overline{p}_{5,10}(n)$ and $\overline{p}_{9,18}(n)$. This study establishes several infinite families of congruences modulo powers of 2 for $\overline{p}_{2,4}(n)$ employing *q*-series identities and iterative computational techniques.

Keywords: Overpartitions, Regular partitions, Congruences, (j,k)-regular overpartitions

1. Introduction

A partition of a positive integer *n* is a non-increasing sequence of positive integers, whose sum equals *n*. For example, $n = 3$ has three partitions, namely, 3, 2 + 1, 1 + 1 + 1. If $p(n)$ denote the number of partitions of *n*, then $p(3) = 3$. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \ \ p(0) = 1,$$

where

$$(a;q)_\infty = \prod_{k=1}^{\infty}(1 - aq^{k-1}), \ \ |q| < 1.$$

Throughout this paper, we use

$$E_k := (q^k;q^k)_\infty,$$

where *k* is any positive integer.

Corteel and Lovejoy [4] introduced the concept of an overpartition. An overpartition of a non-negative integer *n* is a sequence of natural numbers arranged in non-increasing order, adds up to *n*, where the initial appearance of any integer in the sequence may optionally be overlined. For instance, there are 4 possible overpartitions of 2, specifically: 2, $\overline{2}$, $1+1$, $\overline{1}+1$. If $\overline{p}(n)$ represnts the number of overpartitions of *n* with ($\overline{p}(0) = 1$), the generating function is expressed as follows:

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}.$$

Naika et al.[10] introduced a new type of overpartition called (j,k)-regular overpartition $\overline{p}_{2,4}(n)$ of a positive integer n, where none of its parts are congruent to j (mod k). The generating function for $\overline{p}_{j,k}(n)$ with $\overline{p}_{j,k}(0)=1$ expressed as follows:

$$\sum_{n=0}^{\infty} \bar{p}_{j,k}(n)q^n = \frac{(-q;q)_\infty (q^j;q^k)_\infty}{(q;q)_\infty (-q^j;q^k)_\infty}. \tag{1.1}$$

Naika, Harishkumar and Veeranayaka[10], in their study, discovered many infinite families of congruences modulo powers of 2 for $\overline{p}_{3,6}(n)$, $\overline{p}_{5,10}(n)$ and $\overline{p}_{9,18}(n)$, where n is multiple of the primes 2, 3, 5 and 7. Also, Saikia et al.[12] obtained infinite families of congruences modulo 4, 8 ,16 and 32 for $\overline{p}_{4,8}(n)$, modulo 4 and 8 for $\overline{p}_{6,12}(n)$, modulo 16 for $\overline{p}_{8,16}(n)$.

In this paper, we establish many infinite families of congruences modulo powers of 2 for $\overline{p}_{2,4}(n)$. In second section, we list the main results. We present some preliminary results and lemma to prove main results. In the final section, we establish the proofs.

2. Main Results

In this section, we list the main results.

Theorem 2.1. *For any prime $p \geq 3$, $i \in \{1, 2...p-1\}$ and non-negative integers α and n, we have*

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(p^{2\alpha}.8n + p^{2\alpha})\, q^n \equiv 2E_1^3 \pmod 4, \tag{2.1}$$

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(p^{2\alpha+1}.8n + p^{2\alpha+2})\, q^n \equiv 2E_p^3 \pmod 4, \tag{2.2}$$

$$\overline{p}_{2,4}(p^{2\alpha+1}.8(pn+i) + p^{2\alpha+2}) \equiv 0 \pmod 4, \tag{2.3}$$

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(16n+2)\, q^n \equiv \sum_{n=0}^{\infty} \bar{p}_{2,4}(8n+1)\, q^n \pmod 4. \tag{2.4}$$

Theorem 2.2. *For any non-negative integers α_1, α_2, n and $i \in \{1, 2\}$, we have*

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(3^{2\alpha_1}.8n)\, q^n \equiv \frac{E_1^2}{E_2} \pmod 4, \tag{2.5}$$

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(3^{2\alpha_1+1}.8n)\, q^n \equiv \frac{E_3^2}{E_6} \pmod 4, \tag{2.6}$$

$$\overline{p}_{2,4}(3^{2\alpha_1}.24n + 16.3^{2\alpha_1}) \equiv 0 \pmod 4, \tag{2.7}$$

$$\overline{p}_{2,4}(3^{2\alpha_1+1}.8(3n+i)) \equiv 0 \pmod 4, \tag{2.8}$$

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(2^{2\alpha_2}.8n)\, q^n \equiv \frac{E_2}{E^2} \pmod 4, \tag{2.9}$$

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(2^{2\alpha_2+1}.8n)\, q^n \equiv \frac{E_4}{E_2^2} \pmod 4 \tag{2.10}$$

$$\bar{p}_{2,4}(2^{2\alpha_2+1}.8(2n+1)) \equiv 0 \pmod 4. \tag{2.11}$$

Theorem 2.3. *For any prime* $p \geq 3$, $i \in \{1, 2...p-1\}$ *and non-negative integers* α *and* n, *we have*

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(p^{2\alpha}.32n + 4p^{2\alpha})\, q^n \equiv 8E_1^3 \pmod{16}, \tag{2.12}$$

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(p^{2\alpha+1}.32n + 4p^{2\alpha+2})\, q^n \equiv 8E_p^3 \pmod{16}, \tag{2.13}$$

$$\bar{p}_{2,4}(p^{2\alpha+1}.32(pn+i) + 4p^{2\alpha+2}) \equiv 0 \pmod{16}. \tag{2.14}$$

Theorem 2.4. *For any non-negative integers* α_1, α_2, α_3, *we have*

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(32.5^{2\alpha_1}.7^{2\alpha_2}.13^{2\alpha_3}n + 12.5^{2\alpha_1}.7^{2\alpha_2}.13^{2\alpha_3})\, q^n \equiv 16E_1^9 \pmod{32}, \tag{2.15}$$

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(32.5^{2\alpha_1+1}.7^{2\alpha_2}.13^{2\alpha_3}n + 28.5^{2\alpha_1+1}.7^{2\alpha_2}.13^{2\alpha_3})\, q^n \equiv 16qE_5^9 \pmod{32}, \tag{2.16}$$

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(32.5^{2\alpha_1}.7^{2\alpha_2+1}.13^{2\alpha_3}n + 20.5^{2\alpha_1}.7^{2\alpha_2+1}.13^{2\alpha_3})\, q^n \equiv 16q^2E_7^9 \pmod{32}, \tag{2.17}$$

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(32.5^{2\alpha_1}.7^{2\alpha_2}.13^{2\alpha_3+1}n + 28.5^{2\alpha_1}.7^{2\alpha_2}.13^{2\alpha_3+1})\, q^n \equiv 16q^4E_{13}^9 \pmod{32}, \tag{2.18}$$

Corollary 2.1. *For any non-negative integers* n, α_1, α_2, α_3, *and* $i \in \{0, 2, 3, 4\}$, $j \in \{0, 1, 3, 4, 5, 6\}$, $k \in \{0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12\}$, *we have*

$$\bar{p}_{2,4}(32\cdot 5^{2\alpha_1+1}\cdot 7^{2\alpha_2}\cdot 13^{2\alpha_3}(5n+i) + 28\cdot 5^{2\alpha_1+1}\cdot 7^{2\alpha_2}\cdot 13^{2\alpha_3}) \equiv 0 \pmod{32}, \tag{2.19}$$

$$\bar{p}_{2,4}(32\cdot 5^{2\alpha_1}\cdot 7^{2\alpha_2+1}\cdot 13^{2\alpha_3}(7n+j) + 20\cdot 5^{2\alpha_1}\cdot 7^{2\alpha_2+1}\cdot 13^{2\alpha_3}) \equiv 0 \pmod{32}, \tag{2.20}$$

$$\bar{p}_{2,4}(32\cdot 5^{2\alpha_1}\cdot 7^{2\alpha_2}\cdot 13^{2\alpha_3+1}(13n+k) + 28\cdot 5^{2\alpha_1}\cdot 7^{2\alpha_2}\cdot 13^{2\alpha_3+1}) \equiv 0 \pmod{32}. \tag{2.21}$$

Theorem 2.5. *For any prime* $p \geq 3$, $i \in \{1, 2...p-1\}$ *and non-negative integers* α *and* n, *we have*

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(p^{2\alpha}.64n + 8p^{2\alpha})\, q^n \equiv 16E_1^3 \pmod{64}, \tag{2.22}$$

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(p^{2\alpha+1}.64n + 8p^{2\alpha+2})\, q^n \equiv 16E_p^3 \pmod{64}, \tag{2.23}$$

$$\bar{p}_{2,4}\,(p^{2\alpha+1}.64(pn+i) + 8p^{2\alpha+2}) \equiv 0 \pmod{64}. \tag{2.24}$$

Theorem 2.6. *For any non-negative integers* $\alpha_1, \alpha_2, \alpha_3$, *we have*

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(16.\, 5^{2\alpha1}.\, 7^{2\alpha2}.\, 13^{2\alpha3}n + 6.\, 5^{2\alpha1}.\, 7^{2\alpha2}.\, 13^{2\alpha3}) \equiv 16E_1^9 \pmod{64}, \tag{2.25}$$

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(16.5^{2\alpha_1+1}.7^{2\alpha_2}.13^{2\alpha_3}n + 14.5^{2\alpha_1+1}.7^{2\alpha_2}.13^{2\alpha_3})\, q^n \equiv 16qE_5^9 \pmod{64}, \tag{2.26}$$

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(16.5^{2\alpha_1}.7^{2\alpha_2+1}.13^{2\alpha_3}n + 10.5^{2\alpha_1}.7^{2\alpha_2+1}.13^{2\alpha_3})\, q^n \equiv 16q^2E_7^9 \pmod{64} \tag{2.27}$$

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(16.5^{2\alpha_1}.7^{2\alpha_2}.13^{2\alpha_3+1}n + 14.5^{2\alpha_1}.7^{2\alpha_2}.13^{2\alpha_3+1})\, q^n \equiv 16q^4E_{13}^9 \pmod{64} \tag{2.28}$$

Corollary 2.2. *For any non-negative integers n,* α_1, α_2, α_3, *and* $i \in \{0, 2, 3, 4\}$, $j \in \{0, 1, 3, 4, 5, 6\}$, $k \in \{0,1,2,3,5,6,7,8,9,10,11,12\}$, *we have*

$$\bar{p}_{2,4}(16\cdot 5^{2\alpha_1+1}\cdot 7^{2\alpha_2}\cdot 13^{2\alpha_3}(5n+i) + 14\cdot 5^{2\alpha_1+1}\cdot 7^{2\alpha_2}\cdot 13^{2\alpha_3}) \equiv 0 \pmod{64}, \tag{2.29}$$

$$\bar{p}_{2,4}(16\cdot 5^{2\alpha_1}\cdot 7^{2\alpha_2+1}\cdot 13^{2\alpha_3}(7n+j) + 10\cdot 5^{2\alpha_1}\cdot 7^{2\alpha_2+1}\cdot 13^{2\alpha_3}) \equiv 0 \pmod{64}, \tag{2.30}$$

$$\bar{p}_{2,4}(16\cdot 5^{2\alpha_1}\cdot 7^{2\alpha_2}\cdot 13^{2\alpha_3+1}(13n+k) + 14\cdot 5^{2\alpha_1}\cdot 7^{2\alpha_2}\cdot 13^{2\alpha_3+1}) \equiv 0 \pmod{64}. \tag{2.31}$$

3. Preliminary

In order to prove the main theorems, the following lemmas are required.

Lemma 3.1. *The following* 2*-dissections hold:*

$$\frac{1}{E_1^2} = \frac{E_8^5}{E_2^5 E_{16}^2} + 2q\frac{E_4^2 E_{16}^2}{E_2^5 E_8}, \tag{3.1}$$

$$E_1^2 = \frac{E_2 E_8^5}{E_4^2 E_{16}^2} - 2q \frac{E_2 E_{16}^2}{E_8}, \tag{3.2}$$

$$\frac{1}{E_1^4} = \frac{E_4^{14}}{E_2^{14} E_8^4} + 4q \frac{E_4^2 E_8^4}{E_2^{10}}, \tag{3.3}$$

$$E_1^4 = \frac{E_4^{10}}{E_2^2 E_8^4} - 4q \frac{E_2^2 E_8^4}{E_4^2}, \tag{3.4}$$

The identity (3.1) can be obtained from the 2-dissection of $\phi(q)$ in [9, p.1]. The identity (3.2) can be obtained from the 2-dissection of $\phi(q)^2$ in [9, p.1]. By substituting q and $-q$ in (3.1) and (3.2), we can derive (3.2) and (3.4).

Lemma 3.2. *The following* 3*-dissection holds:*

$$E_1^3 = \frac{E_6 E_9^6}{E_3 E_{18}^3} - 3q E_9^3 + 4q^3 \frac{E_3^2 E_{18}^6}{E_6^2 E_9^3}. \tag{3.5}$$

The equation (3.5) is the same as (14.8.5) in [9, p.137].

Lemma 3.3. *The following* 3*-dissection holds:*

$$\frac{E_1^2}{E_2} = \frac{E_9^2}{E_{18}} - 2q \frac{E_3 E_{18}^2}{E_6 E_9}. \tag{3.6}$$

The equation (3.6) is from [5].

Lemma 3.4. *The following* 2*-dissection holds:*

$$\frac{E_2}{E_1^2} = \frac{E_8^5}{E_2^4 E_{16}^2} + 2q \frac{E_4^2 E_{16}^2}{E_2^4 E_8}. \tag{3.7}$$

Lemma 3.4 was obtained by Hirschhorn and Sellers [7, p.2].

Lemma 3.5. *We have*

$$E_1 = E_{25}(R(q^5) - q - q^2 R(q^5)^{-1}), \tag{3.8}$$

where

$$R(q) = \frac{(q^2;q^5)_\infty (q^3;q^5)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty}.$$

The equation (3.8) can be found in [6].

Lemma 3.6. *The following* 7*-dissection holds:*

$$E_1 = E_{49}\left(\frac{A(q^7)}{B(q^7)} - q\frac{C(q^7)}{A(q^7)} - q^2 + q^5\frac{B(q^7)}{C(q^7)}\right), \tag{3.9}$$

where $C(q) = f(-q^3;-q^4)$, $A(q) = f(-q^2;-q^5)$, $B(q) = f(-q;-q^5)$.

The dissection (3.9) can be obtained in [3, Entry 17].

Lemma 3.7. *The following* 13*-dissection holds:*

$$E_1 = E_{169}\ B_1(q^{13}) - qB_2(q^{13}) - q^2B_3(q^{13}) + q^5B_4(q^{13}) + q^7 - q^{12}B_5(q^{13}) + q^{22}B_6(q^{13})), \tag{3.10}$$

where

$B_1(q) = \frac{f(-q^4,-q^9)}{f(-q^2,-q^{11})}$, $B_2(q) = \frac{f(-q^6,-q^7)}{f(-q^3,-q^{10})}$, $B_3(q) = \frac{f(-q^2,-q^{11})}{f(-q,-q^{12})}$, $B_4(q) = \frac{f(-q^5,-q^8)}{f(-q^4,-q^9)}$, $B_5(q) = \frac{f(-q^3,-q^{10})}{f(-q^5,-q^8)}$, $B_6(q) = \frac{f(-q,-q^{12})}{f(-q^6,-q^7)}$.

The dissection (3.10) can be obtained from [3, Entry 8].

Lemma 3.8. *From [1, Lemma 2.3] for any prime* $p \geq 3$, *we have*

$$E_1^3 = p(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} E_{p^2}^3 + \sum_{\substack{k=0 \\ k\neq \frac{\pm p-1}{2}}}^{p-1} (-1)^k q^{\frac{(k^2+k)}{2}} \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1).q^{pn.\frac{pn+2k+1}{2}}. \tag{3.11}$$

Furthermore, for $0 \leq k \leq (p-1)$ *and* $k \neq (p-1)/2$,

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod p.$$

From the binomial theorem and for any positive integers k, m and β, we have

$$E_k^{2^\beta m} \equiv E_k^{2^{\beta-1} m} \pmod{2^\beta}. \tag{3.12}$$

4. Proofs of Theorems

In this section, we prove the main theorems.

4.1 Proof of Theorem 2.1

Substituting j = 2 and $k = 4$ in (1.1), we have

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(n) q^n = \frac{E_2^3 E_8}{E_1^2 E_4^3}. \tag{4.1}$$

Employing Lemma 3.4 into (4.1) and isolating the terms that involve q^{2n} and q^{2n+1}, we get

$$\sum_{n=0}^{\infty}\bar{p}_{2,4}(2n)q^n = \frac{E_4^6}{E_2^3E_1^2E_8^2}, \tag{4.2}$$

$$\sum_{n=0}^{\infty}\bar{p}_{2,4}(2n+1)q^n = 2\frac{E_8^2}{E_2E_1^2}. \tag{4.3}$$

Now applying (3.12) for $\beta = 1$ in (4.3) and isolating the terms involving q^{4n}, we have

$$\sum_{n=0}^{\infty}\bar{p}_{2,4}(8n+1)q^n \equiv 2E_1^3 \pmod 4. \tag{4.4}$$

The congruence (4.4) is the case for $\alpha = 0$ of (2.1). Consider that (2.1) is true for $\alpha \geq 0$. Now utilizing Lemma 3.8 in (2.1) and collecting the terms that involve $q^{pn+\frac{p^2-1}{8}}$, we have

$$\sum_{n=0}^{\infty}\bar{p}_{2,4}(p^{2\alpha+1}.8n+p^{2\alpha+2})q^n \equiv 2E_p^3 \pmod 4) \tag{4.5}$$

Again colleting the terms that involve q^{pn}, we obtain

$$\sum_{n=0}^{\infty}\bar{p}_{2,4}(p^{2\alpha+2}.8n+p^{2\alpha+2})q^n \equiv 2E_1^3 \pmod 4. \tag{4.6}$$

The congruence (4.6) shows that (2.1) is true for $\alpha + 1$. This shows that (2.1) is true for all $\alpha \geq 0$.Employing Lemma 3.8 in (2.1) and isolating the terms that involve $q^{pn+\frac{p^2-1}{8}}$, we obtain (2.2). Collecting the terms that involve q^{pn+i}, from (2.2), we get equation (2.3), where$i \in \{1, 2....p-1\}$.

Again, utilizing (3.1) in (4.2) and isolating terms involving q^{2n} and q^{2n+1}, we get

$$\sum_{n=0}^{\infty}\bar{p}_{2,4}(4n)q^n = \frac{E_2^6E_4^3}{E_1^8E_8^2}, \tag{4.7}$$

and

$$\sum_{n=0}^{\infty}\bar{p}_{2,4}(4n+2)q^n = 2\frac{E_2^8E_8^2}{E_1^8E_4^3}. \tag{4.8}$$

Now applying (3.12) for $\beta = 1$ in (4.8), we get

$$\sum_{n=0}^{\infty}\bar{p}_{2,4}(4n+2)q^n = 2E_4^3 \pmod 4. \tag{4.9}$$

Collecting terms involving q^{4n}, we have

$$\sum_{n=0}^{\infty}\bar{p}_{2,4}(16n+2)q^n = 2E_1^3 \pmod 4. \tag{4.10}$$

Combining (4.4) and (4.10), we have (2.4).

4.2 Proof of Theorem 2.2

Employing (3.12) for $\beta = 2$ in (4.7), we have

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(4n)q^n \equiv \frac{E_1^4}{E_4} \pmod 4. \tag{4.11}$$

Isolating terms involving q^{2n}, we have

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(8n)q^n \equiv \frac{E_1^2}{E_2} \pmod 4. \tag{4.12}$$

The equation (4.12) shows that (2.5) holds for $\alpha_1 = 0$. Consider that (2.5) is true for all $\alpha_1 \geq 0$. Now employing Lemma 3.3 in (2.5) and isolating the terms involving q^{3n}, we have

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(3^{2\alpha_1+1}.8n)q^n \equiv \frac{E_3^2}{E_6} \pmod 4. \tag{4.13}$$

Again extracting q^{3n} from (4.13), we have

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(3^{2\alpha_1+2}.8n)q^n \equiv \frac{E_1^2}{E_2} \pmod 4. \tag{4.14}$$

The equation (4.14) shows that (2.5) is true for $\alpha_1 + 1$. This shows that (2.5) holds for all integer $\alpha_1 \geq 0$.

Employing Lemma 3.3 in (2.5) and extracting terms involving q^{3n}, we obtain (2.6). Extracting the terms involving q^{3n+2}, from (2.5), we get equation (2.7). Isolating the terms involving q^{3n+i}, where $i \in \{1, 2\}$ from (2.6), we obtain (2.8).

The equation (4.12) shows that (2.9) holds for $\alpha_2 = 0$. Consider that (2.9) holds for $\alpha_2 \geq 0$. Employing (3.2) in (2.9) and isolating terms involving q^{2n}, we arrive at

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(2^{2\alpha_2+1}.8n)q^n \equiv \frac{E_4}{E_2^2} \pmod 4. \tag{4.15}$$

Extracting coefficients of q^{2n} from (4.15), we get

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(2^{2\alpha_2+2}.8n)q^n \equiv \frac{E_2}{E_1^2} \pmod 4. \tag{4.16}$$

The equation (4.16) shows that (2.9) is true for $\alpha_2 + 1$. This shows (2.9) is true for all $\alpha_2 \geq 0$. Now employing Lemma 3.4 in (2.9) and collecting the coefficients of q^{2n}, we have (2.10). Isolating terms that involve q^{2n+1} from (2.10), we have (2.11).

4.3 Proof of Theorem 2.3

Employing (3.3) in (4.7) and isolating terms involving q^{2n+1}, we arrive at

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(8n+4)q^n = 8\frac{E_2^{19}}{E_4^2 E_1^{18}}. \quad (4.17)$$

Now applying (3.12) for $\beta = 1$ in (4.17) and isolating the terms involving q^{4n}, we have

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(32n+4)q^n \equiv 8E_1^3 \pmod{16}. \quad (4.18)$$

The congruence (4.18) is the case for $\alpha = 0$ of (2.12). Consider that (2.12) is true for all $\alpha \geq 0$. Now utilizing Lemma 3.8 in (2.12) and isolating the terms that involve $q^{pn+\frac{p^2-1}{8}}$, we get

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(p^{2\alpha+1}.32n + 4p^{2\alpha+2})q^n \equiv 8E_p^3 \pmod{16}. \quad (4.19)$$

Again extracting terms involving q^{pn}, we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(p^{2\alpha+2}.32n + 4p^{2\alpha+2})q^n \equiv 8E_1^3 \pmod{16} \quad (4.20)$$

The equation (4.20) shows that (2.12) is true for α+1. This shows that (2.12) is true for all $\alpha \geq 0$. Employing Lemma 3.8 in (2.12) and collecting the coefficients of $q^{pn+\frac{p^2-1}{8}}$, we obtain (2.13). Isolating the terms that involve q^{pn+i}, from (2.13), we get equation (2.14), where the $i \in \{1, 2....p-1\}$.

4.4 Proof of Theorem 2.4

Employing (3.1) in (4.17) and extracting terms involving q^{2n+1}, we get

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(16n+12)q^n = 16\frac{E_1^6 E_8^2}{E_4}. \quad (4.21)$$

Now applying (3.12) for $\beta = 1$ in (4.21) and isolating the terms involving q^{2n}, we have

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(32n+12)q^n \equiv 16E_1^9 \pmod{32}. \quad (4.22)$$

Equation (4.22) is the case for $\alpha_1 = \alpha_2 = \alpha_3 = 0$ of (2.15). Now assume that (2.15) is true for $\alpha_1 \geq 0$ and $\alpha_2 = \alpha_3 = 0$. Utilizing Lemma 3.5 in (2.15) with $\alpha_2 = \alpha_3 = 0$ and isolating the terms involving q^{5n+4}, we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(32.5^{2\alpha_1+1}n + 28.5^{2\alpha_1+1})q^n \equiv 16qE_5^9 \pmod{32}. \quad (4.23)$$

Again extracting terms involving q^{5n+1}, we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(32.5^{2\alpha_1+2}n + 12.5^{2\alpha_1+2})q^n \equiv 16E_1^9 \pmod{32}. \quad (4.24)$$

Equation (4.24) shows that (2.15) is true for α_1+1 and $\alpha_2=\alpha_3=0$. This implies (2.15) is true for all $\alpha_1\geq 0$. Suppose that (2.15) is true for α_1, $\alpha_2\geq 0$ and $\alpha_3=0$. Now utilizing Lemma 3.6 in (2.15) with $\alpha_3=0$ and isolating the terms involving q^{7n+4}, we get

$$\sum_{n=0}^{\infty}\bar{p}_{2,4}(32.5^{2\alpha_1}.7^{2\alpha_2+1}n+20.5^{2\alpha_1}.7^{2\alpha_2+1})q^n\equiv 16q^2E_7^9 \pmod{32} \tag{4.25}$$

Again extracting terms involving q^{7n+4}, we obtain

$$\sum_{n=0}^{\infty}\bar{p}_{2,4}(32.5^{2\alpha_1}.7^{2\alpha_2+2}n+12.5^{2\alpha_1}.7^{2\alpha_2+2})q^n\equiv 16E_1^9 \pmod{32}. \tag{4.26}$$

Equation (4.26) indicates that (2.15) is true for α_2+1 and $\alpha_3=0$. This implies (2.15) is true for all α_1, $\alpha_2\geq 0$. Now assume that (2.15) is true for α_1, α_2, $\alpha_3\geq 0$. Now utilizing Lemma 3.7 in (2.15) and isolating the terms involving q^{13n+11}, we obtain

$$\sum_{n=0}^{\infty}\bar{p}_{2,4}(32.5^{2\alpha_1}.7^{2\alpha_2}.13^{2\alpha_3+1}n+28.5^{2\alpha_1}.7^{2\alpha_2}.13^{2\alpha_3+1})q^n\equiv 16q^4E_{13}^9 \pmod{32}. \tag{4.27}$$

Again extracting terms involving q^{13n+4}, we obtain

$$\sum_{n=0}^{\infty}\bar{p}_{2,4}(32.5^{2\alpha_1}.7^{2\alpha_2}.13^{2\alpha_3+2}n+12.5^{2\alpha_1}.7^{2\alpha_2}.13^{2\alpha_3+2})q^n\equiv 16E_1^9 \pmod{32}. \tag{4.28}$$

Equation (4.28) indicates that (2.15) is true for α_3+1. This implies (2.15) is true for all α_1, α_2, $\alpha_3\geq 0$.

By applying Lemma 3.5 in (2.15) and isolating terms involving q^{5n+4}, we get (2.16). Utilzing Lemma 3.6 in (2.15) and isolating terms involving q^{7n+4}, we get (2.17). Again using Lemma 3.7 in (2.15) and extracting terms involving q^{13n+11}, we get (2.18).

4.5 Proof of Corollary 2.1

Isolating the terms that involve q^{5n+i} for $i\in\{0, 2, 3, 4\}$ from (2.16), we obtain equation (2.19). From equation (2.17), isolating the terms consisting q^{7n+j} for $j\in\{0, 1, 3, 4, 5, 6\}$, we have (2.20). Collecting the terms that involve q^{13n+k} where $k\in\{0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12\}$ from (2.18), we get equation (2.21).

4.6 Proof of Theorem 2.5

Employing (3.3) in (4.7) and isolating terms that involve q^{2n}, we get

$$\sum_{n=0}^{\infty}\bar{p}_{2,4}(8n)q^n\equiv\left(\frac{E_2^{31}}{E_1^{22}E_4^{10}}+16q\frac{E_2^7E_2^6}{E_1^{14}}\right) \pmod{64}. \tag{4.29}$$

Now applying (3.3) in (4.29) and extracting terms involving q^{2n+1}, we get

$$\sum_{n=0}^{\infty}\bar{p}_{2,4}(16n+8)q^n\equiv 16\left(\frac{E_2^6E_4^5}{E_1^4E_8^2}\right) \pmod{64}. \tag{4.30}$$

Using (3.12) for $\beta = 2$ in (4.30) and isolating the terms involving q^{4n}, we have

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(16n+8)q^n \equiv 16E_4^3 \pmod{64}. \tag{4.31}$$

Isolating the terms involving q^{4n}, we have

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(64n+8)q^n \equiv 16E_1^3 \pmod{64}. \tag{4.32}$$

Equation (4.32) is the case for $\alpha = 0$ of (2.22).

The proof of identities (2.22)-(2.24) follows a similar approach to those of identities (2.1)-(2.3) of Theorem 2.1.

4.7 Proof of Theorem 2.6

Employing (3.1) in (4.8) and extracting terms involving q^{2n+1}, we get

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(8n+6)q^n = 16\frac{E_2^{13}E_4^2}{E_1^{16}}. \tag{4.33}$$

Now applying (3.12) for $\beta = 2$ in (4.33), we get

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(8n+6)q^n = 16E_2^9 \pmod{64}. \tag{4.34}$$

Collecting terms involving q^{2n}, we have

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(16n+6)q^n = 16E_1^9 \pmod{64}. \tag{4.35}$$

Equation (4.35) is the case for $\alpha_1 = \alpha_2 = \alpha_3 = 0$ of (2.25). Now assume that (2.25) is true for all $\alpha_1 \geq 0$ and $\alpha_2 = \alpha_3 = 0$. Now utilizing Lemma 3.5 in (2.25) with $\alpha_2 = \alpha_3 = 0$ and extracting the terms involving q^{5n+4}, we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(32.5^{2\alpha_1+1}n + 14.5^{2\alpha_1+1})q^n \equiv 16qE_5^9 \pmod{64}. \tag{4.36}$$

Again extracting terms involving q^{5n+1}, we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(32.5^{2\alpha_1+2}n + 6.5^{2\alpha_1+2})q^n \equiv 16E_1^9 \pmod{64}. \tag{4.37}$$

Equation (4.37) shows that (2.25) is true for $\alpha_1 + 1$ and $\alpha_2 = \alpha_3 = 0$. This implies (2.25) is true for all $\alpha_1 \geq 0$. Let us assume that (2.25) is true for all α_1, $\alpha_2 \geq 0$ and $\alpha_3 = 0$. Now utilizing Lemma 3.6 in (2.25) with $\alpha_3 = 0$ and extracting the terms involving q^{7n+4}, we get

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(16.5^{2\alpha_1}.7^{2\alpha_2+1}n + 10.5^{2\alpha_1}.7^{2\alpha_2+1})q^n \equiv 16q^2E_7^9 \pmod{64}. \tag{4.38}$$

Again extracting terms involving q^{7n+2}, we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(16.5^{2\alpha_1}.7^{2\alpha_2+2}n + 6.5^{2\alpha_1}.7^{2\alpha_2+2})q^n \equiv 16E_1^9 \pmod{64}. \tag{4.39}$$

Equation (4.39) indicates that (2.25) is true for $\alpha_2 + 1$ and $\alpha_3 = 0$. This implies (2.25) is true for all α_1, $\alpha_2 \geq 0$. Now assume that (2.25) is true for α_1, α_2, $\alpha_3 \geq 0$. Now utilizing Lemma 3.7 in (2.25) and extracting the terms involving q^{13n+11}, we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(32.5^{2\alpha_1}.7^{2\alpha_2}.13^{2\alpha_3+1}n + 14.5^{2\alpha_1}.7^{2\alpha_2}.13^{2\alpha_3+1})q^n \equiv 16q^4E_{13}^9 \pmod{32}. \tag{4.40}$$

Again extracting terms involving q^{13n+4}, we obtain

$$\sum_{n=0}^{\infty} \bar{p}_{2,4}(32.5^{2\alpha_1}.7^{2\alpha_2}.13^{2\alpha_3+2}n + 6.5^{2\alpha_1}.7^{2\alpha_2}.13^{2\alpha_3+2})q^n \equiv 16E_1^9 \pmod{32}. \tag{4.41}$$

Equation (4.41) indicates that (2.25) is true for α_3 +1. This implies (2.25) is true for all α_1, α_2, $\alpha_3 \geq 0$.

By applying Lemma 3.5 in (2.25) and isolating terms involving q^{5n+4}, we get (2.26). Utilzing Lemma 3.6 in (2.25) and isolating terms involving q^{7n+4}, we get (2.27). Again using Lemma3.7 in (2.25) and extracting terms involving q^{13n+11}, we get (2.28).

4.8 Proof of Corollary 2.2

Isolating the terms involving q^{5n+i} for $i \in \{0, 2, 3, 4\}$ from (2.26), we obtain equation (2.29). From equation (2.27), isolating the terms involving q^{7n+j} for $j \in \{0, 1, 3, 4, 5, 6\}$, we have (2.30). Extracting the terms involving q^{13n+k} where $k \in \{0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12\}$ from (2.28), we get equation (2.31).

Data Availability

No datasets were generated or analysed during the current study.

Author Contributions

Both the authors equally contributed for prepairing this manuscript.

Declaration of competing interest

The authors declared that they have no conflicts of interest to this work.

Acknowledgement

The author Yudhisthira Jamudulia is thankful to GMU, Sambalpur for awarding seed fund research project [Order No.4934/GMU] under which this work has been done.

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