Advances in Nonlinear Variational Inequalities
ISSN: 1092-910X
Vol 28 No. 6s (2025)

Certain Subclasses of Bi-Univalent Functions Governed By Generalized Bivariate Fibonacci Polynomials

Sondekola Rudra Swamy[1*], Sibel Yalçın[2], Paduvalapattana Kempegowda Mamatha [3]Gulab Singh Chauhan [4], and Narasimhaiah. Jagadish[5]

[1,5]Department of Information Science and Engineering, Acharya institute of Technology, Bengaluru-560 107, Karnataka, India.
Email:[1]sondekola.swamy@gmail.com [5]jagadish2475@acharya.ac.in

[2] Bursa Uludag University, 16059, Bursa, Turkey. Email:syalcin@uludag.edu.tr

[3] School 0f Mathematics, Alliance University, Central Campus, Bengaluru - 562 106, karnataka, India

[4]Department of computer science and Engineering, Acharya University, Karakul, Uzbekisthan, Email:gulabsingh81@acharya.ac.in

Article History:

Received: 30-11-2024
Revised: 10-1-2025
Accepted: 19-1-2025

Abstract: We study certain subclasses of bi-univalent and regular functions in the open unit disk that are governed by generalized bivariate Fibonacci polynomials. For these subclasses of functions, we derive initial coefficient bounds. Additionally, the Fekete-Szegö problem is handled for the elements of these subclasses. We also discuss relevant connections to previous findings and present some new results.

2020 MSC: 30C45, 05A15, 11B39.

Keywords: Analytic functions, subordination, bi-univalent functions, generalized bivariate Fibonacci polynomials.

1. Preliminaries

A productive area of mathematics within complex analysis is Geometric Function Theory (GFT). In recent years, this sub-branch has succeeded in drawing researchers' attention. let $U=\{\zeta \in C: |\zeta|<1\}$, where C is the complex plane. The class of analytic functions ϕ in U is identified by A and elements of A are of the form

$$\phi(\varsigma)=\varsigma+d_2\varsigma^2+d_3\varsigma^3+\cdots=\varsigma+\sum_{j=2}^{\infty} d_j\varsigma^j, \quad \varsigma\in U, \tag{1.1}$$

and let $S=\{\phi \in A: \phi$ is univalent in $U\}$. In [5], Biebereach conjecture that $|d_j|\le j, j\ge 2$ for every function $\phi\in S$. Numerous new subclasses of S were defined to settle the Biebereach conjecture and a number of results were established. Researchers worked on this conjecture's proof for many years and finally, Luis De Branges solved this conjecture for every $j\ge 2$ in [11]. Another problem in GFT is Fekete-Szegö functional $|d_3-\xi d_2^2|, \xi\in R$, for every function $\phi\in S$. Familiar researchers published many papers on above mentioned problem for functions belonging to subclasses of S. One of the remarkable subclass of S is bi-univalent function class

σ. The notion σ of bi-univalent functions was presented by Levin in his work [23]. These are analytic functions, denoted by ϕ, where both ϕ and $\phi^{-1} = \psi$ are univalent in U. The renowned Koebe theorem (see[14]) states that, each function $\phi \in S$ of the form (1.1) has an inverse given by

$$\phi^{-1}(w) = w - d_2 w^2 + (2d_2^2 - d_3)w^3 - (5d_2^3 - 5d_2 d_3 + d_4)w^4 + \cdots = \psi(w) \qquad (1.2)$$

satisfying $\varsigma = \psi(\phi(\varsigma))$ and $w = \phi(\psi(w))$, $|w| < r_0(\phi)$, $1/4 \le r_0(\phi)$, ς, $w \in U$. The class σ is not an empty set since the functions $\frac{1}{2} log\left(\frac{1+\varsigma}{1-\varsigma}\right)$, $-log(1-\varsigma)$ and $\frac{\varsigma}{1-\varsigma}$ are functions in the σ family. However, $\varsigma - \frac{\varsigma^2}{2}$, $\frac{\varsigma}{1-\varsigma^2}$, and the Koebe function $\frac{\varsigma}{(1-\varsigma)^2}$ are not elements of σ, even though they are in S. For a concise analysis and to discover some of the characteristics of the family σ (see [6,7,25,36]). The article by Srivastava and hi co-authors [30] gave rise to the recent momentum of studies of the bi-univalent function family. Numerous scholars have looked into several fascinating special families of σ since this article brought the subject back to life (see [8,9,12,16,17,37]) and the citation provided in these papers.

In many fields, including number theory, numerical analysis, combinatorics, computer science, physics, and engineering, special polynomials like Faber, Lucas, Chebyshev, Horadam, Bernoulli, Gegenbauer, Lucas-Lehmer, Pell-Lucas, Fibonacci, and their generalizations are crucial. Some generalizations of the Fibonacci polynomials have been defined in the literature due to their widespread use in the applied sciences (see [22]). Researchers have recently focused attention on a specific type of polynomials called Generalized Bivariate Fibonacci Polynomials (GBFP).

It is known that the Fibonacci numbers have the recursive relation $F_j = F_{j-1} + F_{j-2}$, $(F_0 = 0, F_1 = 1, j \ge 2)$. Fibonacci polynomials which generalize Fibonacci numbers is defined as (see [4]) $F_j(\varkappa) = \varkappa F_{j-1}(\varkappa) + F_{j-2}(\varkappa)$, $(F_0(\varkappa) = 0, F_1(\varkappa) = 1, j \ge 2)$. A new generalization of $F_j(\varkappa)$ is introduced in [21] and is expressed as below:

Let $k(\varkappa, y)$ and $l(\varkappa, y)$ be polynomials with real coefficients. For, $j \ge 2$, the GBFP are defined by the recurrence relation:

$$F_j(\varkappa, y) = k(\varkappa, y)F_{j-1}(\varkappa, y) + l(\varkappa, y)F_{j-2}(\varkappa, y), \qquad (1.3)$$

where $F_0(\varkappa, y) = 0$, $F_1(\varkappa, y) = 1$ and $k^2(\varkappa, y) + 4l(\varkappa, y) > 0$. The generating function of GBFP is (see [21])

$$F(\varkappa, y, z) = \sum_{j=0}^{\infty} F_j(\varkappa, y)z^j = \frac{z}{1 - k(\varkappa, y)z - l(\varkappa, y)z^2}. \qquad (1.4)$$

Readers with an interest in GBFP can find a brief history and extensive information in [10] and its references.

The present focus is on functions that belong to a specific σ subfamily and are subordinate to known number sequences or special polynomials. Several researchers have found coefficient estimates and Fekete-Szegö functional $|d_3 - \xi d_2^2|$, $\xi \in R$, for elements of subfamilies of σ subordinate to known special polynomials or number sequences (Refer to [2,3,13,18,20,26,29,31,33,34,38] for additional details). Researchers have recently focused attention on GBFP. For members of specific subclasses of σ associated with GBFP, interesting results have been found in [1,19,27] regarding coefficient estimates and Fekete-Szegö functional.

For brevity, we write hereafter that $k(\varkappa, y) = k$ and $l(\varkappa, y)$ = l. Clearly

$$F_2(\varkappa,y)=k,\ \ F_3(\varkappa,y)=k^2+l,\cdots \tag{1.5}$$

are evident from (1.3).

Remark 1.1. It is possible to infer numerous polynomial sequences from GBFP by specializing k and l (see [39]). They are i). The bivariate Fibonacci polynomials are obtained if $k=\varkappa$ and $l=y$, ii). We arrive at the Pell polynomials if $k=2\varkappa$ and $l=1$, iii). We derive the Jacobsthal polynomials if $k=1$ and $l=\varkappa$, iv). We get the Fermat polynomials, if $k=3\varkappa$ and $l=-2$. v). We have the Chebyshev polynomials of the second kind if $k=2\varkappa$ and $l=-1$ and so forth.

For a_1, $a_2\in A$ analytic in U, a_1 is subordinate to a_2, if there is a Schwarz function $\theta(\varsigma)$ that is analytic in U with $\theta(0)=0$ and $|\theta(\varsigma)|<1$, such that $a_1(\varsigma)=a_2\big(\theta(\varsigma)\big)$, $\varsigma\in U$. This subordination is symbolized as $a_1\prec a_2$ or $a_1(\varsigma)\prec a_2(\varsigma)$ $(\varsigma\in U)$. In case, if $a_2\in S$, $then\ a_1(\varsigma)\prec a_2(\varsigma)\ \Leftrightarrow\ a_1(0)=a_2(0)\quad and\quad a_1(U)\subset a_2(U)$.

We present a few subfamilies of σ that are subordinate to a GBFP $F_j(\varkappa,y)$ as in (1.3), namely $T_\sigma^\nu(\varrho,\tau,F)$ and $Z_\sigma^\nu(\varrho,\tau,F)$. These are motivated by the aforementioned patterns in coefficient-related problems as well as the Fekete-Szegö functional [15] on specific subclasses of σ.

This paper uses the generating function '$F(\varkappa,y,z)$ as in (1.4) and the inverse function $\phi^{-1}(w)=\psi(w)$ as in (1.2), unless otherwise noted.

Definition 1.1. Let $\varrho\in C\setminus\{0\}$, $0\le\tau\le1$, $and\ \nu\ge1$. If $\phi\in\sigma$ satisfies

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{[(\varsigma\phi'(\varsigma))']^\nu}{\phi'(\varsigma)}\right)+(1-\tau)\left(\frac{\varsigma\big(\phi'(\varsigma)\big)^\nu}{\phi(\varsigma)}\right)-1\right)\prec F(\varsigma)=\frac{F(\varkappa,y,\varsigma)}{\varsigma},\ \varsigma\in U, \tag{1.6}$$

and

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{[(w\psi'(w))']^\nu}{\psi'(w)}\right)+(1-\tau)\left(\frac{w\big(\psi'(w)\big)^\nu}{\psi(w)}\right)-1\right)\prec F(w)=\frac{F(\varkappa,y,w)}{w},\ w\in U, \tag{1.7}$$

then we say that $\phi\in T_\sigma^\nu(\varrho,\tau,F)$, where

$$F(z)=\frac{1}{1-kz-lz^2},k\neq0,k^2+4l>0. \tag{1.8}$$

The following subfamilies of σ are obtained for particular choices of τ in the class $T_\sigma^\nu(\varrho,\tau,F)$:

1. $P_\sigma^\nu(\varrho,F)\equiv T_\sigma^\nu(\varrho,0,F)$, $\nu\ge1$, $and\ \varrho\in C\setminus\{0\}$ is a family of functions such that $\phi\in\sigma$ fulfills

$$1+\frac{1}{\varrho}\left(\left(\frac{\varsigma\big(\phi'(\varsigma)\big)^\nu}{\phi(\varsigma)}\right)-1\right)\prec F(\varsigma)=\frac{F(\varkappa,y,\varsigma)}{\varsigma},\ \varsigma\in U,$$

and

$$1+\frac{1}{\varrho}\left(\left(\frac{w\big(\psi'(w)\big)^\nu}{\psi(w)}\right)-1\right)\prec F(w)=\frac{F(\varkappa,y,w)}{w},\ w\in U,$$

where $F(z)$ is as mentioned in (1.8).

2. $Q_\sigma^\nu(\varrho,F)\equiv T_\sigma^\nu(\varrho,1,F)$, $\nu\ge1$, $and\ \varrho\in C\setminus\{0\}$ represents a set of functions $\phi\in\sigma$ that meet

$$1+\frac{1}{\varrho}\left(\left(\frac{\left[\left(\varsigma\phi'(\varsigma)\right)'\right]^{\nu}}{\phi'(\varsigma)}\right)-1\right)\prec F(\varsigma)=\frac{F(\varkappa,y,\varsigma)}{\varsigma},\ \varsigma\in U,$$

and

$$1+\frac{1}{\varrho}\left(\left(\frac{\left[\left(w\psi'(w)\right)'\right]^{\nu}}{\psi'(w)}\right)-1\right)\prec F(w)=\frac{F(\varkappa,y,w)}{w},\ w\in U,$$

where $F(z)$ is as mentioned in (1.8).

Definition 1.2. Let $\nu\geq 1,\ 0\leq\tau\leq 1,\ and\ \varrho\in C\setminus\{0\}$. If $\phi\in\sigma$ satisfies

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{\left[\left(\varsigma\phi'(\varsigma)\right)'\right]^{\nu}}{\phi'(\varsigma)}\right)+(1-\tau)\left(\phi'(\varsigma)\right)^{\nu}-1\right)\prec F(\varsigma)=\frac{F(\varkappa,y,\varsigma)}{\varsigma},\ \varsigma\in U, \tag{1.9}$$

and

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{\left[\left(w\psi'(w)\right)'\right]^{\nu}}{\psi'(w)}\right)+(1-\tau)\left(\psi'(w)\right)^{\nu}-1\right)\prec F(w)=\frac{F(\varkappa,y,w)}{w},\ w\in U, \tag{1.10}$$

then we say that $\phi\in Z_{\sigma}^{\nu}(\varrho,\tau,F)$, where $F(z)$ is as mentioned in (1.8).

The following subfamilies of σ are obtained for particular choices of τ and ν in the class $Z_{\sigma}^{\nu}(\varrho,\tau,F)$:

1. $V_{\sigma}^{\nu}(\varrho,F)\equiv Z_{\sigma}^{\nu}(\varrho,0,F),\ \nu\geq 1,\ and\ \varrho\in C\setminus\{0\}$ is the collection of functions $\phi\in\sigma$ that meet

$$1+\frac{1}{\varrho}\left(\left(\phi'(\varsigma)\right)^{\nu}-1\right)\prec F(\varsigma)=\frac{F(\varkappa,y,\varsigma)}{\varsigma},\ \varsigma\in U,$$

and

$$1+\frac{1}{\varrho}\left(\left(\psi'(w)\right)^{\nu}-1\right)\prec F(w)=\frac{F(\varkappa,y,w)}{w},\ w\in U,$$

where $F(z)$ is as mentioned in (1.8).

2. $H_{\sigma}(\varrho,\tau,F)\equiv Z_{\sigma}^{1}(\varrho,\tau,F),\ 0\leq\tau\leq 1,\ and\ \varrho\in C\setminus\{0\}$ represents the set of functions $\phi\in\sigma$ that satisfies

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{\left(\varsigma\phi'(\varsigma)\right)'}{\phi'(\varsigma)}\right)+(1-\tau)\left(\phi'(\varsigma)\right)-1\right)\prec F(\varsigma)=\frac{F(\varkappa,y,\varsigma)}{\varsigma},\ \varsigma\in U,$$

and

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{\left(w\psi'(w)\right)'}{\psi'(w)}\right)+(1-\tau)\left(\psi'(w)\right)-1\right)\prec F(w)=\frac{F(\varkappa,y,w)}{w},\ w\in U,$$

where $F(z)$ is as mentioned in (1.8).

Remark 1.2. $Z_{\sigma}^{\nu}(\varrho,1,F)\equiv Q_{\sigma}^{\nu}(\varrho,F)$.

In Section 2, we find estimates for $|d_2|$, $|d_3|$, and $|d_3-\xi d_2^2|,\xi\in R$ for functions in the classes $T_{\sigma}^{\nu}(\varrho,\tau,F)$, and $Z_{\sigma}^{\nu}(\varrho,\tau,F)$. In Section 3, interesting results from these classes are presented along with connections to the published findings.

2. Main results

We find the coefficient related estimates for any function $\phi \in T_\sigma^\nu(\varrho, \tau, F)$.

Theorem 2.1. *Let $\varrho \in C \setminus \{0\}, \xi \in R, \nu \geq 1, 0 \leq \tau \leq 1$ and $F(\varsigma)$ is as in (1.8). If $\phi \in \sigma$ is an element of the family $T_\sigma^\nu(\varrho, \tau, F)$, then*

$$|d_2| \leq |\varrho k| \sqrt{\frac{|k|}{|(\nu(\nu-1)(6\tau+1)+\tau+\nu^2)\varrho k^2 - (2\nu-1)^2(\tau+1)^2(k^2+l)|}}, \tag{2.1}$$

$$|d_3| \leq \frac{|\varrho k|^2}{(2\nu-1)^2(\tau+1)^2} + \frac{|\varrho k|}{(3\nu-1)(2\tau+1)}, \tag{2.2}$$

and

$$|d_3 - \xi d_2^2| \leq \begin{cases} \dfrac{|\varrho k|}{(3\nu-1)(2\tau+1)} & ; |1-\xi| \leq ד, \\ \dfrac{|\varrho|^2 |k|^3 \, |1-\xi|}{|(\nu(\nu-1)(6\tau+1)+\tau+\nu^2)\varrho k^2 - (2\nu-1)^2(\tau+1)^2(k^2+l)|} & ; |1-\xi| \geq ד, \end{cases} \tag{2.3}$$

where

$$ד = \left| \frac{(\nu(\nu-1)(6\tau+1)+\tau+\nu^2)\varrho k^2 - (2\nu-1)^2(\tau+1)^2(k^2+l)}{(3\nu-1)(2\tau+1)\varrho k^2} \right|. \tag{2.4}$$

Proof. Let $\phi \in T_\sigma^\nu(\varrho, \tau, F)$. Then, from subordinations (1.6) and (1.7), we can write

$$1 + \frac{1}{\varrho}\left(\tau\left(\frac{[(\varsigma\phi'(\varsigma))']^\nu}{\phi'(\varsigma)}\right) + (1-\tau)\left(\frac{\varsigma(\phi'(\varsigma))^\nu}{\phi(\varsigma)}\right) - 1\right) = F(u(\varsigma)), \ \varsigma \in U, \tag{2.5}$$

and

$$1 + \frac{1}{\varrho}\left(\tau\left(\frac{[(w\psi'(w))']^\nu}{\psi'(w)}\right) + (1-\tau)\left(\frac{w(\psi'(w))^\nu}{\psi(w)}\right) - 1\right) = F(v(w)), \ w \in U, \tag{2.6}$$

where $u(\varsigma) = \sum_{j=1}^\infty u_j \varsigma^j$, *and* $v(w) = \sum_{j=1}^\infty v_j w^j$, $\varsigma, w \in U$ are Schwarz functions with the property (See [14])

$$|u_j| \leq 1, \ and \ |v_j| \leq 1 \ (j \in N). \tag{2.7}$$

By using few fundamental mathematical techniques we can write equations (2.5) and (2.6) as

$$1 + \frac{1}{\varrho}\left(\tau\left(\frac{[(\varsigma\phi'(\varsigma))']^\nu}{\phi'(\varsigma)}\right) + (1-\tau)\left(\frac{\varsigma(\phi'(\varsigma))^\nu}{\phi(\varsigma)}\right) - 1\right) = 1 + \frac{1}{\varrho}(2\nu-1)(\tau+1)d_2\varsigma$$
$$+ \frac{1}{\varrho}([(3\nu-1)(2\tau+1)d_3 + (2\nu^2-4\nu+1)(3\tau+1)d_2^2]\varsigma^2 + \cdots), \tag{2.8}$$

$$F(u(\varsigma)) = 1 + F_2(\varkappa, y)u_1\varsigma + [F_2(\varkappa, y)u_2 + F_3(\varkappa, y)u_1^2]\varsigma^2 + \cdots, \tag{2.9}$$

and

$$1 + \frac{1}{\varrho}\left(\tau\left(\frac{[(w\psi'(w))']^\nu}{\psi'(w)}\right) + (1-\tau)\left(\frac{w(\psi'(w))^\nu}{\psi(w)}\right) - 1\right) = 1 - \frac{1}{\varrho}(2\nu-1)(\tau+1)d_2 w$$

$$+\frac{1}{\varrho}([(3\nu-1)(2\tau+1)(2d_2^2-d_3)+(2\nu^2-4\nu+1)(3\tau+1)d_2^2]w^2+\cdots), \tag{2.10}$$

$$F(v(w))=1+F_2(\varkappa,y)v_1w+[F_2(\varkappa,y)v_2+F_3(\varkappa,y)v_1^2]w^2+\cdots. \tag{2.11}$$

Comparing the terms with the same degree in (2.8) and (2.10), we conclude due to equality (2.5)

$$(2\nu-1)(\tau+1)d_2=\varrho F_2(\varkappa,y)u_1, \tag{2.12}$$

$$(3\nu-1)(2\tau+1)d_3+(2\nu^2-4\nu+1)(3\tau+1)d_2^2=\varrho[F_2(\varkappa,y)u_2+F_3(\varkappa,y)u_1^2], \tag{2.13}$$

Similarly, due to equality (2.6), we draw our conclusion by comparing the terms of the same degree in (2.9) and (2.11)

$$-(2\nu-1)(\tau+1)\,d_2=\varrho F_2(\varkappa,y)v_1, \tag{2.14}$$

and

$$(3\nu-1)(2\tau+1)(2d_2^2-d_3)+(2\nu^2-4\nu+1)(3\tau+1)d_2^2=\varrho[F_2(\varkappa,y)v_2+F_3(\varkappa,y)v_1^2]. \tag{2.15}$$

From (2.12) and (2.14), we get

$$u_1=-v_1, \tag{2.16}$$

and

$$2(2\nu-1)^2(\tau+1)^2d_2^2=\varrho^2(u_1^2+v_1^2)F_2^2(\varkappa,y). \tag{2.17}$$

Addition of (2.13) and (2.15) yield

$$2(\nu(\nu-1)(6\tau+1)+\tau+\nu^2)d_2^2=\varrho F_2(\varkappa,y)(u_2+v_2)+\varrho F_3(\varkappa,y)(u_1^2+v_1^2). \tag{2.18}$$

Replacing $u_1^2+v_1^2$ from (2.17) in (2.18) we get

$$d_2^2=\frac{\varrho^2F_2^3(\varkappa,y)(u_2+v_2)}{2\left[(\nu(\nu-1)(6\tau+1)+\tau+\nu^2)\varrho F_2^2(\varkappa,y)-(\tau+1)^2(2\nu-1)^2F_3(\varkappa,y)\right]}. \tag{2.19}$$

Utilizing (1.5) for $F_2(\varkappa,y)$, $F_3(\varkappa,y)$ and applying (2.7) to u_2,v_2 produces (2.1),

From (2.13) we subtract (2.15) to get the bound on $|d_3|$:

$$d_3=d_2^2+\frac{\varrho F_2(\varkappa,y)(u_2-v_2)}{2(2\tau+1)(3\nu-1)}. \tag{2.20}$$

If we replace d_2^2 from (2.17) in (2.20) we get

$$|d_3|=\frac{\varrho^2F_2^2(\varkappa,y)(u_1^2+v_2^2)}{2(2\nu-1)^2(\tau+1)^2}+\frac{\varrho F_2(\varkappa,y)(u_2-v_2)}{2(2\tau+1)(3\nu-1)}. \tag{2.21}$$

We deduce (2.2) from (2.21) by applying (1.5) and (2.7). Finally, we compute the bound on $|d_3-\xi d_2^2|$ using the values of d_2^2 and d_3 from (2.19) and (2.20), respectively. Consequently, we have,

$$|d_3-\xi d_2^2|$$

$$=\frac{|\varrho||F_2(\varkappa,y)|}{2}\left|\left(\frac{1}{(3\nu-1)(2\tau+1)}+B(\xi,F)\right)u_2-\left(\frac{1}{(3\nu-1)(2\tau+1)}-B(\xi,F)\right)v_2\right|,$$

where

$$B(\xi,F)=\frac{(1-\xi)\varrho F_2^2(\varkappa,y)}{\left[(\nu(\nu-1)(6\tau+1)+\tau+\nu^2)\varrho F_2^2(\varkappa,y)-(2\nu-1)^2(\tau+1^2)F_3(\varkappa,y)\right]}.$$

Clearly

$$|d_3-\xi d_2^2|\ \le \begin{cases}\dfrac{|\varrho||F_2(\varkappa,y)|}{(2\tau+1)(3\nu-1)} & ;\ |B(\xi,F)|\le\dfrac{1}{(2\tau+1)(3\nu-1)}.\\ |\varrho||F_2(\varkappa,y)||B(\xi,F)| & ;\ |B(\xi,F)|\ge\dfrac{1}{(2\tau+1)(3\nu-1)}.\end{cases} \tag{2.22}$$

We derive (2.3) from (2.22), where ٦ is the same as in (2.4). □

Remark 2.1. From Theorem 2.1, we can deduce Theorem 1 and Theorem 3 in [39] by letting $\varrho=1$ and $\nu=1$.

We now provide the coefficient related estimates for functions in the class, $Z_\sigma^\nu(\varrho,\tau,F)$.

Theorem 2. 2. *Let $\xi\in R,\varrho\in C\setminus\{0\},\nu\ge 1,0\le\tau\le 1$, and $F(\varsigma)$ be as in ([1.8). If an element ϕ of σ belongs to the class $Z_\sigma^\nu(\varrho,\tau,F)$, then*

$$|d_2|\le|\varrho k|\sqrt{\frac{|k|}{\left|\varrho k^2\left(2\nu(\nu-1)(3\tau+1)+3\nu-\tau(2\nu-1)\right)-4\left(\nu+\tau(\nu-1)\right)^2(k^2+l)\right|}}, \tag{2.23}$$

$$|d_3|\le\frac{|\varrho k|^2}{4\left(\nu+\tau(\nu-1)\right)^2}+\frac{|\varrho k|}{3\left(\nu+\tau(2\nu-1)\right)}, \tag{2.24}$$

and

$$|d_3-\xi d_2^2|\ \le$$
$$\begin{cases}\dfrac{|\varrho||k|}{3\left(\nu+\tau(2\nu-1)\right)} & ;|1-\xi|\le Z,\\ \dfrac{|\varrho|^2|k|^3\,|1-\xi|}{\left|\varrho k^2\left(2\nu(\nu-1)(3\tau+1)+3\nu-\tau(2\nu-1)\right)-4\left(\nu+\tau(\nu-1)\right)^2(k^2+l)\right|} & ;|1-\xi|\ge Z,\end{cases} \tag{2.25}$$

where

$$Z=\left|\frac{\varrho k^2\left(2\nu(\nu-1)(3\tau+1)+3\nu-\tau(2\nu-1)\right)-4\left(\nu+\tau(\nu-1)\right)^2(k^2+l)}{3\left(\nu+\tau(2\nu-1)\right)\varrho k^2}\right|.$$

Proof. Let $\phi\in Z_\sigma^\nu(\varrho,\tau,F)$. Then, from (1.9) and (1.10), we get

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{\left[(\varsigma\phi'(\varsigma))'\right]^\nu}{\phi'(\varsigma)}\right)+(1-\tau)\left(\phi'(\varsigma)\right)^\nu-1\right)=F\left(u(\varsigma)\right),\ \varsigma\in U, \tag{2.26}$$

and

$$1+\frac{1}{\varrho}\left(\tau\left(\frac{\left[(w\psi'(w))'\right]^\nu}{\psi'(w)}\right)+(1-\tau)\left(\psi'(w)\right)^\nu-1\right)=F\left(v(w)\right),\ w\in U,. \tag{2.27}$$

where $u(\varsigma)=\sum_{j=1}^{\infty} u_j\varsigma^j,\ and\ v(w)=\sum_{j=1}^{\infty} v_j w^j,\ \varsigma,w\in U$ are Schwarz functions with the property (2.7). By using a few fundamental techniques we can write equations (2.26) and (2.27) as

$$1+\frac{2}{\varrho}(\tau(\nu-1)+\nu)d_2\varsigma$$
$$+\frac{1}{\varrho}\left[3(\tau(2\nu-1)+\nu)d_3-2\big(2\tau(2\nu-1)-\nu(\nu-1)(3\tau+1)\big)d_2^2\right]\varsigma^2+\cdots$$
$$=F_1(\varkappa,y)+F_2(\varkappa,y)u_1\varsigma+[F_2(\varkappa,y)u_2+F_3(\varkappa,y)u_1^2]\varsigma^2+\ldots, \tag{2.28}$$

and

$$1-\frac{2}{\varrho}(\tau(\nu-1)+\nu)d_2w$$
$$+\frac{1}{\varrho}\left[3(\tau(2\nu-1)+\nu)(2d_2^2-d_3)-2\big(2\tau(2\nu-1)-\nu(\nu-1)(3\tau+1)\big)d_2^2\right]w^2+\cdots$$
$$=F_1(\varkappa,y)+F_2(\varkappa,y)v_1w+[F_2(\varkappa,y)v_2+F_3(\varkappa,y)v_1^2]w^2+\ldots \tag{2.29}$$

Thus, by comparing the corresponding coefficients in (2.28) and (2.29), it is possible to obtain

$$2(\tau(\nu-1)+\nu)d_2=\varrho F_2(\varkappa,y)u_1, \tag{2.30}$$
$$3(\tau(2\nu-1)+\nu)d_3-2\big(2\tau(2\nu-1)-\nu(\nu-1)(3\tau+1)\big)d_2^2$$
$$=\varrho[(\varkappa,y)u_2+F_3(\varkappa,y)u_1^2], \tag{2.31}$$
$$-2(\tau(\nu-1)+\nu)\,d_2=\varrho F_2(\varkappa,y)v_1, \tag{2.32}$$

and

$$3(\tau(2\nu-1)+\nu)(2d_2^2-d_3)-2\big(2\tau(2\nu-1)-\nu(\nu-1)(3\tau+1)\big)d_2^2.$$
$$=\varrho[F_2(\varkappa,y)v_2+F_3(\varkappa,y)v_1^2]. \tag{2.33}$$

The results (2.23) - (2.25) of this theorem now follow from (2.30) - (2.33), using the same method as found in Theorem 2.1 about (2.12) - (2.15). ◻

Remark 2.2. Regarding the above-described definitions, we can obtain multiple subfamilies of bi-univalent functions associated with GBFP for specific parameters, such as τ *and* ν. Thus, the related results come from the findings presented in the publication. We address some of these in the following section.

3. Special cases

In the scenario where $\tau=0$, Theorem 2.1 would yield the following results:

Corollary 3.1. *Let $\xi\in R,\varrho\in C\setminus\{0\},\nu\geq 1$ and $F(\varsigma)$ be as in (1.8). If $\phi\in P_\sigma^\nu(\varrho,F)$, then*

$$|d_2|\leq|\varrho k|\sqrt{\frac{|k|}{|\nu(2\nu-1)\varrho k^2-(2\nu-1)^2(k^2+l)|}},\qquad |d_3|\leq\frac{|\varrho k|^2}{(2\nu-1)^2}+\frac{|\varrho k|}{3\nu-1},$$

and

$$|d_3-\xi d_2^2| \le \begin{cases} \dfrac{|\varrho k|}{3\nu-1} & ; |1-\xi| \le X \\ \dfrac{|\varrho|^2|k|^3\,|1-\xi|}{\left|\left(\nu(2\nu-1)\right)\varrho k^2-(2\nu-1)^2(k^2+l)\right|} & ; |1-\xi| \ge X. \end{cases}$$

where $X=\left|\frac{(\nu(2\nu-1))\varrho k^2-(2\nu-1)^2(k^2+l)}{(3\nu-1)\varrho k^2}\right|$.

Remark 3.1. Allowing $\varrho=1$ and $\nu=1$ in Corollary 3.1, we get Corollaries 2 and 6 in [39].

Theorem 2.1 would yield the following in the scenario where $\tau=1$.

Corollary 3. 2. *Let $\varrho\in C\setminus\{0\},\xi\in R,\nu\ge 1$ and $F(\varsigma)$ be as in (1.8). If $\phi\in Q_\sigma^\nu(\varrho,F)$, then*

$$|d_2|\le|\varrho k|\sqrt{\frac{|k|}{|(8\nu^2-7\nu+1)\varrho k^2-4(2\nu-1)^2(k^2+l1)|}},\qquad |d_3|\le\frac{|\varrho k|^2}{4(2\nu-1)^2}+\frac{|\varrho k|}{3(3\nu-1)},$$

and

$$|d_3-\xi d_2^2| \le \begin{cases} \dfrac{|\varrho k|}{3(3\nu-1)} & ; |1-\xi| \le \Theta \\ \dfrac{|\varrho|^2|k|^3\,|1-\xi|}{|(8\nu^2-7\nu+1)\varrho k^2-4(2\nu-1)^2(k^2+l)|} & ; |1-\xi| \ge \Theta \end{cases}$$

where $\Theta=\left|\frac{\varrho k^2(8\nu^2-7\nu+1)-4(k^2+l)(2\nu-1)^2}{3(3\nu-1)\varrho k^2}\right|$.

Remark 3.2. Using $\varrho=1$ and $\nu=1$ in Corollary 3.2, we get Corollaries 3 and 7 in [39].

In the case where $\tau=0$, Theorem 2.2 would lead to the following :

Corollary 3. 3. *Let $\varrho\in C\setminus\{0\},\xi\in R,\nu\ge 1$ and $F(\varsigma)$ be as in ([1.8). If $\phi\in V_\sigma^\nu(\varrho,F)$, then*

$$|d_2|\le|\varrho k|\sqrt{\frac{|k|}{|\nu(2\nu+1)\varrho k^2-4\nu^2(k^2+l)|}},\qquad |d_3|\le\frac{|\varrho k|^2}{4\nu^2}+\frac{|\varrho k|}{3\nu},$$

and

$$|d_3-\xi d_2^2| \le \begin{cases} \dfrac{|\varrho k|}{3\nu} & ; |1-\xi| \le \left|\dfrac{\nu(2\nu+1)\varrho k^2-4\nu^2(k^2+l)}{3\nu\varrho k^2}\right| \\ \dfrac{|\varrho|^2|k|^3\,|1-\xi|}{|\nu(2\nu+1)\varrho k^2-4\nu^2(k^2+l)|} & ; |1-\xi| \ge \left|\dfrac{\nu(2\nu+1)\varrho k^2-4\nu^2(k^2+l)}{3\nu\varrho k^2}\right|. \end{cases}$$

In the case where $\nu=1$, Theorem 2.2 would lead to the following:

Corollary 3.4. *Let $\xi\in R,\varrho\in C\setminus\{0\},0\le\tau\le 1$ and $F(\varsigma)$ be as in (1.8). If $\phi\in H_\sigma(\varrho,\tau,F)$, then*

$$|d_2|\le|\varrho k|\sqrt{\frac{|k|}{|(3-\tau)\varrho k^2-4(k^2+l)|}},\qquad |d_3|\le\frac{|\varrho k|^2}{4}+\frac{|\varrho k|}{3(\tau+1)},$$

and $$|d_3-\xi d_2^2| \le \begin{cases} \dfrac{|\varrho k|}{3(\tau+1)} & ; \quad |1-\xi| \le \left|\dfrac{(3-\tau)\varrho k^2-4(k^2+l)}{3(\tau+1)\varrho k^2}\right| \\ \dfrac{|\varrho|^2|k|^3\,|1-\xi|}{|(3-\tau)\varrho k^2-4(k^2+l)|} & ; \quad |1-\xi| \ge \left|\dfrac{(3-\tau)\varrho k^2-4(k^2+l)}{3(\tau+1)\varrho k^2}\right| \end{cases}$$

Remark 3.3. In Corollary 3.4, if we take $\tau = 1$ and $\varrho = 1$, we get the outcomes [9, corollaries 3 and 7].

4. Conclusion

This study contains the upper bounds on $|d_2|\ and\ |d_3|$ for functions that belong to the defined σ subclasses associated with GBFP. Additionally, for functions in these subfamilies, we have determined the Fekete-Szegö functional $|d_3 - \xi d_2^2|, \xi \in R$. Specialization of the parameters applied to our results, as mentioned in Section 3, produces previously unexplored new results. Additionally, pertinent links to the current findings are indicated. Nevertheless, this paper does not address all of the significant subclasses of σ that exist in the literature. For example, authors have examined various subclasses involving (p,q)-operators introduced in (p,q)-calculus. It is recommended that the interested readers review these papers and the associated references. We conclude our study by pointing out to interested readers that these subclasses can be studied for higher order Hankel determinant problems.

Authors's Contributions: Each author contributed equally to the results' derivation and gave their final manuscript approval.

Conflicts of Interest: There are no competing interests with regard to the publishing of this manuscript, the authors reaffirm.

Funding: No External fund is received for this research.

Data Availability Statement: This manuscript uses no data.

References

[1] İ. Aktaş and D. Hamarat, Generalized bivariate Fibonacci polynomial and two new subclasses of bi-univalent functions, Asian-European Journal of Mathematics, 16(8) (2023), 2350147

[2] I. Al-Shbeil, A.K. Wanas, A. Catas, and H.Alaqad, Applications of Horadam polynomials for Bazilevic and λ-pseudo-starlike bi-univalent functions associated with Sakaguchi type functions, Symmetry, 16 (2024), Art. ID 218, 1–11.

[3] Ş. Altınkaya and S. Yalçın, Lucas polynomials and applications to an unified class of bi-univalent functions equipped with (P,Q)-derivative operators, TWMS J. Pure Appl. Math., 11(1) (2020), 100–108.

[4] M. Bicknell, Introduction to Fibonacci polynomials and their divisibility properties, Fibonacci Quart., 8 (1970), 407–420.

[5] L Bieberbach, Uber die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln, Sitzungsberichte Preussische Akademie der Wissenschaften, 138 (1916), 940–955.

[6] D.A. Brannan and J.G. Clunie, Aspects of contemporary complex analysis, Proceedings of the NATO Advanced study institute held at University of Durhary, New York Academic press, (1979).

[7] D.A. Brannan and T.S. Taha, On some classes of bi-univalent functions, Studia Univ. Babes-Bolyai Math., 31(2) (1986), 70–77.

[8] S. Bulut, Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions, C R Acad. Sci. Paris Sér I 352, (2014), 479–484.

[9] M. Çağlar, E. Deniz, and H. M. Srivastava, Sėcond Hankel determinant for certain subclasses of bi-univalent functions, Turk. J. Math., 41 (2017), 694–706.

[10] M. Catalani, Generalized Bivariate Fibonacci Polynomials, arXiv:math 0211366v2 [math.CO] 4 June (2004).

[11] L. De Branges, A proof of the Bieberbach conjecture, Acta Mathematica, 154(1-2) (1985), 137–152.

[12] E. Deniz, Certain subclasses of bi-univalent functions satisfying subordinate conditions, J. Classical Ana., 2(1) (2013), 49–60.

[13] E. Deniz, M. Kamali, and S. Korkmaz, A certain subclass og bi-univalent functions associated with Bell numbers and q-Srivastava Attiya operator, AIMS Mathematics, 5(6) (2020), 7259–7271.

[14] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, 1983.

[15] M. Fekete and G. Szegö, Eine Bemerkung über ungerade schlichte Funktionen, J. Lond. Math. Soc., 89 (1933), 85–89.

[16] B.A. Frasin, Coefficient bounds for certain classes of bi-univalent functions, Hacet. J. Math. stat., 43(3) (2014), 383–389.

[17] B.A. Frasin and M.K. Aouf, New subclasses of bi-univalent functions, Appl. Math., 24 (2011), 1569–1573.

[18] B.A. Frasin, S.R. Swamy, and I. Aldawish, A comprehensive family of bi-univalent functions defined by k-Fibonacci numbers, J. Func. Spaces., 2021(1), 4249509, (2021).

[19] B.A. Frasin, S.R. Swamy, A. Amourah, J. Salah, and R.H. Maheshwarappa, A family of bi-univalent functions defined by (p, q)-derivative operator subordinate to a generalized bivariate Fibonacci polynomials, European Journal of Pure and Applied Mathematics, 17(4) (2024) 3801–3814.

[20] H. Ö. Guney, G. Murugusundaramoorthy, and J. Sokol, Certain subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers, Acta Univ. Sapientiae Math., 10 (2018), 70–84.

[21] E.G. Kocer and S. Tuncez, Bazilevic Fibonacci and Lucas like polynomials, Gazi Univ.J. Sci., 29 (2016), 109–113.

[22] T. Koshy, Fibonacci and Lucas numbers with applications, Wiley, NewYork, (2011).

[23] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc., 18 (1967), 63–68 .

[24] A. Motamednezhad and S. Salehian, New subclasses of Bi-univalent functions by (p;q)-derivative operator, Honam Mathematical J., 41(2) (2019), 381–390.

[25] E. Netenyahu, The minimal distance of the image boudary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Ration. Mech. Anal., 32 (1969), 100–112.

[26] H. Orhan, P. K. Mamatha, S. R. Swamy, N. Magesh, and J. Yamini, Certain classes of bi-univalent functions associated with the Horadam polynomials, Acta Univ. Sapieniae, Mathematica., 13(1) (2021), 258–272.

[27] R. Öztürk and İ. Aktaş, Coefficient Estimate and Fekete-Szegö Problems for certain new subclasses of Bi-univalent functions defined by generalized bivariate Fibonacci polynomial, Sahand Communications in Mathematical Analysis, 21(3) (2024), 35–53. https://doi.org/10.22130/scma.2023.1988826.124

[28] S. Santhiya and K. Thilagavathi, Geometric properties of analytic functions defined by the (p, q)-derivative operator involving the Poisson distribution, Journal of Mathematics, Volume 2023, Article ID 2097976, 12 pages. https://doi.org/10.1155/2023/2097976

[29] H.M. Srivastava, Ş. Altınkaya, and S. Yalçın, Certain Subclasses of bi-univalent functions associated with the Horadam polynomials, Iran J. Sci. Technol. Trans. Sci., 43 (2019), 1873–1879. doi.org/10.1007/s 40995-018-0647-0

[30] H.M. Srivastava, A.K. Mishra, and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., 23 (2010), 1188–1192.

[31] S.R. Swamy, D. Breaz, V. Kala, P.K. Mamatha, L-I.Cotîrlă, and E. Rapeanu, Initial Coefficient bounds analysis for novel subclasses of bi-univalent functions linked with Lucas-Balancing polynomials, Mathematics, 12 (2024), 1325. https://doi.org/10.3390/math12091325

[32] S.R. Swamy, B.A. Frasin, D. Breaz, and L.-I. Cotîrlă, Two families of bi-Univalent functions associating the (p, q)-derivative with generalized bivariate Fibonacci polynomials, Mathematics, 12 (2024), 3933.

[33] S.R. Swamy and S. Yalçin, Coefficient bounds for regular and bi-univalent functions linked with Gegenbauer polynomials, Prbl. Anal. Issues Anal., 11(1) (2022), 133–144. DoI:10.15393/j3.art.2022.10351

[34] S.R. Swamy and A.K. Wanas, A comprehensive family of bi-univalent functions defined by (m, n) - ucas polynomials, Bol. Soc. Mat. Mex., 28, 34 (2022). https://doi.org/10.1007/s40590-022-00411-0

[35] S.R. Swamy, A.K. Wanas, P.K. Mamatha, G.S. Chauhan, and Y. Sailaja, Bi-univalent function subfamilies associated with the (p, q)-derivative operator subordinate to Lucas-Balancing polynomials, Earthline Journal of Mathematical Sciences, 15(3)(2025), 273–287

[36] D.L. Tan, Coefficient estimates for bi-univalent functions, Chin. Ann. Math. Ser. A, 5 (1984), 559–568.

[37] H. Tang, G. Deng, and S. Li, Coefficient estimates for new subclasses of Ma-Minda bi-univalent functions, J. Ineq. Appl., 2013, Art.317, 10 pages (2013).

[38] A.K. Wanas and L-I. Cotîrlă, Applications of (M,N)-Lucas Polynomials on a certain family of bi-univalent functions, Mathematics, 10(4) (2022), 595.

[39] N. Yilmaz and I. Aktas, On some new subclasses of bi-univalent functions defined by generalized bivariate Fibonacci polynomials, Afr. Matematika, 3(59) (2022), 3–12.