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Construction of Perfect Squares and Pythagorean Triples using Continued Fractions

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Abstract:

Continued fractions play a major role in Number theory and solving Diophantine equations. With the help of Continued fractions and Pell's equations, we have proved some interesting results like obtaining Pythagorean triples whose smaller sides differ by some particular integer and also we have provided methods to generate primitive Pythagorean triples satisfying a specific condition. Perfect squares are one of the most well-known class of numbers in number theory. In this paper we introduced ways to produce natural numbers that are related to perfect squares with some specified conditions.

Keywords: Continued fractions, Primitive Pythagorean Triple,Perfect square, Pell's equation.

1. Introduction:

Since ancient times, the study of Pythagorean triples has generated a great interest among the mathematicians. In this paper, in section 3, we have derived a result to generate Pythagorean triples whose smaller sides differ by k where k is a particular integer, using Pell's equation and continued fractions. In section 4, using continued fractions we proved a result to obtain primitive Pythagorean triples whose smallersides differ by a number of the form$2n^2 - 1$ where $n \in \mathbb{N}$.

In mathematics perfect squares are the numbers of the form $n^2, n \in \mathbb{N}$. In this paper, we proved a theorem that, there are infinitely many integers n such that both $n - k$ and $\frac{n-1}{k}$ are perfect squares for some specific non-perfect square number k satisfying certain conditions. **2. Preliminaries**

In this section let us make some basic definitions and prove a useful lemma needed to prove further results.

2.1. Definition

A finite continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}} \qquad (2.1)$$

where $a_0, a_1, \ldots a_n$ are real numbers and $a_1, a_2, \ldots a_n$ are positive. The a_i's are called the partial quotients of the continued fraction. If the partial quotients are all integers, then the continued fraction is simple. We use the notation $[a_0; a_1, \ldots a_n]$ to represent the continued fraction given in equation (2.1). When $n = 0$ we write $[a_0]$

2.2. Definition

For $1 \leq k \leq n$, the k^{th} convergent C_K of a continued fraction $[a_0; a_1, a_{2,} \dots . a_n]$ is the continued fraction

$$C_k = [a_0; a_1, a_2, \dots \dots a_k]$$

We extend this definition to include $k = 0$ and so we set $C_0 = a_0$.

2.3. Definition

A Pythagorean triple consists of three positive integers a, b and c such that $a^2 + b^2 = c^2$. Primitive Pythagorean triple is a Pythagorean triple (a, b, c) such that GCD$(a, b, c) = 1$, where GCD is the greatest common divisor.

2.4. Definition

A Pell's equation is a quadratic Diophantine equation of the form$x^2 - dy^2 = 1$ where $x, y \in \mathbb{Z}$ and d is a given natural number which is not a square.

2.5 Lemma 1

Consider the quadratic Diophantine equation $ax^2 - by^2 = c$ (2.2)

andlet $D = ab$ and $u^2 - Dv^2 = 1$ (2.3)

Let (x_0, y_0) be the initial solutions of (2.2). Then $x = x_0u + b\beta v$, $y = \beta u + \alpha v$ (2.4)

and $x = |-x_0u + b\beta v|$, $y = |\beta u - \alpha v|$ (2.5) are the solutions of equation (2.2) where $(u, v)$are solutions of equation (2.3),

$\alpha = ax_0, \beta = y_0$ (2.6)

Proof:

From the hypothesis, we have $ax_0^2 - by_0^2 = c$. Using (2.3),(2.4) and (2.6), we have

$$ax^2 - by^2 = a(x_0u + by_0v)^2 - b(y_0u + ax_0v)^2 = u^2(ax_0^2 - by_0^2) - abv^2(ax_0^2 - by_0^2)$$
$$= (u^2 - abv^2)(ax_0^2 - by_0^2) = 1(c) = c$$

Hence the values of x and y from (2.4) forms solutions to (2.2). Now using (2.5), we have

$$ax^2 - by^2 = a(-x_0u + by_0v)^2 - b(y_0u - ax_0v)^2 = u^2(ax_0^2 - by_0^2) - abv^2(ax_0^2 - by_0^2)$$
$$= (u^2 - abv^2)(ax_0^2 - by_0^2) = 1(c) = c$$

Thus the values of x and y from (2.5) forms solutions to (2.2).

Also we notice that if (x_0, y_0) is one of the solutions to (2.2), then we can generate infinitely many solutions of (2.2), using (2.4) and (2.5).

In the following sections we prove some interesting results.

3. Pythagorean Triples

Pythagorean triples has several interesting properties. In this section, we will prove a result to generate Pythagorean triples satisfying a specific property.

3.1 Theorem 1

There exists infinitely many Pythagorean triples such that the two smaller numbers differ by k, where $k \in \mathbb{N}$.

Proof:

Let x and $x+k$ be the smaller sides of a right - angled triangle, which differ by k, where $k \in \mathbb{N}$ and let y be its hypotenuse as shown in figure 1.

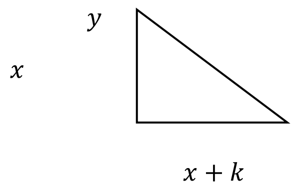

Figure 1

Using Pythagoras theorem for right triangle, we get,

$$x^2+(x+k)^2=y^2$$

$$2x^2+2xk+k^2-y^2=0$$

$$x=\frac{-2k\pm\sqrt{4k^2-8(k^2-y^2)}}{4}$$

$$4x+2k=\pm\sqrt{4k^2-8(k^2-y^2)}$$

$$(2x+k)^2-2y^2=-k^2 \qquad (3.1)$$

Now consider, $X=2x+k$, then equation (3.1) becomes $X^2-2y^2=-k^2$ (3.2)

one of the solutions for equation (3.2) is $(X_0,y_0)=(k,k)$.

Comparing equation (3.2) with (2.2) we get, $a=1, b=2$ and $c=-k^2$

$$\therefore D=ab=2$$

Now consider, $u^2-2v^2=1$ (3.3)

The trivial solution of equation (3.3) is $(u,v)=(1,0)$. The successive convergents of the continued fraction expansion of $\sqrt{2}=[1;\bar{2}]$are given by

$$1,\frac{3}{2},\frac{7}{5},\frac{17}{12},\frac{41}{29},\frac{99}{70},\frac{239}{169},\frac{577}{408},\ldots$$

Considering the trivial solution and the alternative successive convergents of above values, the solutions to (3.3) are given by

$(u,v)=(1,0),(3,2),(17,12),(99,70),(577,408),\ldots$ (3.4)

From (2.6) of lemma 1, $\alpha=aX_0=k$ and $\beta=y_0=k$ and by (2.4) of lemma1 we obtain,

$$X=X_0u+b\beta v=ku+2kv=k(u+2v) \qquad (3.5)$$

$$y=\beta u+\alpha v=k(u+v) \qquad (3.6)$$

From equations (3.5) and (3.6), for every pair of (u,v) given by (3.4), we obtain the solutions of equation (3.2).

i.e. $(X,y)=(k,k),(7k,5k),(41k,29k),(239k,169k),(1393k,985k),\ldots$

Since $x=\frac{X-k}{2}$, we obtain,

$(x,y)=(0,k),(3k,5k),(20k,29k),(119k,169k),(696k,985k),\ldots$ (3.7)

By (2.5) of lemma1,we obtain,

$$X = |-X_0 u + b\beta v| = |-ku + 2kv| = |k(-u + 2v)| \quad (3.8)$$

$$y = |\beta u - \alpha v| = |k(u - v)| \quad (3.9)$$

From equations (3.8) and (3.9), for every pair of (u, v) given by (3.4), we obtain the solutions of equation (3.2) as given below,

$$(X, y) = (k, k), (k, k), (7k, 5k), (41k, 29k), (239k, 169k), (1393k, 985k), \dots$$

Since $x = \frac{X-k}{2}$, we obtain,

$(x, y) = (0, k), (0, k), (3k, 5k), (20k, 29k), (119k, 169k), (696k, 985k), \dots$ (3.10)

Notice that (3.7) and (3.10) produces the same values, so we can omit (3.10) and also we can omit the pair $(0, k)$ since numbers forming Pythagorean triples must be positive.

Hence using (3.7), we obtain the following list of right triangles whose two smaller sides differ by k.

$\boldsymbol{x}$	$\boldsymbol{x+k}$	$\boldsymbol{y}$
$3k$	$4k$	$5k$
$20k$	$21k$	$29k$
$119k$	$120k$	$169k$
$696k$	$697k$	$985k$
...	...	...

Table 1

From table 1, we can observe that the Pythagorean triples we obtained are multiples of k**.** Since we can obtain infinitely many solutions through (3.7), there exists infinitely many Pythagorean Triples such that the two smaller numbers differ by k where $k \in \mathbb{N}$.

3.2. Corollary 1

There exists infinitely many Pythagorean Triples whose two smaller numbers differ by $k = 1,2$.

Solution:

Since $k = 1,2$ are natural numbers, by theorem 1, there exists infinitely many Pythagorean triples whose smaller sides differ by $k = 1,2$.

Hence we obtain the following Pythagorean triples whosesmaller numbers differ by 1 by substituting $k = 1$ in table 1 of theorem 1.

$(3,4,5), (20,21,29), (119,120,169), (696,697,985), \dots$

and we get the following Pythagorean triples whose smaller numbers differ by 2 by substituting $k = 2$ in table 1 of theorem 1.

$(6,8,10), (40,42,58), (238,240,338), (1392,1394,1970), \dots$

4. Primitive Pythagorean Triples

In this section we provided a nice continued fraction expansion for $\sqrt{2n^2 - 1}, n \in \mathbb{N}$ and also we are focusing on obtaining primitive Pythagorean triples satisfying a particular condition using the convergents of $\sqrt{2n^2 - 1}$.

4.1. Theorem 2

If n is a natural number, then the continued fraction expansion for $\sqrt{2n^2 - 1}$ is given by,

$$n\sqrt{2} - \cfrac{1}{2n\sqrt{2} - \cfrac{1}{2n\sqrt{2} - \cfrac{1}{2n\sqrt{2} - \dots}}}$$

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Proof :

For a natural number n, consider, $\left(\sqrt{2n^2-1}-n\sqrt{2}\right)\left(\sqrt{2n^2-1}+n\sqrt{2}\right)=-1$

$$\sqrt{2n^2-1}-n\sqrt{2} = \frac{-1}{\sqrt{2n^2-1}+n\sqrt{2}} \qquad (4.1)$$

$$= \frac{-1}{\sqrt{2n^2-1}+n\sqrt{2}+n\sqrt{2}-n\sqrt{2}}$$

$$= \frac{-1}{2n\sqrt{2}-\frac{1}{\sqrt{2n^2-1}+n\sqrt{2}}} \quad \text{by equation (4.1)}$$

...

$$\sqrt{2n^2-1} = n\sqrt{2} - \frac{1}{2n\sqrt{2}-\frac{1}{2n\sqrt{2}-\frac{1}{2n\sqrt{2}-\dots}}} \qquad (4.2)$$

Using definition 2.2, the convergents of the continued fraction (4.2) are given below,

$$C_0 = n\sqrt{2}$$

$$C_1 = \frac{4n^2-1}{2n\sqrt{2}}$$

$$C_2 = \frac{(8n^3-3n)\sqrt{2}}{8n^2-1}$$

$$C_3 = \frac{32n^4-16n^2+1}{(16n^3-4n)\sqrt{2}}$$

$$C_4 = \frac{(64n^5-40n^3+5n)\sqrt{2}}{64n^4-24n^2+1}$$

$$C_5 = \frac{256n^6-192n^4+36n^2-1}{(128n^5-64n^3+6n)\sqrt{2}}$$

...

Using the convergents of the continued fraction expansion of $\sqrt{2n^2-1}$, we will provide some ways to generate primitive Pythagorean triples satisfying specific property in the following theorems.

4.2. Theorem 3

There exists infinitely many primitive Pythagorean triples whose smaller sides differ by the number of the form $2n^2-1$ where $n \in \mathbb{N}$.

Proof:

Let $x, x+(2n^2-1)$ be the smaller sides of the right triangle where $n \in \mathbb{N}$ and let y be the hypotenuse as shown in Figure 2

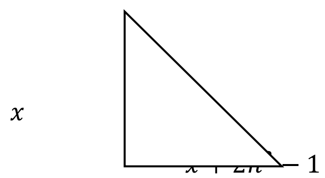

Figure 2

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From the denominator of the convergent C_1 of theorem 2, choose the coefficient of $\sqrt{2}$ that is $2n$ and consider one of the smaller side of the right triangle to be $2n \times (n+1)$ where $n \in \mathbb{N}$.

Since two smaller sides must be differ by $2n^2 - 1$, we can find the other smaller side of a right triangle i.e $x = 2n(n+1) - (2n^2 - 1) = 2n + 1$

By Pythagorean Theorem, $y^2 = (2n+1)^2 + \left(2n(n+1)\right)^2$

$$= 4n^2 + 4n + 1 + \left(2n(n+1)\right)^2 = \left(2n(n+1)\right)^2 + 4n(n+1) + 1 = (2n(n+1)+1)^2$$

Therefore $y = 2n(n+1) + 1$.

Hence, $(2n+1, 2n(n+1), 2n(n+1)+1)$, $n \in \mathbb{N}$ (4.3) are the Pythagorean triple whose smaller sides differ by $2n^2 - 1$.

Now let us prove that the Pythagorean triple (4.3) is primitive. Let $d \geq 1$ be the common factor of Pythagorean triple (4.3), then $d|2n+1, d|2n(2n+1)$ and $d|2n(n+1)+1$.

This implies $d|2n(n+1) + 1 - 2n(2n+1) = 1$, therefore $d = 1$. Hence the Pythagorean triple (4.3) is primitive.

Hence $2n+1, 2n(n+1), 2n(n+1)+1$, $n \in \mathbb{N}$ forms infinitely many primitive Pythagorean triples whose smaller sides differ by the number of the form $2n^2 - 1$ where$n \in \mathbb{N}$. Using (4.3) we obtain the following primitive Pythagorean triple whose smaller side differ by the number of the form $2n^2 - 1, n \in \mathbb{N}$.

n	Difference between smaller sides $2n^2 - 1$	$x = 2n + 1$	$x + 2n^2 - 1 = 2n(n+1)$	Hypotenuse $y = 2n(n+1) + 1$
1	1	3	4	5
2	7	5	12	13
3	17	7	24	25
4	31	9	40	41
5	49	11	60	61
...	...	...	...	...

Table 2

4.3. Theorem 4

There exists infinitely many primitive Pythagorean triples whose smaller sides differ by the number of the form $16n^2(2n^2 - 1) + 1$where $n \in \mathbb{N}$.

Proof:

Let x, $x + 16n^2(2n^2 - 1) + 1$ be the smaller sides of the right triangle where $n \in \mathbb{N}$ and let$y$ be the hypotenuse as shown in Figure 3

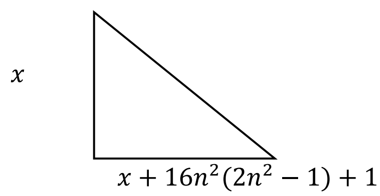

Figure 3

From the denominator of the convergent C_3 of theorem 2, choose the coefficient of $\sqrt{2}$ i.e.$16n^3 - 4n$. Also from the denominator of the convergent C_1 of theorem 2, choose the coefficient of $\sqrt{2}$ i.e.$2n$.

Now consider one of the smaller sides of the right triangle to be $(16n^3 - 4n)(2n)$, $n \in \mathbb{N}$.Since two smaller sides must be differ by $16n^2(2n^2 - 1) + 1$, we can find the other smaller side of a right triangle i.e $x = (16n^3 - 4n)(2\text{n}) - (16n^2(2n^2 - 1) + 1) = 8n^2 - 1$.

We can observe that the value of x is nothing but the denominator of the convergent C_2 of theorem 2.

By Pythagorean theorem, $y^2 = \left((16n^3 - 4n)(2n)\right)^2 + (8n^2 - 1)^2$

$$= \left((16n^3 - 4n)(2n)\right)^2 + 64n^4 - 16n^2 + 1$$

$$= \left((16n^3 - 4n)(2n) + 1\right)^2$$

Therefore $y = (16n^3 - 4n)(2n) + 1, n \in \mathbb{N}$.

Hence $(8n^2 - 1, (16n^3 - 4n)(2n), (16n^3 - 4n)(2n) + 1), n \in \mathbb{N}$ (4.4)

are the Pythagorean triple whose smaller sides differ by $16n^2(2n^2 - 1) + 1$.

Now let us prove that the Pythagorean triple (4.4) is primitive. Let $d \geq 1$ be the common factor of Pythagorean triple (4.4), then $d|8n^2 - 1, d|(16n^3 - 4n)(2n)$ and $d|(16n^3 - 4n)(2n) + 1$.

This implies $d|(16n^3 - 4n)(2n) + 1 - (16n^3 - 4n)(2n) = 1$,therefore $d = 1$. Hence the Pythagorean triple (4.4) forms infinitely many primitive Pythagorean triples whose smaller sides differ by $16n^2(2n^2 - 1) + 1, n \in \mathbb{N}$.

Using (4.4) we obtain the following primitive Pythagorean triple whose smaller side differ by the number of the form $16n^2(2n^2 - 1) + 1, n \in \mathbb{N}$.

n	Difference between smaller sides $16n^2(2n^2 - 1) + 1$	$x = 8n^2 - 1$	$x + 16n^2(2n^2 - 1) + 1 = (16n^3 - 4n)(2n)$	Hypotenuse, $y = (16n^3 - 4n)(2n) + 1$
1	17	7	24	25
2	449	31	480	481
3	2449	71	2520	2521
4	7937	127	8064	8065
5	19601	199	19800	19801
...	...	...	...	...

Table 3

4.4. Theorem 5

There exists infinitely many primitive Pythagorean triples whose smaller sides differ by the number of the form $2048n^8 - 1536n^6 + 288n^4 - 1$ where $n \in \mathbb{N}$.

Proof:

Let x, $x + 2048n^8 - 1536n^6 + 288n^4 - 1$ be the smaller sides of the right triangle where $n \in \mathbb{N}$ and let$y$ be the hypotenuse as shown in figure 4

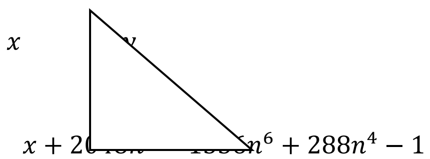

Figure 4

From the denominator of the convergent C_5 of theorem 2, choose the coefficient of $\sqrt{2}$ i.e.$128n^5 - 64n^3 + 6n$. Also from the denominator of the convergent C_3 of theorem 2, choose the coefficient of $\sqrt{2}$ i.e.$16n^3 - 4n$.

Now consider one of the smaller side of the right triangle to be

$$(128n^5 - 64n^3 + 6n)(16n^3 - 4n),\ n \in \mathbb{N}.$$

Since two smaller sides must be differ by $2048n^8 - 1536n^6 + 288n^4 - 1$, we can find the other smaller side of a right triangle i.e ,

$$x = (128n^5 - 64n^3 + 6n)(16n^3 - 4n) - 2048n^8 - 1536n^6 + 288n^4 - 1$$

$= 64n^4 - 24n^2 + 1.$

We can observe that the value of x is nothing but the denominator of the convergent C_4 of theorem 2.

By Pythagorean theorem,

$$y^2 = \left((128n^5 - 64n^3 + 6n)(16n^3 - 4n)\right)^2 + (64n^4 - 24n^2 + 1)^2$$

$$= \left((128n^5 - 64n^3 + 6n)(16n^3 - 4n)\right)^2$$

$$+2(128n^5 - 64n^3 + 6n)(16n^3 - 4n) + 1$$

$$= \left((128n^5 - 64n^3 + 6n)(16n^3 - 4n) + 1\right)^2$$

Therefore $y = (128n^5 - 64n^3 + 6n)(16n^3 - 4n) + 1, n \in \mathbb{N}$.

Hence $(64n^4 - 24n^2 + 1, (128n^5 - 64n^3 + 6n)(16n^3 - 4n),$

$(128n^5 - 64n^3 + 6n)(16n^3 - 4n) + 1), n \in \mathbb{N}$ (4.5)

are the Pythagorean triple whose smaller sides differ by $2048n^8 - 1536n^6 + 288n^4 - 1$.

Now let us prove that the Pythagorean triple (4.5) is primitive.Let $d \geq 1$ be the common factor of Pythagorean triple (4.5), then$d | 64n^4 - 24n^2 + 1$,

$$d |\ (128n^5 - 64n^3 + 6n)(16n^3 - 4n) \text{ and } d |\ (128n^5 - 64n^3 + 6n)(16n^3 - 4n) + 1$$

$$\Rightarrow d |\ (128n^5 - 64n^3 + 6n)(16n^3 - 4n) + 1 - (128n^5 - 64n^3 + 6n)(16n^3 - 4n) = 1$$

Therefore $d = 1$. Hence the Pythagorean triple (4.5) forms infinitely many primitive Pythagorean triples whose smaller sides differ by $2048n^8 - 1536n^6 + 288n^4 - 1$, $n \in \mathbb{N}$.

Using (4.5) we obtain the following primitive Pythagorean triple whose smaller side differ by the number of the form $2048n^8 - 1536n^6 + 288n^4 - 1, n \in \mathbb{N}$.

n	Difference between smaller sides $2048n^8 - 1536n^6 + 288n^4 - 1$	$x = 64n^4 - 24n^2 + 1$	$x + 2048n^8 - 1536n^6 + 288n^4 - 1 = (128n^5 - 64n^3 + 6n)(16n^3 - 4n)$	Hypotenuse, $y = (128n^5 - 64n^3 + 6n)(16n^3 - 4n) + 1$
1	799	41	840	841
2	430591	929	431520	431521
3	12340511	4969	12345480	12345481
4	127999999	16001	128016000	128016001
...	...	...	...	...

Table 4

5. Perfect Squares

In this section we are concentrating to prove some results relating to perfect squares using continued fractions.

5.1. Theorem 6

There exists infinitely many $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect squares for any non - perfect square $k \in \mathbb{N}$ where $k+1$ is perfect square.

Proof:

Since $n-k$ and $\frac{n-1}{k}$ are perfect square, let us assume

$n-k = x^2$ and $\frac{n-1}{k} = y^2$ (5.1)

From equations (5.1), $n = x^2 + k$ (5.2) and $n = ky^2 + 1$ (5.3)

Now from equation (5.2) and (5.3) we get, $x^2 + k = ky^2 + 1$

$$x^2 - ky^2 = -(k-1) \quad (5.4)$$

One of the solution of equation (5.4) is $(x_0, y_0) = (1,1)$ and comparing equation (5.4) with (2.2) of lemma 1, we get, $a = 1, b = k$ and $c = -(k-1)$

$\therefore D = ab = k$ is not a perfect square.

Now consider, $u^2 - kv^2 = 1$ (5.5)

The trivial solution of equation (5.5) is $(u, v) = (1,0)$

$$\alpha = ax_0 = 1 \text{ and } \beta = y_0 = 1$$

By (2.4) of lemma 1 we obtain,

$$x = x_0u + b\beta v = u + kv \quad (5.6)$$

$$y = \beta u + \alpha v = u + v \quad (5.7)$$

and by (2.5) of lemma 1 we obtain,

$$x = |-x_0u + b\beta v| = |-u + kv| \quad (5.8)$$

$$y = |\beta u - \alpha v| = |u - v| \quad (5.9)$$

To solve equation (5.5), let us expand $\sqrt{k}$ as continued fraction in terms of $\sqrt{k+1}$.

Consider $\left(\sqrt{k+1} - \sqrt{k}\right)\left(\sqrt{k+1} + \sqrt{k}\right) = 1$

$$\sqrt{k+1} - \sqrt{k} = \frac{1}{\sqrt{k+1} + \sqrt{k}} \quad (5.10)$$

$$= \frac{1}{\sqrt{k+1} + \sqrt{k} + \sqrt{k+1} - \sqrt{k+1}}$$

$$= \frac{1}{2\sqrt{k+1} - \frac{1}{\sqrt{k+1} + \sqrt{k}}} \quad \text{(by equation (5.10))}$$

...

$$= \frac{1}{2\sqrt{k+1} - \frac{1}{2\sqrt{k+1} - \frac{1}{2\sqrt{k+1} - \ddots}}}$$

$$\sqrt{k} = \sqrt{k+1} - \frac{1}{2\sqrt{k+1} - \frac{1}{2\sqrt{k+1} - \frac{1}{2\sqrt{k+1} - \ddots}}} \quad (5.11)$$

Using definition 2.2, the convergents of the continued fraction (5.11) are given below,

$$C_0 = \sqrt{k+1}$$

$$C_1 = \frac{2k+1}{2\sqrt{k+1}}$$

$$C_2 = \frac{(4k+1)\sqrt{k+1}}{4k+3}$$

$$C_3 = \frac{8k^2+8k+1}{(8k+4)\sqrt{k+1}}$$

$$C_4 = \frac{(16k^2+12k+1)\sqrt{k+1}}{16k^2+20k+5}$$

...

Since $k+1$ is perfect square, let $k+1=\lambda_1^2$. Then $\lambda_1 = \sqrt{k+1}$ is apositive integer.

Considering the trivial solution and the above convergent values, the solution of the equation (5.5) are given below,

$(u,v) = (1,0),(\lambda_1,1),(2k+1,2\lambda_1),\big((4k+1)\lambda_1,4k+3\big),$

$(8k^2+8k+1,(8k+4)\lambda_1),\big((16k^2+12k+1)\lambda_1,16k^2+20k+5\big),\ldots$ (5.12)

From equations (5.6) and (5.7), for every pair of (u,v) given by (5.12), we obtain the solutions of equation (5.4) as follows,

$(x,y) = (1,1),(\lambda_1+k,\lambda_1+1),(2k+1+2k\lambda_1,2k+1+2\lambda_1),$

$\big((4k+1)\lambda_1+4k^2+3k,(4k+1)\lambda_1+4k+3\big),$

$(8k^2+8k+1+(8k^2+4k)\lambda_1,8k^2+8k+1+(8k+4)\lambda_1),\ldots$(5.13)

From equations (5.8) and (5.9), for every pair of (u,v) given by (5.12), we obtain the solutions of equation (5.4) as follows,

$(x,y) = (1,1),(|-\lambda_1+k|,|\lambda_1-1|),(|-(2k+1)+2k\lambda_1|,|2k+1-2\lambda_1|),$

$(|-(4k+1)\lambda_1+4k^2+3k|,|(4k+1)\lambda_1-(4k+3)|),$

$(|-(8k^2+8k+1)+(8k^2+4k)\lambda_1|,|8k^2+8k+1-(8k+4)\lambda_1|),\ldots$(5.14)

Combining (5.13) and (5.14) we obtain,

$(x,y) = (1,1),(|-\lambda_1+k|,|\lambda_1-1|),(\lambda_1+k,\lambda_1+1),$

$(|-(2k+1)+2k\lambda_1|,|2k+1-2\lambda_1|),(2k+1+2k\lambda_1,2k+1+2\lambda_1),$

$(|-(4k+1)\lambda_1+4k^2+3k|,|(4k+1)\lambda_1-(4k+3)|),\ldots$(5.15)

From (5.2), for every values of x in (5.15) we obtain the values of n as follows,

$$n = 1+k,(-\lambda_1+k)^2+k,(\lambda_1+k)^2+k,(-(2k+1)+2k\lambda_1)^2+k,(2k+1+2k\lambda_1)^2 +k,(-(4k+1)\lambda_1+4k^2+3k)^2+k,$$

$\big((4k+1)\lambda_1+4k^2+3k\big)^2+k,\ldots$ (5.16)

Notice $n\in\mathbb{N}$, since k and λ_1 are natural numbers.

Hence every values of n in equation (5.16) are both $n-k$ and $\frac{n-1}{k}$ are perfect square when $k+1$ is a perfect square.

Observe that, there exists infinitely many $n\in\mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for any $k\in\mathbb{N}$ of the form $k=m^2+4m+3, m\geq 0$.

5.2. Illustration 1

Find $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for $k=3,8$.

Solution:

By theorem 6, there exists infinitely many $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for $k=3,8$ since $k=3,8$ are natural numbers, non – perfect squares and $k+1$ is a perfect square.

Substituting $k=3$ and $\lambda_1=\sqrt{k+1}=2$ in (5.16) of theorem 6 we get,

$$n=4,4,28,28,364,364,5044,\dots$$

And substituting $k=8$ and $\lambda_1=\sqrt{k+1}=3$ in (5.16) of theorem 6 we get,

$$n=9,33,129,969,\dots$$

Verification:

k	n	$n-k$	$\frac{n-1}{k}$
3	4	1	1
	28	$25=5^2$	$9=3^2$
	364	$361=19^2$	$121=11^2$
	5044	$5041=71^2$	$1681=41^2$
	…	…	…
8	9	1	1
	33	$25=5^2$	$4=2^2$
	129	$121=11^2$	$16=4^2$
	969	$961=31^2$	$121=11^2$
	…	…	…

Table 5

5.3. Theorem 7

There exists infinitely many $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for any non - perfect square $k \in \mathbb{N}$ satisfying $k-1$ is perfect square.

Proof:

Since $n-k$ and $\frac{n-1}{k}$ are perfect square, from theorem 6, it is enough to solve (5.5).

To solve equation (5.5), let us expand $\sqrt{k}$ as continued fraction in terms of $\sqrt{k-1}$.

Consider $\left(\sqrt{k}-\sqrt{k-1}\right)\left(\sqrt{k}+\sqrt{k-1}\right)=1$

$$\sqrt{k}-\sqrt{k-1}=\frac{1}{\sqrt{k}+\sqrt{k-1}} \qquad (5.17)$$

$$=\frac{1}{\sqrt{k}+\sqrt{k-1}+\sqrt{k-1}-\sqrt{k-1}}$$

$$=\frac{1}{2\sqrt{k-1}+\frac{1}{\sqrt{k}+\sqrt{k-1}}} \qquad \text{(by equation (5.17))}$$

…

$$=\frac{1}{2\sqrt{k-1}+\frac{1}{2\sqrt{k-1}+\frac{1}{2\sqrt{k-1}+\ddots}}}$$

$$\sqrt{k} = \sqrt{k-1} + \cfrac{1}{2\sqrt{k-1} + \cfrac{1}{2\sqrt{k-1} + \cfrac{1}{2\sqrt{k-1} + \ddots}}} \qquad (5.18)$$

Using definition 2.2, the convergents of the continued fraction (5.18) are given below,

$$C_0 = \sqrt{k-1}$$

$$C_1 = \frac{2k-1}{2\sqrt{k-1}}$$

$$C_2 = \frac{(4k-1)\sqrt{k-1}}{4k-3}$$

$$C_3 = \frac{8k^2-8k+1}{(8k-4)\sqrt{k-1}}$$

$$C_4 = \frac{(16k^2-12k+1)\sqrt{k-1}}{16k^2-20k+5}$$

...

Since $k-1$ is perfect square, let $k-1=\lambda_2^2$. Then $\lambda_2 = \sqrt{k-1}$ is a positive integer.

Considering the trivial solution and the above successive alternativeconvergent values, the solutions of the equation (5.5) are given below

$(u, v) = (1,0), (2k-1, 2\lambda_2), (8k^2-8k+1, (8k-4)\lambda_2), \dots$ (5.19)

From equations (5.6) and (5.7), for every pair of (u, v) given by (5.19), we obtain the solutions of equation (5.4) as follows

$(x, y) = (1,1), (2k-1+2k\lambda_2, 2k-1+2\lambda_2),$

$(8k^2-8k+1+(8k^2-4k)\lambda_2, 8k^2-8k+1+(8k-4)\lambda_2), \dots$(5.20)

From equations (5.8) and (5.9), for every pair of (u, v) given by (5.19), we obtain the solutions of equation (5.4) as follows

$(x, y) = (1,1), (|-(2k-1)+2k\lambda_2|, |2k-1-2\lambda_2|),$

$(|-(8k^2-8k+1)+(8k^2-4k)\lambda_2|, |8k^2-8k+1-(8k-4)\lambda_2|), \dots$(5.21)

Combining (5.20) and (5.21) we obtain,

$(x, y) = (1,1), (|-(2k-1)+2k\lambda_2|, |2k-1-2\lambda_2|), (2k-1+2k\lambda_2, 2k-1+2\lambda_2),$

$(|-(8k^2-8k+1)+(8k^2-4k)\lambda_2|, |8k^2-8k+1-(8k-4)\lambda_2|), \dots$(5.22)

From (5.2), for every values of x in (5.22) we obtain the values of n as follows,

$n = 1+k, (-(2k-1)+2k\lambda_2)^2+k, (2k-1+2k\lambda_2)^2+k,$

$(-(8k^2-8k+1)+(8k^2-4k)\lambda_2)^2+k, \dots$(5.23)

Notice $n \in \mathbb{N}$, since k and λ_2 are natural numbers.

Hence every values of n in equation (5.23) are both $n-k$ and$\frac{n-1}{k}$are perfect square when $k-1$ isperfect square.

Observe that, there exists infinitely many $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for any $k \in \mathbb{N}$ of the form $k = m^2+2m+2, m \geq 0$.

5.4. Illustration 2

Find $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for $k = 5,10$.

Solution:

By theorem 7, there exists infinitely many $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for $k = 5,10$ since $k = 5,10$ are natural numbers, non – perfect squares and $k-1$ is a perfect square.

Substituting $k = 5$ and $\lambda_2 = \sqrt{k-1} = 2$ in (5.23) of theorem 7 we get,

$$n = 6, 126, 846, 39606, \ldots$$

And substituting $k = 10$ and $\lambda_2 = \sqrt{k-1} = 3$ in (5.23) of theorem 7 we get,

$$n = 11, 1691, 6251, 2430491, \ldots$$

Verification:

k	n	$n-k$	$\frac{n-1}{k}$
5	6	1	1
	126	$121 = 11^2$	$25 = 5^2$
	846	$841 = 29^2$	$169 = 13^2$
	39606	$39601 = 199^2$	$7921 = 89^2$
	...	...	...
10	11	1	1
	1691	$1681 = 41^2$	$169 = 13^2$
	6251	$6241 = 79^2$	$625 = 25^2$
	2430491	$2430481 = 1559^2$	$243049 = 493^2$
	...	...	...

Table 6

5.5. Theorem 8

There exists infinitely many $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for any non - perfect square $k \in \mathbb{N}$ satisfying $k+2$ is perfect square.

Proof:

Since $n-k$ and $\frac{n-1}{k}$ are perfect square, from theorem 6, it is enough to solve (5.5).

To solve equation (5.5), let us expand $\sqrt{k}$ as continued fraction in terms of $\sqrt{k+2}$.

Consider $\left(\sqrt{k+2}-\sqrt{k}\right)\left(\sqrt{k+2}+\sqrt{k}\right) = 2$

$$\sqrt{k+2}-\sqrt{k} = \frac{2}{\sqrt{k+2}+\sqrt{k}} \qquad (5.24)$$

$$= \frac{2}{\sqrt{k+2} + \sqrt{k} + \sqrt{k+2} - \sqrt{k+2}}$$

$$= \frac{2}{2\sqrt{k+2} - \frac{2}{\sqrt{k+2}+\sqrt{k}}} \qquad \text{(by equation (5.24))} \qquad \ldots$$

$$= \frac{2}{2\sqrt{k+2} - \frac{2}{2\sqrt{k+2} - \frac{2}{2\sqrt{k+2} - \ddots}}}$$

$$\sqrt{k} = \sqrt{k+2} - \cfrac{2}{2\sqrt{k+2} - \cfrac{2}{2\sqrt{k+2} - \cfrac{2}{2\sqrt{k+2} - \ddots}}} \quad (5.25)$$

Using definition 2.2, the convergents of the continued fraction (5.25) are given below,

$$C_0 = \sqrt{k+2}$$

$$C_1 = \frac{k+1}{\sqrt{k+2}}$$

$$C_2 = \frac{(2k+1)\sqrt{k+2}}{2k+3}$$

$$C_3 = \frac{2k^2+4k+1}{2(k+1)\sqrt{k+2}}$$

$$C_4 = \frac{(4k^2+6k+1)\sqrt{k+2}}{4k^2+10k+5}$$

...

Since $k+2$ is perfect square, let $k+2 = \lambda_3^2$. Then $\lambda_3 = \sqrt{k+2}$ is a positive integer.

Considering the trivial solution and the above successive alternative convergent values, the solution of the equation (5.5) are given below

$(u,v) = (1,0), (k+1, \lambda_3), (2k^2+4k+1, 2(k+1)\lambda_3), ...$ (5.26)From equations (5.6) and (5.7), for every pair of (u,v) given by (5.26), we obtain the solutions of equation (5.4).

$(x,y) = (1,1), (k+1+k\lambda_3, k+1+\lambda_3),$

$(2k^2+4k+1+2k(k+1)\lambda_3, 2k^2+4k+1+2(k+1)\lambda_3), ...$(5.27)

From equations (5.8) and (5.9), for every pair of (u,v) given by (5.26), we obtain the solutions of equation (5.4) as follows

$(x,y) = (1,1), (|-(k+1)+k\lambda_3|, |k+1-\lambda_3|),$

$(|-(2k^2+4k+1)+2k(k+1)\lambda_3|, |2k^2+4k+1-2(k+1)\lambda_3|), ...$(5.28)

Combining (5.27) and (5.28) we get,

$(x,y) = (1,1), (|-(k+1)+k\lambda_3|, |k+1-\lambda_3|), (k+1+k\lambda_3, k+1+\lambda_3),$

$(|-(2k^2+4k+1)+2k(k+1)\lambda_3|, |2k^2+4k+1-2(k+1)\lambda_3|), ...$(5.29)

From (5.2), for every values of x in (5.29) we obtain the values of n as follows,

$n = 1+k, (-(k+1)+k\lambda_3)^2+k, (k+1+k\lambda_3)^2+k,$

$(-(2k^2+4k+1)+2k(k+1)\lambda_3)^2+k, ...$ (5.30)

Notice $n \in \mathbb{N}$, since k and λ_3 are natural numbers.

Hence every values of n in equation (5.30) are both $n-k$ and$\frac{n-1}{k}$are perfect square when $k+2$ is perfect square.

Observe that, there exists infinitely many $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for any $k \in \mathbb{N}$ of the form $k = m^2+4m+2, m \geq 0$.

5.6. Illustration 3

Find $n \in \mathbb{N}$ such that both $n-k$ and$\frac{n-1}{k}$ are perfect square for$k = 2,7$.

Solution:

By theorem 8, there exists infinitely many $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for$k = 2,7$ since $k = 2,7$ are natural numbers, non – perfect squares and $k+2$ is a perfect square.

Substituting $k = 2$ and$\lambda_3 = \sqrt{k+2} = 2$ in (5.30) of theorem 8 we get,

$$n = 3, 3, 51, 51, 1683, \ldots$$

And substituting $k = 7$ and$\lambda_3 = \sqrt{k+2} = 3$ in (5.30) of theorem 8 we get,

$$n = 8, 176, 848, 214376, \ldots$$

Verification:

k	n	$n-k$	$\frac{n-1}{k}$
2	3	1	1
	51	$49 = 7^2$	$25 = 5^2$
	1683	$1681 = 41^2$	$841 = 29^2$
	…	…	…
7	8	1	1
	176	$169 = 13^2$	$25 = 5^2$
	848	$841 = 29^2$	$121 = 11^2$
	214376	$214369 = 463^2$	$30625 = 175^2$
	…	…	…

Table 7

5.7. Theorem 9

There exists infinitely many $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for any non - perfect square $k \in \mathbb{N}$ satisfying $k-2$ is perfect square.

Proof:

Since $n-k$ and $\frac{n-1}{k}$ are perfect square, from theorem 6, it is enough to solve (5.5).

To solve equation (5.5), let us expand $\sqrt{k}$ as continued fraction in terms of$\sqrt{k-2}$.

Consider $\left(\sqrt{k}-\sqrt{k}-2\right)\left(\sqrt{k}+\sqrt{k-2}\right) = 2$

$$\sqrt{k}-\sqrt{k-2} = \frac{2}{\sqrt{k}+\sqrt{k-2}} \tag{5.31}$$

$$= \frac{2}{\sqrt{k}+\sqrt{k-2}+\sqrt{k-2}-\sqrt{k-2}}$$

$$= \frac{2}{2\sqrt{k-2}+\frac{2}{\sqrt{k}+\sqrt{k-2}}} \quad \text{(by equation (5.31))}$$

…

$$= \frac{2}{2\sqrt{k-2}+\frac{2}{2\sqrt{k-2}+\frac{2}{2\sqrt{k-2}+\ddots}}}$$

$$\sqrt{k} = \sqrt{k-2}+\frac{2}{2\sqrt{k-2}+\frac{2}{2\sqrt{k-2}+\frac{2}{2\sqrt{k-2}+\ddots}}} \tag{5.32}$$

Using definition 2.2, the convergents of the continued fraction (5.32) are given below,

$$C_0 = \sqrt{k-2}$$

$$C_1 = \frac{k-1}{\sqrt{k-2}}$$

$$C_2 = \frac{(2k-1)\sqrt{k-2}}{2k-3}$$

$$C_3 = \frac{2k^2-4k+1}{2(k-1)\sqrt{k-2}}$$

$$C_4 = \frac{(4k^2-6k+1)\sqrt{k-2}}{4k^2-10k+5}$$

...

Since $k-2$ is perfect square, let $k-2=\lambda_4^2$. Then $\lambda_4=\sqrt{k-2}$ is a positive integer.

Considering the trivial solution and the above successive alternative convergent values, the solution of the equation (5.5) are given below

$(u,v) = (1,0), (k-1,\lambda_4), (2k^2-4k+1, 2(k-1)\lambda_4), \ldots$ (5.33) From equations (5.6) and (5.7), for every pair of (u,v) given by (5.33), we obtain the solutions of equation (5.4) as follows,

$(x,y) = (1,1), (k-1+k\lambda_4, k-1+\lambda_4)$,

$(2k^2-4k+1+2k(k-1)\lambda_4, 2k^2-4k+1+2(k-1)\lambda_4), \ldots$(5.34)

From equations (5.8) and (5.9), for every pair of (u,v) given by (5.33), we obtain the solutions of equation (5.4) as follows,

$(x,y) = (1,1), (|-(k-1)+k\lambda_4|, |k-1-\lambda_4|)$,

$(|-(2k^2-4k+1)+2k(k-1)\lambda_4|, |2k^2-4k+1-2(k-1)\lambda_4|), \ldots$(5.35)

Combining (5.34) and (5.35) we get,

$(x,y) = (1,1), (|-(k-1)+k\lambda_4|, |k-1-\lambda_4|), (k-1+k\lambda_4, k-1+\lambda_4)$,

$(|-(2k^2-4k+1)+2k(k-1)\lambda_4|, |2k^2-4k+1-2(k-1)\lambda_4|), \ldots$(5.36)

From (5.2), for every values of x in (5.36) we obtain the values of n as follows,

$n = 1+k, (-(k-1)+k\lambda_4)^2+k, (k-1+k\lambda_4)^2+k$,

$(-(2k^2-4k+1)+2k(k-1)\lambda_4)^2+k, \ldots$(5.37)

Notice $n \in \mathbb{N}$, since k and λ_4 are natural numbers.

Hence every values of n in equation (5.37) are both $n-k$ and $\frac{n-1}{k}$ are perfect square when $k-2$ is perfect square.

Observe that, there exists infinitely many $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for any $k \in \mathbb{N}$ of the form $k = m^2+2m+3, m \geq 0$.

5.8. Illustration 4

Find $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for $k = 6,11$.

Solution:

By theorem 9, there exists infinitely many $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for $k = 6,11$ since $k = 6,11$ are natural numbers, non – perfect squares and $k-2$ is a perfect square.

Substituting $k = 6$ and$\lambda_4 = \sqrt{k-2} = 2$ in (5.37) of theorem 9 we get,

$$n = 7, 55, 295, 5047, \dots$$

And substituting $k = 11$ and$\lambda_4 = \sqrt{k-2} = 3$ in (5.37) of theorem 9 we get,

$$n = 12, 540, 1860, 212532, \dots$$

Verification:

k	n	$n-k$	$\frac{n-1}{k}$
6	7	1	1
	55	$49 = 7^2$	$9 = 3^2$
	295	$289 = 17^2$	$49 = 7^2$
	5047	$5041 = 71^2$	$841 = 29^2$
	...	...	...
11	12	1	1
	540	$529 = 23^2$	$49 = 7^2$
	1860	$1849 = 43^2$	$169 = 13^2$
	212532	$212521 = 461^2$	$19321 = 139^2$
	...	...	...

Table 8

5.9. Theorem 10

There exists infinitely many $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for any non - perfect square $k \in \mathbb{N}$ satisfying $k+4$ is perfect square.

Proof:

Since $n-k$ and $\frac{n-1}{k}$ are perfect square, from theorem 6, it is enough to solve (5.5).

To solve equation (5.5), let us expand $\sqrt{k}$ as continued fraction in terms of$\sqrt{k+4}$.

Consider $\left(\sqrt{k+4} - \sqrt{k}\right)\left(\sqrt{k+4} + \sqrt{k}\right) = 4$

$$\sqrt{k+4} - \sqrt{k} = \frac{4}{\sqrt{k+4}+\sqrt{k}} \qquad (5.38)$$

$$= \frac{4}{\sqrt{k+4} + \sqrt{k} + \sqrt{k+4} - \sqrt{k+4}}$$

$$= \frac{4}{2\sqrt{k+4} - \frac{4}{\sqrt{k+4}+\sqrt{k}}} \qquad \text{(by equation (5.38))}$$

...

$$= \frac{4}{2\sqrt{k+4} - \frac{4}{2\sqrt{k+4} - \frac{4}{2\sqrt{k+4} - \ddots}}}$$

$$\sqrt{k} = \sqrt{k+4} - \frac{4}{2\sqrt{k+4} - \frac{4}{2\sqrt{k+4} - \frac{4}{2\sqrt{k+4} - \ddots}}} \qquad (5.39)$$

Using definition 2.2, the convergents of the continued fraction (5.39) are given below,

$$C_0 = \sqrt{k+4}$$

$$C_1 = \frac{k+2}{\sqrt{k+4}}$$

$$C_2 = \frac{(k+1)\sqrt{k+4}}{k+3}$$

$$C_3 = \frac{k^2+4k+2}{(k+2)\sqrt{k+4}}$$

...

Since $k+4$ is perfect square, let $k+4=\lambda_5^2$. Then $\lambda_5 = \sqrt{k+4}$ is a positive integer.

Considering the trivial solution and the above successive alternative convergent values, the solution of the equation (5.5) are given below

$(u,v) = (1,0), \left(\frac{k+2}{2}, \frac{\lambda_5}{2}\right), \left(\frac{k^2+4k+2}{2}, \frac{(k+2)\lambda_5}{2}\right), \ldots$ (5.40) From equations (5.6) and (5.7), for every pair of (u,v) given by (5.40), we obtain the solutions of equation (5.4).

$$(x,y) = (1,1), \left(\frac{1}{2}(k+2+k\lambda_5), \frac{1}{2}(k+2+\lambda_5)\right),$$

$$\left(\frac{1}{2}(k^2+4k+2+k(k+2)\lambda_5), \frac{1}{2}(k^2+4k+2+(k+2)\lambda_5)\right), \ldots (5.41)$$

From equations (5.8) and (5.9), for every pair of (u,v) given by (5.40), we obtain the solutions of equation (5.4).

$$(x,y) = (1,1), \left(\frac{1}{2}|-(k+2)+k\lambda_5|, \frac{1}{2}|k+2-\lambda_5|\right),$$

$$\left(\frac{1}{2}|-(k^2+4k+2)+k(k+2)\lambda_5|, \frac{1}{2}|k^2+4k+2-(k+2)\lambda_5|\right), \ldots \quad (5.42)$$

Combining (5.41) and (5.42) we get,

$$(x,y) = (1,1), \left(\frac{1}{2}|-(k+2)+k\lambda_5|, \frac{1}{2}|k+2-\lambda_5|\right), \left(\frac{1}{2}(k+2+k\lambda_5), \frac{1}{2}(k+2+\lambda_5)\right),$$

$$\left(\frac{1}{2}|-(k^2+4k+2)+k(k+2)\lambda_5|, \frac{1}{2}|k^2+4k+2-(k+2)\lambda_5|\right), \ldots (5.43)$$

From (5.2), for every values of x in (5.43) we obtain the values of n as follows,

$$n = 1+k, \frac{1}{4}(-(k+2)+k\lambda_5)^2+k, \frac{1}{4}(k+2+k\lambda_5)^2+k,$$

$$\frac{1}{4}(-(k^2+4k+2)+k(k+2)\lambda_5)^2+k, \ldots (5.44)$$

Notice $n \in \mathbb{N}$, since k and λ_5 are natural numbers and both numerators of u and v in (5.40) are either even or odd, since square of an even integer is even and square of an odd integer is odd.

Hence every values of n in equation (5.44) are both $n-k$ and $\frac{n-1}{k}$ are perfect square when $k+4$ is perfect square.

Observe that, there exists infinitely many $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for any $k \in \mathbb{N}$ of the form $k = m^2+6m+5, m \geq 0$.

5.10. Illustration 5

Find $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for $k = 5,12$.

Solution:

By theorem 10, there exists infinitely many $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for$k = 5{,}12$ since $k = 5{,}12$ are natural numbers, non – perfect squares and $k+4$ is a perfect square.

Substituting $k = 5$ and$\lambda_5 = \sqrt{k+4} = 3$ in (5.44) of theorem 10 we get,

$$n = 6, 21, 126, 846, \ldots$$

And substituting $k = 12$ and$\lambda_5 = \sqrt{k+4} = 4$ in (5.44) of theorem 10 we get,

$$n = 13, 301, 973, 57133, \ldots$$

Verification:

k	n	$n-k$	$\frac{n-1}{k}$
5	6	1	1
	21	$16 = 4^2$	$4 = 2^2$
	126	$121 = 11^2$	$25 = 5^2$
	846	$841 = 29^2$	$1\ 69 = 13^2$
	...	...	...
12	13	1	1
	301	$289 = 17^2$	$25 = 5^2$
	973	$961 = 31^2$	$81 = 9^2$
	57133	$57121 = 239^2$	$4761 = 69^2$
	...	...	...

Table 9

5.11. Theorem 11

There exists infinitely many $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for any non - perfect square $k \in \mathbb{N}$ satisfying $k-4$ is perfect square.

Proof:

Since $n-k$ and $\frac{n-1}{k}$ are perfect square, from theorem 6, it is enough to solve (5.5).

To solve equation (5.5), let us expand $\sqrt{k}$ as continued fraction in terms of$\sqrt{k-4}$.

Consider $\left(\sqrt{k}-\sqrt{k-4}\right)\left(\sqrt{k}+\sqrt{k-4}\right) = 4$

$$\sqrt{k}-\sqrt{k-4} = \frac{4}{\sqrt{k}+\sqrt{k-4}} \quad (5.45)$$

$$= \frac{4}{\sqrt{k} + \sqrt{k-4} + \sqrt{k-4} - \sqrt{k-4}}$$

$$= \frac{4}{2\sqrt{k-4} + \frac{4}{\sqrt{k}+\sqrt{k-4}}} \quad \text{(by equation (5.45))}$$

...

$$= \frac{4}{2\sqrt{k-4} + \frac{4}{2\sqrt{k-4} + \frac{4}{2\sqrt{k-4} + \ddots}}}$$

$$\sqrt{k} = \sqrt{k-4} + \frac{4}{2\sqrt{k-4} + \frac{4}{2\sqrt{k-4} + \frac{4}{2\sqrt{k-4} + \ddots}}} \quad (5.46)$$

Using definition 2.2, the convergents of the continued fraction (5.46) are given below,

$$C_0 = \sqrt{k-4}$$

$$C_1 = \frac{k-2}{\sqrt{k-4}}$$

$$C_2 = \frac{(k-1)\sqrt{k-4}}{k-3}$$

$$C_3 = \frac{k^2-4k+2}{(k-2)\sqrt{k-4}}$$

...

Since $k-4$ is perfect square, let $k-4=\lambda_6^2$. Then $\lambda_6=\sqrt{k-4}$ is a positive integer.

Considering the trivial solution and the above successive alternative convergent values, the solution of the equation (5.5) are given below

$(u,v) = (1,0), \left(\frac{k-2}{2}, \frac{\lambda_6}{2}\right), \left(\frac{k^2-4k+2}{2}, \frac{(k-2)\lambda_6}{2}\right), \ldots$ (5.47) From equations (5.6) and (5.7), for every pair of (u,v) given by (5.47), we obtain the solutions of equation (5.4) as follows,

$$(x,y) = (1,1), \left(\frac{1}{2}(k-2+k\lambda_6), \frac{1}{2}(k-2+\lambda_6)\right),$$

$$\left(\frac{1}{2}(k^2-4k+2+k(k-2)\lambda_6), \frac{1}{2}(k^2-4k+2+(k-2)\lambda_6)\right), \ldots (5.48)$$

From equations (5.8) and (5.9), for every pair of (u,v) given by (5.48), we obtain the solutions of equation (5.4) as follows,

$$(x,y) = (1,1), \left(\frac{1}{2}|-(k-2)+k\lambda_6|, \frac{1}{2}|k-2-\lambda_6|\right),$$

$$\left(\frac{1}{2}|-(k^2-4k+2)+k(k-2)\lambda_6|, \frac{1}{2}|k^2-4k+2-(k-2)\lambda_6|\right), \ldots (5.49)$$

Combining (5.48) and (5.49) we get,

$$(x,y) = (1,1), \left(\frac{1}{2}|-(k-2)+k\lambda_6|, \frac{1}{2}|k-2-\lambda_6|\right), \left(\frac{1}{2}(k-2+k\lambda_6), \frac{1}{2}(k-2+\lambda_6)\right),$$

$$\left(\frac{1}{2}|-(k^2-4k+2)+k(k-2)\lambda_6|, \frac{1}{2}|k^2-4k+2-(k-2)\lambda_6|\right), \ldots (5.50)$$

From (5.2), for every values of x in (5.50) we obtain the values of n as follows,

$$n = 1+k, \frac{1}{4}(-(k-2)+k\lambda_6)^2+k, \frac{1}{4}(k-2+k\lambda_6)^2+k,$$

$$\left(\frac{1}{4}(-(k^2-4k+2)+k(k-2)\lambda_6)\right)^2+k, \ldots (5.51)$$

Notice $n \in \mathbb{N}$, since k and λ_6 are natural numbers and both numerators of u and v in (5.47) are either even or odd, since square of an even integer is even and square of an odd integer is odd.

Hence every values of n in equation (5.51) are both $n-k$ and $\frac{n-1}{k}$ are perfect square when $k-4$ is perfect square.

Observe that, there exists infinitely many $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for any $k \in \mathbb{N}$ of the form $k = m^2+2m+5, m \geq 0$.

5.12. Illustration 6

Find $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for $k = 5,13$.

Solution:

By theorem 11, there exists infinitely many $n \in \mathbb{N}$ such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for $k = 5,13$ since $k = 5,13$ are natural numbers, non – perfect squares and $k-4$ is a perfect square.

Substituting $k = 5$ and $\lambda_6 = \sqrt{k-4} = 1$ in (5.51) of theorem 11 we get,

$$n = 6, 21, 126, \ldots$$

And substituting $k = 13$ and $\lambda_6 = \sqrt{k-4} = 3$ in (5.51) of theorem 11 we get,

$$n = 14, 209, 638, 24038, \ldots$$

Verification:

k	n	$n-k$	$\frac{n-1}{k}$
5	6	1	1
	21	$16 = 4^2$	$4 = 2^2$
	126	$121 = 11^2$	$25 = 5^2$
	…	…	…
13	14	1	1
	209	$196 = 14^2$	$16 = 4^2$
	638	$625 = 25^2$	$49 = 7^2$
	24038	$24025 = 155^2$	$1849 = 43^2$
	…	…	…

Table 10

6. Conclusion:

In this paper, in theorem 1, we constructed Pythagorean triples whose smaller sides differ by an integer k as presented in table 1 and notice that all such triples are multiples of k.

In theorem 2, we have given a nice continued fraction expansion for $\sqrt{2n^2-1}$.Using the convergents of $\sqrt{2n^2-1}$, we have generated primitive Pythagorean triples whose smaller sides differ by the number of the form $2n^2-1, 16n^2(2n^2-1)+1$ and $2048n^8-1536n^6+288n^4-1$ for $n \in \mathbb{N}$ in theorems 3,4 and 5 respectively.

In section 5, we proved there are infinitely many positive integer n such that both $n-k$ and $\frac{n-1}{k}$ are perfect square for particular non – square integer k. Examples for generating perfect squares are provided after theorems 6 to 11 by considering $k = 2,3,5,6,7,8,10,11,12,13$. These values verify the results obtained in theorems proved.

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