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# Generalized Padovan Sequences and Figurate Numbers

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**Abstract:**

Generalized Padovan Sequences are more general class of sequences which are viewed as general form of the classical Padovan sequences named after English mathematician Richard Padovan. Figurate numbers are special class of numbers which occur in various problems in number theory. In this paper, we discuss the limiting ratio of the recurrence relation formed by generalizing Padovan sequence and considering Figurate numbers as coefficients in two possible cases. We find very interesting numbers turning out to be limiting ratios under our assumptions.

**Keywords**: Padovan Sequence, Figurate Numbers, Recurrence Relation, Characteristic Equation, Limiting Ratio, Plastic Number.

## 1. Introduction

In 1989, Richard Padovan mentioned a sequence similar to that of the Fibonacci sequence. While in Fibonacci sequence, except the initial two terms, each term is equal to the sum of two previous terms, in Padovan sequence, except for the three initial terms, each term is equal to the sum of one but previous two terms. With this definition, we notice that Padovan sequence differs quite significantly compared to classic Fibonacci sequence. The limiting ratio of successive terms of Padovan sequence is a number called Plastic Ratio. This number is similar to that of Golden Ratio to Fibonacci sequence. The idea of Figurate numbers exists from ancient times and their properties continue to fascinate even today. In this paper, we generalize the usual Padovan sequence by including Figurate numbers as coefficients and study the limiting ratios of resulting sequences.

## 2. Definitions

### 2.1 Padovan Sequence

Let $P(1) = P(2) = P(3) = 1$ and $P(n+3) = P(n+1) + P(n), \quad n \geq 1 \quad (2.1)$ .The sequence of terms defined by $P(n)$ for $n$ = 1, 2, 3, 4, 5, . . . is called Padovan Sequence. The first few terms of the Padovan sequence are given by 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, . . . (2.2)

### 2.2 Generalized Padovan Sequence

Let $P(1) = 1, P(2) = 1, P(3) = k$ where $k$ is a positive real number. The sequence defined through the recurrence relation $P(n+3) = k\,P(n+1) + P(n), n \geq 1 \quad (2.3)$ is called generalized Padovan sequence.

The first few terms of the Generalized Padovan sequence are given by 1, 1, $k$, 2, $k + 1$, $3k$, $k^2 + k + 2$, $3k^2 + k + 1$, $k^3 + k^2 + 5k$, . . . (2.4)

Note that if $k = 1$, then we get the usual Padovan sequence defined in (2.1)

**2.3 Figurate Numbers**

The Figurate numbers of order $m \geq 3$ with respect to any natural number *k* is defined as $F_m(k) = \frac{(m-2)k^2 - (m-4)k}{2}$ (2.5)

We notice that if *m* = 3, then from (2.5), we get $F_3(k) = \frac{k^2+k}{2} = \frac{k(k+1)}{2}$ (2.6). The numbers generated through (2.6) for each positive integer *k*, are called Triangular Numbers.

Similarly if *m* = 4, then from (2.6), we get the square numbers given by $F_4(k) = k^2$ (2.7).

In general for each value of *m* = 3, 4, 5, 6, 7, … we get numbers representing polygons of that many sides in a plane, leading to the alternating name Polygonal numbers.

**2.4 Limiting Ratio**

The ratio of (*n* + 1)th term to *n*th term of any sequence (if it exists) is defined as the Limiting Ratio of that sequence. We denote the limiting ratio by $\lambda$. Thus if $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers then $\lambda = \lim \frac{s_{n+1}}{s_n}$ as $n \to \infty$ (2.8). Using the definition of limiting ratio, for any natural number *r* we have $\lim \frac{s_{n+r}}{s_n} = \lim\left(\frac{s_{n+r}}{s_{n+r-1}} \times \frac{s_{n+r-1}}{s_{n+r-2}} \times \cdots \times \frac{s_{n+1}}{s_n}\right) = \lambda \times \lambda \times \cdots \times \lambda = \lambda^r$ (2.9)

**3. Limiting Ratios of Generalized Padovan Sequence**

The characteristic equation of the Generalized Padovan sequence defined in (2.3) is given by $m^3 - km - 1 = 0$ (3.1). Since *k* is positive, by Descarte's rule of signs, equation (3.1) posses one positive real root.

Now from (2.3) we have $\frac{P(n+3)}{P(n)} = k\frac{P(n+1)}{P(n)} + 1$ (3.2). Thus, if $\lambda$ is the limiting ratio of the Generalized Padovan sequence, then using (2.9), equation (3.2) can be rewritten as $\lambda^3 = k\lambda + 1$ (3.3). From (3.1) and (3.3), we see that the limiting ratio is the desired root of the characteristic equation of the Genralized Padovan sequence.

Now we determine the limiting ratio $\lambda$ when *k* is very large, i.e. as $k \to \infty$. From (3.3) we have $\lambda^2 = k + \frac{1}{\lambda}$ (3.4). If $\lambda = O\left(\sqrt{k}\right)$ then $\frac{1}{\lambda} = \frac{1}{O\left(\sqrt{k}\right)} \to 0$ as $k \to \infty$. Hence as $k \to \infty$, from (3.4) we have $\lambda^2 = k$ from which we get $\lambda = \sqrt{k}$. Thus as $k \to \infty$, the limiting ratio of the Generalized Padovan sequence is $\sqrt{k}$ (3.5)

We now determine the limiting ratios of Generalized Padovan sequence for specific values of *k* say *k* = 1, 2, 4, 9, 64.

**3.1** If *k* = 1, then the limiting ratio $\lambda$ satisfies the equation $\lambda^3 - \lambda - 1 = 0$ (3.6). We notice that if $p(\lambda) = \lambda^3 - \lambda - 1$ then $p(1) = -1 < 0, p(2) = 5 > 0$. Hence the positive real root of (3.6) lies between 1 and 2. Using Newton-Raphson method, we find that the root of (3.6) is 1.32471 approximately. Thus the limiting ratio of Padovan sequence defined in (2.1) is approximately 1.32471. We call this number as Plastic Number.

**3.2** If *k* = 2, then the limiting ratio $\lambda$ satisfies the equation $\lambda^3 - 2\lambda - 1 = 0 \ (3.7)$. We know that the limiting ratio of Fibonacci sequence is a number called Golden Ratio given by $\varphi = \frac{1+\sqrt{5}}{2}$. This number is the positive root of the quadratic equation $\varphi^2 = \varphi + 1$. From this, we get $\varphi^3 = \varphi^2 + \varphi = (\varphi + 1) + \varphi = 2\varphi + 1$. Thus, the Golden Ratio $\varphi$ satisfies $\varphi^3 - 2\varphi - 1 = 0 \ (3.8)$. Hence from (3.7) and (3.8) we get $\lambda = \varphi$. Thus, when *k* = 2, the limiting ratio of the Generalized Padovan sequence is the Golden Ratio $\varphi = \frac{1+\sqrt{5}}{2}$ which is 1.618 approximately.

**3.3** If *k* = 4, then the limiting ratio $\lambda$ satisfies the equation $\lambda^3 - 4\lambda - 1 = 0 \ (3.9)$. We notice that if $p(\lambda) = \lambda^3 - 4\lambda - 1$ then $p(2) = -1 < 0, p(3) = 14 > 0$. Hence the positive real root of (3.9) lies between 2 and 3. Using Newton-Raphson method, we find that the root of (3.9) is 2.115 approximately. Thus, when *k* = 4, the limiting ratio of Generalized Padovan sequence is approximately 2.115.

**3.4** If *k* = 9, then the limiting ratio $\lambda$ satisfies the equation $\lambda^3 - 9\lambda - 1 = 0 \ (3.10)$. We notice that if $p(\lambda) = \lambda^3 - 9\lambda - 1$ then $p(3) = -1 < 0, p(4) = 27 > 0$. Hence the positive real root of (3.10) lies between 3 and 4. Using Newton-Raphson method, we find that the root of (3.10) is 3.054 approximately. Thus, when *k* = 9, the limiting ratio of Generalized Padovan sequence is approximately 3.054.

**3.5** If *k* = 64, then the limiting ratio $\lambda$ satisfies the equation $\lambda^3 - 64\lambda - 1 = 0 \ (3.11)$. We notice that if $p(\lambda) = \lambda^3 - 64\lambda - 1$ then $p(8) = -1 < 0, p(9) = 152 > 0$. Hence the positive real root of (3.11) lies between 8 and 9. Using Newton-Raphson method, we find that the root of (3.11) is 8.008 approximately. Thus, when *k* = 64, the limiting ratio of Generalized Padovan sequence is approximately 8.008.

We notice that the limiting ratio obtained in sections **3.4** and **3.5** verifies our limiting ratio calculation obtained in equation (3.5).

**4. Limiting Ratios of Generalized Padovan Sequence and Figurate Numbers**

We now consider two possible cases of constructing Generalized Padovan sequence using Figurate numbers.

**4.1** Let $P(1) = 1, P(2) = 1, P(3) = k$ where *k* is a positive real number. Let us define a generalized Padovan sequence using Figurate numbers (in equation (2.5)) through the following recurrence relation $P(n+3) = k^4 P(n+1) + F_m(k) P(n),\ n \geq 1 \ (4.1)$.

Now from (4.1), we get $\frac{P(n+3)}{P(n)} = k^4 \frac{P(n+1)}{P(n)} + F_m(k) \ \ (4.2)$

If $\lambda$ is the limiting ratio of the Generalized Padovan sequence as in (4.1), then using (2.9), equation (4.2) can be expressed as $\lambda^3 = k^4 \lambda + F_m(k) \ \ (4.3)$.

From this, we get $\lambda^2 = k^4 + \frac{F_m(k)}{\lambda} \ (4.4)$. From (2.5) by the definition of Figurate numbers, we note $F_m(k)$ is a second degree polynomial in *k*. Hence from (4.4) if the limiting ratio should exist, we should have $\lambda = O(k^2)$.

In particular if $\lambda = k^2$ then $\frac{F_m(k)}{\lambda} = \frac{1}{k^2}\left[\frac{(m-2)k^2 - (m-4)k}{2}\right] \to \frac{m-2}{2}$ as $k \to \infty$.

Hence as $k \to \infty$, equation (4.4) becomes $\lambda^2 = k^4 + \frac{m-2}{2}$ from which $\lambda = \sqrt{\frac{2k^4+m-2}{2}}$.

Thus, as $k \to \infty$, the limiting ratio of Generalized Padovan sequence and Figurate numbers as defined through (4.1) is $\sqrt{\frac{2k^4+m-2}{2}}$ (4.5)

As a verification of our computations done above, let us consider *k* = 12, *m* = 5. Then from (4.3), we get $\lambda^3 - 20736\lambda - 210 = 0$ (4.6). The limiting ratio is the positive root of (4.6). By Newton- Raphson method we find that the positive root of (4.6) i.e. the limiting ratio is 144.00506 which is approximately 144. We notice that the limiting ratio according to (4.5) is given by 144.0052 which agree to the value obtained through direct process to three decimal places. This verifies our computation of limiting ratio in (4.5).

We also note that for finite values of *m*, the limiting ratio of the recurrence relation (4.1) is approximately $k^2$ (4.7)

**4.2** Let $P(1)=1, P(2)=1, P(3)=k$ where *k* is a positive real number. Let us define a generalized Padovan sequence using Figurate numbers (in equation (2.5)) through the following recurrence relation $P(n+3) = F_m(k)P(n+1) + k\,P(n),\ n \geq 1$ (4.8).

From (4.8) we get $\frac{P(n+3)}{P(n)} = F_m(k)\frac{P(n+1)}{P(n)} + k$ (4.9)

If $\lambda$ is the limiting ratio of the Generalized Padovan sequence as in (4.8), then using (2.9), equation (4.9) can be expressed as $\lambda^3 = F_m(k)\lambda + k$ (4.10). From this we get $\lambda^2 = F_m(k) + \frac{k}{\lambda}$ (4.11). From (2.5) we know that the Figurate numbers is a polynomial of second degree in k. Hence if the limiting ratio $\lambda$ exists, then from (4.11), we should have $\lambda = O(k)$.

In particular, if $\lambda = k$ then we have $\frac{k}{\lambda} = \frac{k}{k} = 1$.

Thus (4.11) becomes $\lambda^2 = F_m(k) + 1$ from which $\lambda = \sqrt{F_m(k)+1}$ as $k \to \infty$

Hence, as $k \to \infty$, the limiting ratio of Generalized Padovan sequence and Figurate numbers as defined through (4.8) is $\sqrt{F_m(k)+1}$ (4.12).

As verification of our result, let us assume that *k* = 50, *m* = 12. Then from (2.5) we have $F_{12}(50) = 24600$. Hence from (4.10) we get $\lambda^3 - 24600\lambda - 50 = 0$ (4.13). The limiting ratio which is the positive root of (4.13) is found to be 156.84489. The limiting ratio according to (4.12) must be $\sqrt{24600+1} = 156.847$ approximately. We see that this agree to the exact computation for first two decimal places. Hence this example, verifies our computation for the recurrence defined in (4.8).

**5. Conclusions**

In section 3, we proved that the limiting ratio of Generalized Padovan sequence is $\sqrt{k}$ as $k \to \infty$. If *k* = 1, we found that the limiting ratio is plastic number which behaves the same way for Padovan sequence as the Golden Ratio does for Fibonacci sequence. If *k* = 2, we found that the limiting ratio is Golden Ratio itself. In section 4, by considering Figurate numbers as coefficients in two possible ways we found the limiting ratios are approximately $k^2$ and $\sqrt{F_m(k)+1}$ respectively. We see that these expressions generate limiting ratios for each positive value of $k$.

If we consider other general recurrence relations apart from what we have considered in section 4 through equations (4.1) and (4.8) then the limiting ratios may not even exist. Moreover, the computation of limiting ratios in each case is verified using suitable examples. This paper thus discuss interesting ways in which Generalized Padovan sequences can be interacted with Figurate numbers to provide amusing answers as limiting ratios.

**REFERENCES**

[1] R. Sivaraman, Generalized Padovan and Polynomial Sequences, J. Math. Comput. Sci., 11 (2021), No. 1, pp. 219 – 226.

[2] R. Sivaraman, Properties of Padovan Sequence, Turkish Journal of Computer and Mathematics Education, Vol. 12, No. 2, (2021), 3098 – 3101.

[3] R. Sivaraman, Pythagorean Triples and Generalized Recursive Sequences, Mathematical Sciences International Research Journal, Volume 10, Issue 2, July 2021, pp. 1 – 5.

[4] R. Sivaraman, Limiting Ratios of Generalized Recurrence Relations, Asian Journal of Mathematical Sciences, Volume 5, Issue 3, Jul – Sep 2021, pp. 1 – 4.

[5] R. Sivaraman, P.N. Vijayakumar, J. Suganthi, Limiting Behavior of Limiting Ratios of Generalized Recursive Relations, International Journal of Early Childhood Special Education, Vol 14, Issue 03, 2022, 3671 – 3676.

[6] A. Dinesh Kumar, R. Sivaraman, Asymptotic Behavior of Limiting Ratios of Generalized Recurrence Relations, Journal of Algebraic Statistics, Volume 13, No. 2, 2022, 11 – 19.

[7] A. Dinesh Kumar, R. Sivaraman, Analysis of Limiting Ratios of Special Sequences, Mathematics and Statistics, Vol. 10, No. 4, (2022), pp. 825 – 832

[8] N. Calkin, H.S. Wilf, Recounting the Rationals, American Mathematical Monthly 107 (4) (2000) 360-363

[9] R. Sivaraman, Recognizing Ramanujan’s House Number Puzzle, German International Journal of Modern Science, 22, November 2021, pp. 25 – 27

[10] R. Sivaraman, J. Suganthi, A. Dinesh Kumar, P.N. Vijayakumar, R. Sengothai, On Solving an Amusing Puzzle, Specialusis Ugdymas/Special Education, Vol 1, No. 43, 2022, 643 – 647.

[11] A. Dinesh Kumar, R. Sivaraman, On Some Properties of Fabulous Fraction Tree, Mathematics and Statistics, Vol. 10, No. 3, (2022), pp. 477 – 485.

[12] R. Sengothai, R. Sivaraman, Solving Diophantine Equations using Bronze Ratio, Journal of Algebraic Statistics, Volume 13, No. 3, 2022, 812 – 814.

[13] P.N.Vijayakumar, R. Sivaraman, On Solving Euler’s Quadratic Diophantine Equation, Journal of Algebraic Statistics, Volume 13, No. 3, 2022, 815 – 817.

[14] R. Sivaraman, Generalized Lucas, Fibonacci Sequences and Matrices, Purakala, Volume 31, Issue 18, April 2020, pp. 509 – 515.

[15] R. Sivaraman, J. Suganthi, P.N. Vijayakumar, R. Sengothai, Generalized Pascal’s Triangle and its Properties, NeuroQuantology, Vol. 22, No. 5, 2022, 729 – 732.

[16] Andreescu, T., D. Andrica, and I. Cucurezeanu, An introduction to Diophantine equations: A problem-based approach, Birkhäuser Verlag, New York, 2010.