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Line k-Domination in Graphs

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Abstract:

In this paper, we present a novel concept "Line k-Domination in Graphs". The line k-domination number $\gamma_{kl}(G)$ is the minimum cardinality among all k-dominating set of line graph $L(G)$ derived from the graph G. Throughout this investigation, we obtain various bounds on $\gamma_{kl}(G)$ by considering different parameters of the graph G such as vertices, edges, and other different domination parameters.

Keywords: Graph**,** Line graph, Domination number, k-domination number, Line k-domination number.

1. Introduction

In study, we examine a simple connected graph $G = (V, E)$, where V represents the vertex set and E the edge set. Utilizing the notation introduced by Harary [8], the order and size of the graph G are denoted by $|V(G)| = p$ and $|E(G)| = q$, respectively. For a vertex $u \in G$, the degree of a vertex u, denoted as deg (u), refers to the number of edges incident on u. The minimum and maximum degree of a vertex u are denoted by δ and Δ, respectively. Similarly, for an edge $e = uv$ in the graph G, the degree of the edge e is expressed as deg(e) = deg(u) + deg(v) − 2, and the minimum (maximum) degree of an edge e is denoted by $\delta'(\Delta')$. The open neighborhood of a vertex u is represented as $N(u)$ and it is the set of vertices adjacent to u. Furthermore, the closed neighborhood of vertex u, denoted as $N[u]$, is defined as $N[u] = N(u) \cup \{u\}$. Additional information regarding the concept of neighborhood can be found in the reference [10]. The vertex(edge) covering number, denoted as $\alpha_0(G)(\alpha_1(G))$, in graph G is defined as the minimum cardinality of a vertex(edge) cover. An independent set M of vertices is independent if none of its vertices are adjacent. The independence number $\beta_0(G)$ of G is the maximum Cardinality of an independent vertex set. Similarly, a set N of edges is independent if no two edges in N share an adjacent, and the edge independence number $\beta_1(G)$ represents the maximum cardinality of an independent edge set. The graph's diameter, $diam(G)$, is defined as the maximum distance between any pair of vertices. For further notation, one can refer [11].

A line graph, denoted as $L(G)$, is generated from a graph G where vertices in $L(G)$ corresponding to edges of G, In $L(G)$, two vertices are considered adjacent if the corresponding edges in G share a common vertex, in the original graph (see [3]).

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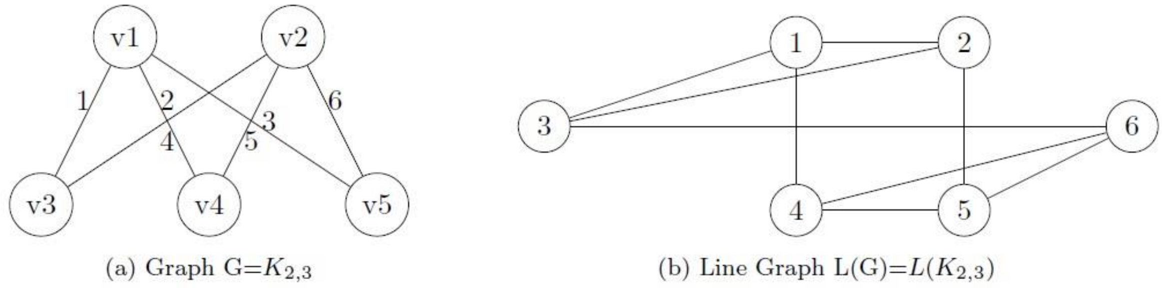

Figure 1. Graph and its Line graph.

We recalling fundamental definitions within domination theory.

A set $D \subseteq V(G)$ is a dominating set if, for every vertex $u \in V(G) - D, N(u) \cap D \neq \phi$. The domination number $\gamma(G)$ represents the minimum cardinality of a dominating set in G. In scenarios where each vertex in $V(G)$ has at least one neighbor in D, the subset $D \subseteq V(G)$ is a total dominating set. The $\gamma_t(G)$ denotes total domination number which is minimum cardinality of total dominating set, as detail in [2]. Further, a subset $D \subseteq V(G)$ called an independent dominating set if the induced subgraph $< D >$ has no edges. The $\gamma_i(G)$ denotes independence domination number which is minimum cardinality independence dominating set (refer to [9]).

Generalizing the concept of k-domination in [5-6], Fink and Jacobson from both domination and independence. A subset $D \subseteq V(G)$ is k-dominating in G if every vertex in $V(G) - D$ is connected to at least k vertices in D. The $\gamma_k(G)$ denotes the k-domination number which is the minimum cardinality of k-dominating set of G. Various bounds on k-domination number discussed in [2-4, 7, 14]. The study by author in [13], the investigation of set-domination in line graph $L(G)$ of a graph G. This paper establishes bounds of $\gamma_{sd}(L(G))$ and provides exact values for certain standard graphs. Similarly, in [12] studied line double domination in graphs and found various bounds on $\gamma_{ddl}(G)$ in relation to vertices, edges and other distinct parameters of G.

This paper applies concept of k-domination of graph G to its line graph $L(G)$. The line k-domination number $\gamma_{kl}(G)$ is the minimum cardinality among all k-dominating sets of line graph $L(G)$. Within this investigation, numerous bounds on $\gamma_{kl}(G)$ have been derived in terms of vertices, edges and other parameters of G. Furthermore, we establish relations between $\gamma_{kl}(G)$ and different domination parameters.

2. Preliminaries

Theorem 2.1.[15] If $G = (p, q)$ is any graph, $\gamma(G) \geq \frac{p}{1+\Delta(G)}$.

Theorem 2.2.[1] If $G = (p, q)$ is any graph, $\gamma(G) \geq p - q$.

3. Main Results

In this section, numerous upper and lower bounds on $\gamma_{kl}(G)$ have been derived, considering vertices, edges, and various parameters of G. Additionally, relationships between $\gamma_{kl}(G)$ and distinct domination parameters are established.

I) Lower bounds on $\gamma_{kl}(G)$

Theorem 3.1. For any graph $G = (p, q)$, $\gamma_{kl}(G) \geq \frac{q}{\Delta'(G)+1}$.

Proof. Consider the vertex set $A = \{w_1, w_2, \dots, w_q\}$ of the line graph $L(G)$. Then there exists a minimal set $A_1 = \{w_1, w_2, \dots, w_s\} \subseteq A$ such that every vertex in $V(L(G)) - A_1$ is adjacent to at least k vertices of. Consequently, A_1 stands as the minimal k-dominating set of $L(G)$. Clearly, $|A_1| = \gamma_{kl}(G)$. Since each vertex in $V(L(G)) - A_1$ is adjacent to at least k vertices of A_1, every vertex in $V(L(G)) - A_1$ contributes at least one to the sum of degree of vertices of A_1. Hence, $|V(L(G)) - A_1| \leq \sum_{w_i \in A_1} \deg(w_i)$. For any graph G, there exists at least one edge $e \in E(G)$, which corresponds to a vertex in $L(G)$, such that $\deg(e) = \Delta'(G)$.

We have, $q - \gamma_{kl}(G) = |V(L(G)) - A_1| \leq \sum_{w_i \in A_1} \deg(w_i)$.

$$\leq \Delta'(G) \cdot \gamma_{kl}(G)$$

Thus, $\gamma_{kl}(G) \geq \frac{q}{\Delta'(G)+1}$.

Corollary 3.2. For any graph $G = (p, q)$, $\gamma_{kl}(G) \geq \frac{p-1}{\Delta'(G)+1}$.

Proof. from Theorem 3.1, $\gamma_{kl}(G) \geq \frac{q}{\Delta'(G)+1}$.

As we know, if G is connected graph, $q \geq p - 1$.

Using this, we get required result.

Theorem 3.3. If $G = (p, q)$ is any graph with $2 \leq k \leq \Delta(G)$, $\gamma_{kl}(G) \geq \gamma(G) + k - 2$.

Proof. Consider the vertex set $A = \{u_1, u_2, \dots, u_p\}$ of a graph G which contains a minimal set D that covers all the vertices of G. Then D is minimal dominating set of G. Let $S \subseteq V(L(G))$ be a minimum k-dominating set of $L(G)$. Take $w \in V(L(G)) - S$ and $w_1, w_2, \dots, w_k$ as distinct vertices in S that dominates w. As S is a k-dominating set, every vertex in $V(L(G)) - S$ is dominated by at least one vertex in $S - \{w_2, w_3, \dots, w_k\}$.

Therefore, since w dominates each vertex in $\{w_2, w_3, \dots, w_k\}$, we can form $S' = S - \{w_2, w_3, \dots, w_k\} \cup \{w\}$ as dominating set of $L(G)$ that satisfies,

$$\gamma(G) \leq |S'| = |S - \{w_2, w_3, \dots, w_k\} \cup \{w\}|$$

$$= \gamma_{kl}(G) - k + 2$$

Thus, $\gamma_{kl}(G) \geq \gamma(G) + k - 2$.

Theorem 3.4. For any graph $G=(p,q)$ with $k\geq 2$, $\gamma_{kl}(G)\geq \frac{p}{\Delta(G)}$.

Proof. Let set $A=\{u_1,u_2,\dots,u_p\}$ represent vertex set of a graph G then $|A|=|V(G)|=p$. Now, consider set $E=\{e_1,e_2,\dots,e_q\}$ be the edges incident on vertices of A, and let $B=\{w_1,w_2,\dots,w_q\}$ be the vertices of $L(G)$ corresponding to elements of E. Consequently, there exists a minimal set $B_1=\{w_1,w_2,\dots,w_s\}\subseteq B$ such that every vertex in $V(L(G))-B_1$ is adjacent to at least k vertices of B_1. Thus, B_1 serves as the minimal k-dominating set of $L(G)$.

In this scenario where $k\geq 2$, we take $A_1=\{u_1,u_2,\dots,u_m\}\subseteq A$ as the set containing all non-end vertices in G. In B_1, an edge is included, which is incident on a vertex u with maximum degree in G. This construction ensures that, $|B_1|\cdot \deg(u)\geq p$. Therefore, $\gamma_{kl}(G)\cdot\Delta(G)\geq p$.

Thus, we get result.

Theorem 3.5. For any graph $G=(p,q)$ and k be any positive integer with $k\geq\Delta(L(G))$, $\gamma_{kl}(G)\geq q-s$, where s is number of vertices of degree $\Delta(L(G))$.

Proof. The lict graph has q vertices corresponding to edges incident on vertices of G. Let $S=\{w_1,w_2,\dots,w_s\}$ contains a vertices of degree $\Delta(L(G))$. The proof is divided into two cases.

Case (1): If $|S|=1$. In this case, the result hold trivially.

Case (2): If $|S|>1$,

Subcase (1). For $k\geq\Delta(L(G))$, if the members of S are adjacent to each other: In this scenario, a k-dominating set contain $|S|-1$ from S. We get required result.

Subcase (2). For $k\geq\Delta(L(G))$, if the members of S are not adjacent to each other: In this case, there exists k members from $V(L(G)$ other than member of S adjacent to every vertex of S. we get, $\gamma_{kl}(G)\geq q-s$.

II) Upper bounds on $\gamma_{kl}(G)$

Theorem 3.6. If $G=(p,q)$ is any graph with $k\leq\delta(G)$, $\gamma_{kl}(G)+\gamma(L(G))\leq q$**.**

Proof. For every $k\leq\delta(G)$, there exist set $S=\{w_1,w_2,\dots,w_r\}\subseteq V(L(G))$, be the minimal k-dominating set of $L(G)$. Let's assume that there exists vertex $w\in S$ that is not adjacent to any vertices in $V(L(G))-S$. This would imply that w is adjacent to at least k vertices in set S. Consequently, $S-\{w\}$ is a minimal k-dominating set and $|S-\{w\}|<|S|$. However, this contradicts our initial that S is minimal k-dominating set. Hence, every vertex in S must be adjacent to at least one vertex in $V(L(G))-S$.

Thus, $V(L(G))-S$ is the dominating set. Therefore, there exists a minimal set $A\subseteq V(L(G))-S$ that covers all the vertices in $V(L(G))$. This implies that A is γ-set of $L(G)$ and which conclude that, $|A|+|S|\leq|E(G)|$. Hence, $\gamma_{kl}(G)+\gamma(L(G))\leq q$**.**

Corollary 3.7. For any graph $G=(p,q)$ with $k\leq\delta(G)$, $\gamma_{kl}(G)\leq\frac{q\cdot\Delta(L(G))}{1+\Delta(L(G))}$.

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Proof. From Theorem 2.1 for any graph G, $\gamma(G) \geq \frac{p}{1+\Delta(G)}$, it also holds for line graph $L(G)$ of G. So, we get $\gamma(L(G)) \geq \frac{q}{1+\Delta(L(G))}$.

Using this in Theorem 3.6, we obtain

$$\gamma_{kl}(G) + \frac{q}{1+\Delta(L(G))} \leq q.$$

Thus, $\gamma_{kl}(G) \leq \frac{q \cdot \Delta(L(G))}{1+\Delta(L(G))}$.

Corollary 3.8. For any graph $G = (p, q)$ with $k \leq \delta(G)$, $\gamma_{kl}(G) \leq \frac{1}{2}\sum_{i=1}^{p} d_i^2 - q$.

Proof. From Theorem 2.2 for any graph G, $\gamma(G) \geq q - p$, it also holds for line graph $L(G)$ of G. So, we get $\gamma\big(L(G)\big) \leq q - \left[\frac{1}{2}\sum_{i=1}^{p} d_i^2 - q\right]$, where d_i's are degree of vertices in G.

Therefore, $\gamma\big(L(G)\big) \leq 2q - \frac{1}{2}\sum_{i=1}^{p} d_i^2$.

Using this in Theorem 3.6, we have

$$\gamma_{kl}(G) + 2q - \frac{1}{2}\sum_{i=1}^{p} d_i^2 \leq q.$$

Thus, $\gamma_{kl}(G) \leq \frac{1}{2}\sum_{i=1}^{p} d_i^2 - q$.

Theorem 3.9. If $G = (p, q)$ is any graph, $\gamma_{kl}(G) \leq \alpha_0(G) + \beta_0(G) + \gamma(L(G))$.

Proof. For any graph $G = (p, q)$, let $\alpha_0(G)$ is the size of the minimum vertex cover of G, which ensures every edge in G is incident to at least one vertex in this set. This implies that the vertices in the line graph $L(G)$ corresponding to these edges are dominated. Let $\beta_0(G)$ be the size of the largest independent set in G, where no two vertices are adjacent, meaning the edges incident to these vertices are not directly connected in G. The vertices in $L(G)$ corresponding to these independent edges can be included without redundancy. Let $\gamma(L(G))$ be the domination number of $L(G)$.

Combining these, we construct a k-dominating set for $L(G)$ by considering the vertices corresponding to edges incident to the vertex cover $\alpha_0(G)$, the independent set $\beta_0(G)$, and the dominating set in $L(G)$ $(\gamma(L(G)))$. The vertices from the vertex cover ensure coverage of edges, the independent set adds non-redundant vertices, and the domination set in $L(G)$ ensures every vertex in $L(G)$ is sufficiently dominated. Since the k-domination number $\gamma_{kl}(G)$ requires that every vertex in $L(G)$ not in the dominating set is adjacent to at least k vertices in the set, the combined set from $\alpha_0(G)$, $\beta_0(G)$, and $\gamma(L(G))$ ensures all vertices in $L(G)$ are sufficiently covered. Therefore, we have $\gamma_{kl}(G) \leq \alpha_0(G) + \beta_0(G) + \gamma(L(G)$.

Theorem 3.10. If $G = (p, q)$ is any graph, $\gamma_{kl}(G) \leq p + \beta_0(G)$.

Proof. Case (1): If G is a tree, then $p \geq q - 1$, and the result holds clearly.

Case (2): If G is not a tree.

Consider the vertex set $V(G) = \{u_1, u_2, \dots, u_p\}$ of graph G. There exists a maximum set $A = \{u_1, u_2, \dots, u_t\} \subseteq V(G)$ of vertices such that $N(u) \cap N(v) = \{y\}, \forall u, v \in A$ and $y \in V(G) - A$. This implies that A is maximum independent set of vertices in $V(G)$ and $|A| = \beta_0(G)$. Since A is independent, no two vertices in A are adjacent. Consequently, in $L(G)$, the vertices corresponding to edges incident to vertices in A are not adjacent. This property of A ensures that adding these vertices to our k-dominating set does not create redundancy.

To construct a k-dominating set in $L(G)$, let B be the set of vertices in $L(G)$ corresponding to edges incident to any vertex in G. Clearly, $|B| = p$, as each vertex in G can be covered by considering the edges incident to it. Adding the vertices corresponding to the independent set A without redundancy, we form the set $D_k = A \cup B$ as the k-dominating set.

Thus, $\gamma_{kl}(G) \leq p + \beta_0(G)$.

Theorem 3.11. If $G = (p, q)$ is any graph, $\gamma_{kl}(G) \leq \alpha_0(G) + p$.

Proof. Case (1): If G is a tree, then $p \geq q - 1$, and the result holds clearly.

Case (2): If G is not a tree.

Let C be a minimum vertex cover of C with $|C| = \alpha_0(G)$. Each edge in G is thus represented in $L(G)$ by a vertex connected to at least one vertex corresponding to an edge incident to a vertex in C. The k-dominating set in $L(G)$ can be formed by including vertices corresponding to all edges incident to vertices in the vertex cover C and vertices corresponding to the remaining edges incident on vertices of G. Each vertex in C contributes to covering edges incident to it. If a vertex in G is not in the vertex cover, it still has its incident edges covered by other vertices in the cover set. This ensures sufficient coverage for k-domination set of $L(G)$.

Since each edge in G is either incident to a vertex in C or contributes a vertex in G, this guarantees that the total size of the k-dominating set of L(G) does not exceed $\alpha_0(G) + p$. Thus, $\gamma_{kl}(G) \leq \alpha_0(G) + p$.

Theorem 3.12. If $G = (p, q)$ is any graph, $\gamma_{kl}(G) \leq diam(G) + \beta_0(G) + \Delta'(G)$.

Proof. Let $A = \{e_i \mid 1 \leq i \leq s\}$ be the edges lying on the longest path between two vertices u and v of G. Additionally, let $B = \{u_1, u_2, \dots, u_t\}$ be maximum independent set of G and $C = \{e_j \mid 1 \leq j \leq m\}$ be edges such that e_j incident on u_j, for all $u_j \in B$. Set $D = \{e_r \mid 1 \leq r \leq l\}$ consist of edges adjacent to an edge of maximum degree other than the element of A and C such that $|D| \leq \Delta'(G)$. The elements of A, C and D are the member of $L(G)$. These sets provide sufficient coverage for the k-dominating set for $L(G)$.

Consider the set $A_1 \cup C_1 \cup D_1$, where $A_1 \subseteq A$, $C_1 \subseteq C$ and $D_1 \subseteq D$ forms a minimal k-dominating set of $L(G)$. It follows that, $|A_1 \cup C_1 \cup D_1| \leq |A| + |C| + |D|$. This implies that, $\gamma_{kl}(G) \leq diam(G) + \beta_0(G) + \Delta'(G)$.

III) Relations of $\gamma_{kl}(G)$ with different domination parameter

Theorem 3.13. For any graph $G = (p, q)$, $\gamma_{kl}(G) + \gamma_i\big(L(G)\big) < q + \beta_1(G)$.

Proof. Let $E = \{e_1, e_2, \dots, e_q\}$ be the edge set of G then there is maximum set $E_1 \subseteq E$ such that $N(e_i) \cap N(e_j) = \phi$ for all e_i, $e_j \in E_1$. This implies E_1 forms a maximal independent set of edges and $|E_1| = \beta_1(G)$. The elements of E_1 corresponds to the vertices of $L(G)$, then there exists minimal subset E_2 of E_1 such that every element of $V\big(L(G)\big) - E_2$ is adjacent to at least one vertex of E_2, and E_2 is an independent set. This implies that E_2 is an independent dominating set of $L(G)$ with $|E_2| \leq |\,E_1|$. Consequently, we have $\gamma_i\big(L(G)\big) \leq \beta_1(G)$ and we know that, $\gamma_{kl}(G) < q$. Thus, we get required result.

Theorem 3.14. For any graph $G = (p, q)$, $\gamma_{kl}(G) + \gamma(G) \geq \gamma_t(G)$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_p\}$ be the vertex set of G, then there exists minimal dominating set $A = \{v_1, v_2, \dots, v_s\}$ of G such that $|A| = \gamma(G)$.

The proof is divided into two parts.

Case (1): If $\gamma(G) = \gamma_t(G)$, result hold trivially.

Case (2): If $\gamma(G) \neq \gamma_t(G)$, the set A ensures that all vertices in G are adjacent to at least one vertex in A. However, this does not guarantee that A total dominating set, as vertices in A need to be dominated as well. The set D_k ensures that each vertex in $L(G)$ (representing an edge in G) is adjacent to at least k vertices in D_k. This property helps in ensuring that edges (and their corresponding vertices) in G are sufficiently added to the set A to obtain total dominating set.

Therefore, the total domination number $\gamma_t(G)$ is bounded above by the sum of the domination number and the k-domination number of the $L(G)$.

4. Conclusion

In this paper, k-domination in line graph is defined. Theorems related to line k-domination are derived and the relation between other different domination parameters. Also, obtained many bounds on $\gamma_{kl}(G)$ in terms of vertices, edges and other different parameters of G.

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