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Nonlinear Elliptic Equations on the Sierpiński Carpet

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Abstract:

This paper deals with a semilinear elliptic equation for a properly defined Laplace operator on the Sierpiński carpet. We investigate the existence of solutions, under Dirichlet boundary conditions, for the problem

$$\Delta u + a(x)u = f(x,u).$$

This work is motivated by previos studies established for such problems on the Sierpiński gasket by many authors: Kenneth J. Falconer, Jiaxin Hu, G.Bonanno and others (more details can be found in Introduction).

The approach used in this study is based on variational methods, more specifically, the the mountain pass theorem of Ambrosetti and Rabinowit, and Sobolev type inequality, which was of principal use for achieving other results.

Keywords: Sierpiński carpet, Weak Laplacian, Weak solutions, Fractal domains, Mountain pass theorem, Sobolev-type inequality, Nonlinear elliptic equations, Dirichlet form.

1 Introduction

We consider the existence of nontrivial solutions for a class of nonlinear elliptic problem

$$(S_{f,a}) \quad \begin{cases} \Delta u + a(x)u = f(x,u) & x \in SC \backslash \partial SC, \\ u_{|\partial SC} = 0, \end{cases}$$

where SC is the Sierpiński carpet in $\mathbb{R}^2$, ∂SC is its boundary, and Δ is the Laplacian operator on the fractal domain SC.
The coefficient of linear term is $a : SC \to \mathbb{R}$, satisfies (a_*), and
$f : SC \times \mathbb{R} \to \mathbb{R}$ is the nonlinear part, that is continuous and satisfies some growth restrictions near zero and at infinity; see (f_*) and (f_{**}) below

(f_*) : There are positive constants t_0 and β such that

$$\eta(t_0) \equiv \max_{\substack{x\in SC \\ |t|\leq t_0}} |f(x,t)| \leq \frac{t_0}{2(\beta+1)(19C)^2}.$$

(f_{**}) : There are positive constants $\nu > 2$ and $r \geq 0$ such that for $|t| \geq r$,

$$tf(x,t) \leq \nu F(x,t) < 0,$$

where $F(x,u) = \displaystyle\int_0^u f(x,t)\mathrm{d}t$, and $a \in L^1(SC)$ satisfied

(a_*) : let $a(x) \leq 0$ for all $x \in SC$ such as $\displaystyle\int_{SC} |a(x)|d\mu < 1/(19C)^2$.
The problem $(S_{f,a})$ is called the Dirichlet problem with zero boundary values.

The non-linear problem $(S_{f,a})$ is strongly associated with physical phenomena such as reaction-diffusion problems, elastic property in fractal media, and flow through fractal domains. See for example [19] and [2].

There is a comprehensive theory for studying nonlinear elliptic equations $(S_{f,a})$ over classical domains, i.e. over sets of $\mathbb{R}^N$ using variational approachs and others. A natural question is how to create an acceptable framework for studying the problem $(S_{f,a})$ on the Sierpiński carpet SC.

As we know the concept of differentiation such as the operator Laplacian in fractal fields does not have a meaning like in the classical domains, $\mathbb{R}^N$ for example. Laplacian's concept was first constructed by Kigami [15] using graph theory and by Kusuoka [16] using probabilistic approach, and later this was generalized to P.C.F (post-critically-finite) self-similar sets [14]. Then, Barlow [3] and Kusuoka [17] constructed Laplacian on the Serpinski carpet in two different ways. And in 2009, Borlow and Bass [6] showed that the two constructions are equivalent and there is uniqueness up to a constant multiple. We may introduce a Hilbert space structure through Laplacian and then construct compactness theorems that will enable us the study of the problem $(S_{f,a})$.

In this paper we focus on a specific non-P.C.F fractal, the Sierpiński carpet SC in $\mathbb{R}^2$; see Figure 1, SC is distinct from PCF fractals such as the Sierpiński gasket, which has a simpler Laplacian construction [22]. SC's geometry is not as homogeneous as that of Sierpiński gasket and other PCF fractals. In this sense, Alexander Grigor'yan and Meng Yang [9] have done an excellent job of demonstrating some important inequalities. This work is characterised by the absence of the probabilistic approach, for more details one can see [9].

We will show the existence of non-trivial solutions for $(S_{f,a})$ over SC. In Section 2, we investigate [17]'s Laplacian definition on the Sierpiński carpet SC and we establish a Sobolev-type inequality see (lemma 2.2), after, we define an energy form belongig to the Hilbert space $H_0^1(SC)$ of finite energy functions. Finally, in Section 3, we show the existence of nontrivial weak solutions of $(S_{f,a})$ using the mountain pass theorem.

Notice that this work is motivated by [10, 11, 13], where Kenneth J. Falconer and Jiaxin Hu obtained the existence of nontrivial weak solutions for the problem $(S_{f,a})$ in a fractal called the Sierpiński gasket.

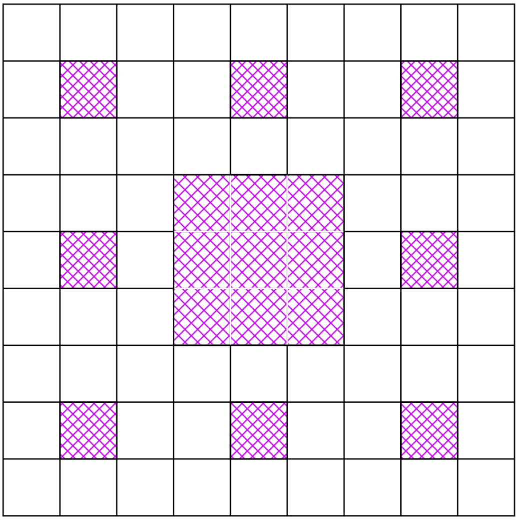

Fig. 1 The standard Sierpiński carpet.

2 Preliminaries and Sobolev-type inequality on the Sierpiński Carpet

We begin this section by a definition of the Sierpiński Carpet SC in $\mathbb{R}^2$, and after we introduce the energy form $\mathcal{E}(u)$ for $u : SC \to \mathbb{R}$, which allows us stating an important Sobolev type inequality (see lemma) 2.2, and defining the Hilbert space $H_0^1(SC)$ where we state our main results.

The Sierpiński carpet SC, is the self-similar fractal in the plane given by the identity

$$SC = \bigcup_{i=0}^{7} F_i(SC),$$

where F_i are contractions given by the following equations :

$$F_0(x) = \frac{1}{3}x + \frac{2}{3}q_0 \quad F_1(x) = \frac{1}{3}x + \frac{2}{3}q_1 \quad F_2(x) = \frac{1}{3}x + \frac{2}{3}q_2 \quad F_3(x) = \frac{1}{3}x + \frac{2}{3}q_3$$
$$F_4(x) = \frac{1}{3}x + \frac{2}{3}q_4 \quad F_5(x) = \frac{1}{3}x + \frac{2}{3}q_5 \quad F_6(x) = \frac{1}{3}x + \frac{2}{3}q_6 \quad F_7(x) = \frac{1}{3}x + \frac{2}{3}q_7 \quad .$$

where

$$q_0 = (0,0),\ q_1 = \left(0, \frac{1}{2}\right),\ q_2 = (0,1),\ q_3 = \left(\frac{1}{2}, 1\right)$$
$$q_4 = (1,1),\ q_5 = \left(1, \frac{1}{2}\right),\ q_6 = (1,0),\ q_7 = \left(\frac{1}{2}, 0\right).$$

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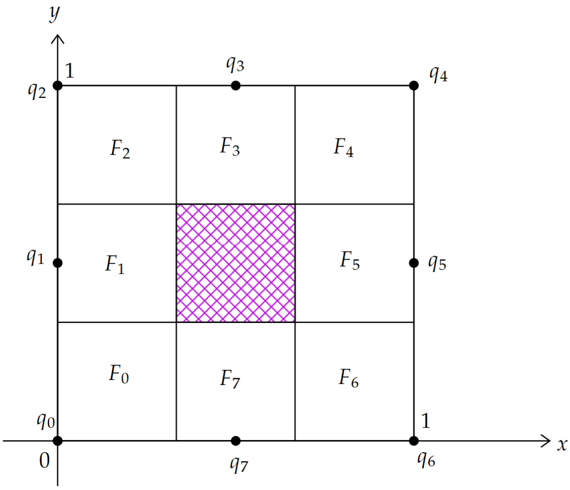

Fig. 2 First iteration of SC with all eight cells.

The boundary of the Sierpiński carpet ∂SC; is defined by convention as the boundary of the unit square containing all SC.

Now we define t he m -th a pproximation o f t he S ierpiński c arpet b y SC_n a nd w e call $F_i(SC_n)$ an n-cell, where SC_n denotes the graph whose vertices are the $n-$cells and whose edges connect two $n-$cells that intersect along an edge.
Where

$$\begin{cases} SC_{n+1} = \bigcup_{i=0}^{7} F_i\left(SC_n\right) & \text{for all } n \geq 0, \\ SC_0 = \{q_0, q_1, ..., q_7\}, \end{cases}$$

then $\{SC_n\}_{n\geq 0}$ is an increasing sequence of finite sets and SC is the closure of $\bigcup_{n=0}^{+\infty} SC_n$.

As in [17], let us consider $u : SC_* \to \mathbb{R}$ and

$$\mathcal{E}_m(u) = \rho^m \mathcal{W}_m(u) = \rho^m \sum_{x \sim_m y} (u(y) - u(x))^2. \tag{1}$$

Where $SC_* = \bigcup_{m\geq 0} SC_m$, $\mathcal{W}_m$ is called the graph energy and the notation $x \sim_m y$ significate that two cells shared a boundary edge, with $1.25147 < \rho < 1.25149$ has been experimentally determined; see [4] and [5].
For $m \geq 0$, the energy $\mathcal{E}\left(u\right)$ is defined for any $u : SC_* \to \mathbb{R}$ as

$$\mathcal{E}\left(u\right) = \lim_{m\to\infty} \mathcal{E}_m\left(u\right) \tag{2}$$

(possibly $\mathcal{E}\left(u\right) = +\infty$); see [7].

The left schema in Figure 3 illustrate the graphe $\mathcal{W}_2$ and the right one is SC_2. For more details one can see [9].

This definition of $\mathcal{E}\left(u\right)$ is justified by the lemma below.

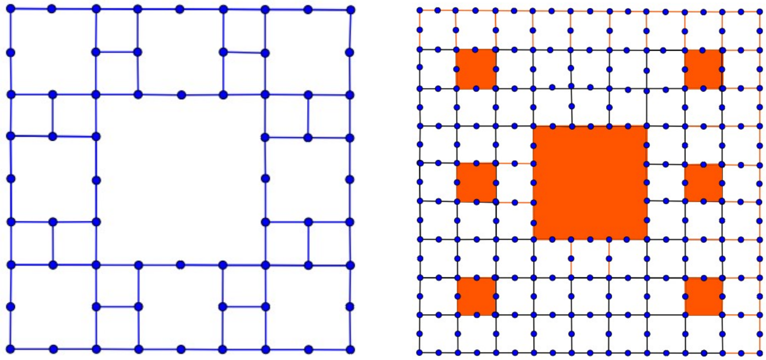

Fig. 3

Lemma 2.1. *$\mathcal{E}_m(u)$ defined by (1) is weakly nondecreasing for all $u : SC_* \to \mathbb{R}$. In other words, there is a constant C such that*

$$\forall n, m \geq 1, \quad \mathcal{E}_n(u) \leq C\mathcal{E}_{n+m}(u),$$

with $u \in L^2(SC)$.

Proof. See [17, Propo5.2] and [9].

We also need the following

Notation If $\Lambda_1$and Λ_2 are defined functions for $f \in \mathcal{C}$ (space of continuous functions). The notation

$$\Lambda_1(f) \asymp \Lambda_2(f)$$

significate the existence of constants $k, K > 0$ such that for all $f \in \mathcal{C}$,

$$k\Lambda_1(f) \leq \Lambda_2(f) \leq K\Lambda_1(f).$$

Theorem 2.1. *(See [9, Theorem 2.5])*

$$\mathcal{E}(u,u) \asymp \sup_{n\geq 1} 3^{(\beta^*-\alpha)n} \sum_{w\in W_n} \sum_{\substack{p,q\in SC_w \\ |p-q|=2^{-1}.3^{-n}}} (u(p)-u(q))^2.$$

$$\mathcal{F} = \left\{ u \in C(SC) : \sup_{n\geq 1} 3^{(\beta^*-\alpha)n} \sum_{w\in W_n} \sum_{\substack{p,q\in SC_w \\ |p-q|=2^{-1}.3^{-n}}} (u(p)-u(q))^2 < +\infty \right\},$$

$$where \begin{cases} \alpha = \frac{Log(8)}{Log(3)} \\ \beta^* = \frac{Log(8\rho)}{Log(3)} \\ \rho = 3^{\beta^*-\alpha} \approx 1.25148, \end{cases} \quad and \begin{cases} W_n = \{w = w_1 \ldots w_n : w_i = 0, 1, ..., 7;\ i = 1, \ldots, n\} \\ W_0 = \{\emptyset\}. \end{cases}$$

Lemma 2.2 (Sobolev-type Inequality). *We pose*

$$E_n(u,u) := \sup_{n\geq 1} 3^{(\beta^*-\alpha)n} \sum_{w\in W_n} \sum_{\substack{p,q\in SC_w \\ |p-q|=2^{-1}\cdot 3^{-n}}} (u(p)-u(q))^2,$$

for all $u : SC_* \to \mathbb{R}$,

$$\sup_{x,y\in SC_n} \frac{|u(x)-u(y)|}{|x-y|^{\frac{\beta^*-\alpha}{2}}} \leq 19\sqrt{E(u)}.$$

With

$$\frac{\beta^*-\alpha}{2} = \frac{Log(\rho)}{2Log(3)} \approx 0.102.$$

Proof. We assume that

$$E(u) = \sup_{n\geq 1} E_n(u) < +\infty \quad and \quad |x-y| < 1,$$

we have

$$\forall x,y \in SC_* = \bigcup_{m\geq 0} SC_m,\ \exists\ m = m(x,y) : \left(2^{-1}.3^{-(m+1)}\right)^{-1} \leq |x-y| \leq \left(2^{-1}.3^{-m}\right)^{-1}.$$

By choosing a sequence $(x_n)_n$ satisfying $x_n \xrightarrow[n\to+\infty]{} x$, and

$$x_n, x_{n+1} \in SC_{n+m+1},\ \ |x_{n+1}-x_n| = \left(2^{-1}.3^{-(n+m+1)}\right)^{-1}.$$

We have by definition of E_n

$$|u(x_n) - u(x_{n+1})| \leq 3^{-\frac{(\beta^*-\alpha)(n+m+1)}{2}} \sqrt{E_{n+m+1}(u)}.$$

On the other hand, we have

$$\begin{aligned}
|u(x) - u(x_0)| &= \lim_{n\to+\infty} |u(x_{n+1}) - u(x_0)| \\
&\leq \lim_{n\to+\infty} \sum_{j=0}^{n} |u(x_{j+1}) - u(x_j)| \\
&\leq 3^{\frac{-(\beta^*-\alpha)(m+1)}{2}} \times \frac{1}{1-\sqrt{3^{-(\beta^*-\alpha)}}} \sqrt{E(u)}.
\end{aligned}$$

And similary

$$|u(y) - u(y_0)| \leq 3^{\frac{-(\beta^*-\alpha)(m+1)}{2}} \times \frac{1}{1-\sqrt{3^{-(\beta^*-\alpha)}}} \sqrt{E(u)},$$

therefore

$$|u(x)-u(y)| \leq 3^{\frac{-(\beta^*-\alpha)(m+1)}{2}} \times \frac{2}{1-\sqrt{3^{-(\beta^*-\alpha)}}}\sqrt{E(u)}$$
$$\leq |x-y|^{\frac{\beta^*-\alpha}{2}} \times \frac{2}{1-\sqrt{3^{-(\beta^*-\alpha)}}}\sqrt{E(u)}$$
$$\leq 19|x-y|^{\frac{\beta^*-\alpha}{2}}\sqrt{E(u)}.$$

Since

$$3^{\frac{-(\beta^*-\alpha)(m+1)}{2}} \leq \left(2^{-1}.3^{-(m+1)}\right)^{-\frac{(\beta^*-\alpha)}{2}} \leq |x-y|^{\frac{\beta^*-\alpha}{2}} \quad and \quad \frac{2}{1-\sqrt{3^{-(\beta^*-\alpha)}}} \approx 18.87 \ ,$$

whence by passing to the supremum we have

$$\sup_{x,y\in SC_n} \frac{|u(x)-u(y)|}{|x-y|^{\frac{\beta^*-\alpha}{2}}} \leq 19\sqrt{E(u)}.$$

□

Remark 2.1. *In Reference [1](Theorem 4.11), Grigor'yan-Hu-Lau introduced an alternative approach distinct from proving Lemma 2.2, using the heat kernel estimate when the walk dimension is strictly less than the Hausdorff dimension.*

Let $C(SC)$ be the space of continuous functions on SC and

$$C_0(SC) = \left\{u \mid u \in C(SC), u_{|\partial SC} = 0\right\}.$$

We define

$$H_0^1(SC) = \{u \mid u \in C_0(SC), \mathcal{E}(u) < \infty\},$$

which is equipped with the norm

$$\|u\|_{\mathcal{E}} = \sqrt{\mathcal{E}(u)}. \tag{3}$$

Remark 2.2. *The norm $\|u\|_E$ is equivalent to the norm $\|u\|_{\mathcal{E}}$. Indeed, since*

$$E_n(u,v) := \sup_{n\geq 1} 3^{(\beta^*-\alpha)n} \sum_{w\in W_n} \sum_{\substack{x,y\in SC_w \\ |x-y|=2^{-1}\cdot 3^{-n}}} (u(x)-u(y))(v(x)-v(y)),$$

then by lemma 2.2, for all $u \in H_0^1(SC)$

$$|u(x)-u(y)| \leq 19|x-y|^{\frac{\beta^*-\alpha}{2}}\|u\|.$$

Let $y = q_0 = (0,0)$ and $C > 0$. Since $E(u) \asymp \mathcal{E}(u)$ then

$$|u(x)| \leq 19\|u\|_E \leq 19C\|u\|_{\mathcal{E}}. \tag{4}$$

Lemma 2.2. *Let $T : \mathbb{R} \to \mathbb{R}$ be a Lipschitz mapping with the Lipschitz constant $K \geq 0$ such that $T(0) = 0$. Then, for every $u \in H_0^1(V)$, we have $T \circ u \in H_0^1(V)$ and $\|T \circ u\| \leq K\|u\|$.*

Proof: It is clear that $T\circ u\in C_0(SC)$. By (1) and the Lipschitz property of T, we have that for any $m\in\mathbb{N}$.

$$\begin{aligned}\mathcal{E}_m(T\circ u)&=\rho^m\mathcal{W}_m(T\circ u)\\&=\rho^m\sum_{y\sim_m x}((T\circ u)(y)-(T\circ u)(x))^2\\&\le\rho^m K^2\sum_{y\sim x}(u(y)-u(x))^2\\&=K^2\mathcal{E}_m(u).\end{aligned}$$

Hence $\sqrt{\mathcal{E}\left(T\circ u\right)}\le K\sqrt{\mathcal{E}\left(u\right)}$, according to (2). Thus $T\circ u\in H_0^1\left(SC\right)$ and $\|T\circ u\|_{\mathcal{E}}\le K\left\|u\right\|_{\mathcal{E}}$.

We now define the Laplacian on the Sierpiński carpet SC.
Let $H^{-1}(SC)$ be the closure of $L^2(SC,\mu)$ with consideration to the pre-norm.

$$\|v\|_{-1}=\sup_{\substack{g\in H_0^1(SC)\\ \|g\|=1}}\langle v,g\rangle,$$

where

$$\langle v,g\rangle=\int_{SC}vgd\mu,\qquad v\in L^2(SC,\mu),\qquad g\in H_0^1(SC).$$

Here μ is the restriction to SC of normalized $\frac{\log 3}{\log 8}$-dimensional Hausdorff measure on SC , so that $\mu(SC)=1$.

$H^{-1}(SC)$ equipped with the inner product defined above is a Hilbert space. Using the notation $\mathcal{E}(u,v)$ for the inner product in $\in H_0^1(SC)$, we have

$$-\mathcal{E}(u,v)=\langle\Delta u,v\rangle=\int_{SC}(\Delta u)vd\mu\quad\text{ for all }\quad v\in H_0^1(SC),\tag{5}$$

defines an operator $\triangle u\in H^{-1}(SC)$ uniquely for all $u\in H_0^1(SC)$, and the term $\triangle$ is the weak Laplacian on SC; see [7] and [6].

We say that $u\in H_0^1(SC)$ is a weak solutions of $\left(S_{f,a}\right)$ if and only if

$$\mathcal{E}(u,v)-\int_{SC}a(x)uvd\mu+\int_{SC}f(x,u)vd\mu=0,\quad\forall v\in H_0^1(SC).\tag{6}$$

Proposition 2.1. *(The Palais-Smale condition. See [18],[8]).*
We say that a continuously differentiable functional in the sense of Fréchet $I\in C^1(H,\mathbb{R})$ of a Hilbert space H with values in $\mathbb{R}$ satisfies the Palais-Smale (in short (PS)-condition) condition if any sequence $(u_n)_{n\ge1}\subset H$ satisfying
- $\{I\left[u_k\right]\}_{k=1}^{\infty}$ *is bounded,*
- $\lim_{n\to\infty}\left\|I'\left(u_n\right)\right\|_{H*}=0,$

has a convergent subsequence in H.

Proposition 2.2. *(Mountain Pass Theorem. See [21] and [18]).*
Let E be a real Banach space and $I\in C^1(E,\mathbb{R})$ satisfy the Palais-Smale condition. Assume that $I(0)=0$ and
(I_1) there exist two positive constants, ρ and α, such that $I_{|\partial B_\rho}\ge\alpha$, and
(I_2) there is $e\in E\backslash\overline{B_\rho}$ such that $I(e)\le0$.
Then I has a critical point $c\ge\alpha$, which can be characterized as

$$c=\inf_{g\in\Gamma}\max_{u\in g([0,1])}I(u),$$

where

$$\Gamma = \{g \in C([0,1], E) \mid g(0) = 0 \ et \ g(1) = e\}$$

and

$$\overline{B_\rho} = \{u \in E, \ \|u\| \leq \rho\}.$$

3 Existence theorems using the mountain pass theorem

Let $\mathcal{B}$ be a Banach space. In what follows, denotes $C^1(\mathcal{B}, \mathbb{R})$ the set of functions that are Fréchet differentiable with continuous Fréchet derivatives on $\mathcal{B}$.

Define $F : SC \times \mathbb{R} \to \mathbb{R}$ by $F(x,u) = \int_0^u f(x,t)\mathrm{d}t$, for all $u \in \mathbb{R}$.

Let $I : H_0^1(SC) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\|u\|_{\mathcal{E}}^2 - \frac{1}{2}\int_{SC} a(x)u^2(x)\mathrm{d}\mu + \int_{SC} F(x,u)\mathrm{d}\mu, \forall u \in H_0^1(SC). \tag{7}$$

Proposition 3.1. *(Variational Principle).*

Assume $f(x,t) \in C(SC \times \mathbb{R}, \mathbb{R})$ and $\int_{SC} |a(x)|d\mu < \infty$. The map I given by (7) belongs to $C^1\left(H_0^1(SC), \mathbb{R}\right)$.
Moreover

$$I'(u)v = \mathcal{E}(u,v) - \int_{SC} a(x)uvd\mu + \int_{SC} f(x,u)vd\mu, \forall v \in H_0^1(SC), \ \forall u \in H_0^1(SC). \tag{8}$$

In particular, u is a weak solution of $(S_{f,a})$ if and only if $I'(u) = 0$.

Proof .
For all $u \in H_0^1(SC)$, let

$$L(u) = \int_{SC} F(x,u)\,d\mu. \tag{9}$$

We need verify the following result

$$L'(u)\,v = \int_{SC} f(x,u)\,vd\mu, \quad v \in H_0^1(SC). \tag{10}$$

By (4), for all $u, v \in H_0^1(SC)$ and $0 \leq \eta \leq 1$, we have

$$\begin{aligned}\left|L(u+v)-L(u)-\int_{SC} f(x,u)vd\mu\right| &= |\int_{SC}\left[\int_0^{u+v} f(x,t)dt-\int_0^{u} f(x,t)dt\right]d\mu \\ &\quad -\int_{SC} f(x,u)vd\mu| \\ &= \left|\int_{SC}\left[\int_u^{u+v} f(x,t)dt - f(x,t)v\right]d\mu\right| \\ &\leq \int_{SC}\left|\int_u^{u+v} dt - f(x,t)v\right|d\mu \\ &\leq \int_{SC}\left|\sup_{t\in[u,u+v]} f(x,t)\int_u^{u+v} dt - f(x,t)v\right|d\mu \\ &\leq \int_{SC}|f(x,u+\eta v)-f(x,t)||v|d\mu \\ &\leq 19C\|v\|_{\mathcal{E}}\int_{SC}|f(x,u+\eta v)-f(x,t)|d\mu.\end{aligned}$$

Since $f(x,t)$ is continuous on $SC\times\mathbb{R}$, it follows that

$$\begin{aligned}\lim_{\|v\|_{\mathcal{E}}\to 0}\frac{\left|L(u+v)-L(u)-\int_{SC} f(x,u)vd\mu\right|}{\|v\|_{e}} &\leq 19C\lim_{\|v\|_{\mathcal{E}}\to 0}\int_{SC}|f(x,u+\eta v)-f(x,u)|d\mu \\ &= 0.\end{aligned}$$

Therefore $L(u)$ is Fréchet differentiable and its derivative is given by (10) .
Since $f(x,t)\in C(SC\times\mathbb{R};\ \mathbb{R})$, the continuity of $L'(u)$ is easy to see using (4).
Similarly, it can easily be shown that the map $\frac{1}{2}\int_{SC} a(x)u^2d\mu \in C^1\left(H_0^1(SC),\mathbb{R}\right)$ provided that $\int_{SC}|a(x)|d\mu<\infty$, and its derivative in the sense of Fréchet is the second term on the right side of (8). It is clear that the application $\frac{1}{2}\|u\|_{\mathcal{E}}^2\in C^1\left(H_0^1(SC),\mathbb{R}\right)$ and its Fréchet derivative is given by $\mathcal{E}(u,v)$. The last remark follows from (6). □

The proposition that follows simplifies the proof of the condition PS.

Proposition 3.2. *Let I be given by (7). If the sequence $(u_n)_{n\geq 1}$ is bounded in $H_0^1(SC)$ and $I'(u_n)\xrightarrow[n\to\infty]{} 0$, then $(u_n)_{n\geq 1}$ is a Cauchy sequence in $H_0^1(SC)$ and has a convergent subsequence.*

Proof .
We have by (4) and the Ascoli-Arzela theorem; see [23]

$$H_0^1(SC)\hookrightarrow C_0(SC)$$

is compact.
So if $(u_n)_n$ is bounded in $H_0^1(SC)$ there exists a subsequence, which we denote by $(u_n)_n$ and a function $u\in C_0(SC)$ such that $u_n\to u$ in $C_0(SC)$.

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By (8) and (4), it follows that

$$\begin{aligned}|u_i-u_j| &= \sup_{\|v\|_{\mathcal{E}}\leq 1}|\mathcal{E}(u_i-u_j,v)| \\ &= \sup_{\|v\|_{\mathcal{E}}\leq 1}\left|I'(u_i-u_j)-\int_{SC}a(x)(u_i-u_j)vd\mu+\int_{SC}f(u_i-u_j)vd\mu\right| \\ &= \sup_{\|v\|_{\mathcal{E}}\leq 1}|\, I'(u_i)v-I'(u_j)v+\int_{SC}a(x)(u_i-u_j)vd\mu \\ &-\int_{SC}(f(x,u_t)-f(x,u_j))vd\mu\,| \\ &\leq \|I'(u_t)\|+\|I'(u_j)\| \\ &+19C\max_{x\in SC}|u_i(x)-u_j(x)|\int_{SC}|a(x)|d\mu \\ &+19C\int_{SC}|f(x,u_t)-f(x,u_j)|d\mu \\ &\underset{i,j\to+\infty}{\longrightarrow} 0.\end{aligned}$$

This holds since $I'(u_n)\to 0$ and from the Lebesgue's dominated convergence theorem.□

In order to use the proposition 2.2, we suppose that $f\in C(SC\times\mathbb{R})$ satisfies (7).

Theorem 3.1. *If f satisfies (f_*) and (f_{**}) and satisfied (a_*), then $(S_{f,a})$ has a non-trivial weak solution.*

Proof.
The proof use the classical approach, see [20].
Let $E=H_0^1(SC)$ and let I be defined by (7). According to the proposition 3.1, $I\in C^1(E,\mathbb{R})$; the weak solution of $(S_{f,a})$ will be obtained as a critical point of I, using the proposition 2.2. By (a_*) and (4), we have

$$\|u\|_\varepsilon\leq\|u\|_{\varepsilon,a}=\sqrt{\mathcal{E}(u,u)-\int_{SC}a(x)u^2d\mu}\leq\sqrt{2}\|u\|_\varepsilon,$$

then $\|u\|_{\varepsilon,a}$ is a norm equivalent to $\sqrt{\mathcal{E}(u)}=\|u\|_{\mathcal{E}}$ on $H_0^1(SC)$, a notation that we will adopte for the rest of the proof. So in (7)

$$\begin{aligned}I(u) &= \frac{1}{2}\mathcal{E}(u,u)-\frac{1}{2}\int_{SC}a(x)u^2d\mu+\int_{SC}F(x,u)d\mu \\ &= \frac{1}{2}\|u\|^2_{\varepsilon,a}+\int_{SC}F(x,u)d\mu.\end{aligned} \tag{11}$$

First, we verify that I satisfies (I_1). We take t_0 in (f_*), (9) gives, using (4),

$$|F(x,t)| \leq \int_0^{|u|} |f(x,t)|dt \leq \int_0^{|u|} \max_{\substack{x\in SC\\|t|\leq t_0}} |f(x,t)|dt$$
$$\leq \int_0^{|u|} \eta(t_0)\,dt$$
$$= |u|\eta(t_0)$$
$$\leq t_0\eta(t_0).$$

Hence

$$-t_0\eta(t_0) \leq F(x,t) \leq t_0\eta(t_0).$$

Therefore, for $u \in E$ with $\|u\|_{\varepsilon,a} = \rho = \dfrac{t_0}{19C}$.
Using (11) and $F(x,t) \geq -t_0\eta(t_0)$ and (f_*), we obtain

$$\begin{aligned} I(u) &\geq \frac{1}{2}\rho^2 - t_0\eta(t_0) \\ &\geq \frac{1}{2}\frac{t_0^2}{(19C)^2} - \frac{t_0^2}{2(\beta+1)(19C)^2} \\ &= \frac{\beta}{\beta+1}\cdot\frac{t_0^2}{2(19C)^2}, \end{aligned}$$

and so (I_1) is proved with $\rho = \dfrac{t_0}{(19C)^2}$ et $\alpha = \dfrac{\beta}{\beta+1}.\dfrac{{t_0}^2}{2(19C)^2}$.

To check (I_2) , we integrate the inequality (f_{**}), we have

$$\frac{f(x,t)}{F(x,t)} \leq \frac{\nu}{t} \Rightarrow \int_{SC}\frac{f(x,t)}{F(x,t)}dt \leq \int_{SC}\frac{\nu}{t}dt$$
$$\Rightarrow \text{Ln}\,|F(x,t)| + c \leq Ln|t|^\nu + c'$$

Moreover $F(x,t) \leq 0$, therefore there exists two positive constants c_1 and c_2 such as

$$F(x,t) \leq -c_1|t|^\nu + c_2 \ \ , for\ all\ \ x \in SC\ and\ \ t \in \mathbb{R}.$$

Then

$$L(u) = \int_{SC} F(x,t)\,d\mu \leq -c_1\int_{SC}|u|^\nu d\mu + c_2 \tag{12}$$

for all $u \in E$. Choosing $u \in E$ with $\|u\|_{\varepsilon,a} = 1$ and for $\sigma = \int_{SC}|u|^\nu d\mu > 0$,

$$\begin{aligned} I(su) &= \frac{1}{2}\|su\|^2_{\varepsilon,a} + L(su) \\ &= \frac{1}{2}s^2 + \int_{SC} F(x,su)\,d\mu \\ &\leq \frac{1}{2}s^2 - c_1\sigma s^\nu + c_2 \\ &\xrightarrow[s\to+\infty]{} -\infty. \end{aligned}$$

Since $\nu > 2$; so (I_2) is proved.

It remains to verify that I satisfies PS.
Suppose that $|I(u_n)| \leq b$ and $I'(u_n) \xrightarrow[n\to+\infty]{} 0$. We have for n sufficiently large, by using (11),

$$\begin{aligned} b+\nu^{-1}\|u_n\|_{\varepsilon,a} &\geq I(u_n)-\nu^{-1}I'(u_n)u_n \\ &= \left(\frac{1}{2}-\frac{1}{\nu}\right)\|u_n\|^2_{\varepsilon,a}+\nu^{-1}\int_{SC}(\nu F(x,u_n)-f(x,u_n)u_n)\,d\mu \\ &= \left(\frac{1}{2}-\frac{1}{\nu}\right)\|u_n\|^2_{\varepsilon,a} \\ &+\nu^{-1}\int_{\{x\in SC;\ |u_n|<r\}}(\nu F(x,u_n)-f(x,u_n)u_n)\,d\mu \\ &+\nu^{-1}\int_{\{x\in SC;\ |u_n|\geq r\}}(\nu F(x,u_n)-f(x,u_n)u_n)\,d\mu. \end{aligned} \tag{13}$$

The third term of (13) is non-negative because of (f_{**}), whereas the second term is bounded by a constant independent of n. Since $\frac{1}{2}-\frac{1}{v}>0$, (13) implies that $\{u_n\}$ is in $H^1_0(SC)$.

Therefore, by proposition 3.2, I satisfies PS.
Finally, since $I(0)=0$, then there is a critical point u which verify $I(u)\geq\alpha>0$ according to proposition 2.2. As a result, u is a nontrivial weak solution of $(S_{f,a})$. □

4 Conclusion

In this paper we establish the existance of non-trivial solutions using the mountain pass theorem on the Sierpiński carpet. Without doubt, techniques and results obtained in this study are applicable to a considerably broader class of fractal domains. The geometry of the Sierpiński carpet is very different from that of other fractal sets of type PCF, particularly around the boundaries, this made difficult the formulation of an explicit definition of Laplacian. However, Barlow and Kusuoka [4],[16] were able to construct a definition of Laplacian in two different ways.

Note that the work of Grigor'yan and Yang's [9], which is distinguished by its use of what is known as the purely analytic [12] construction of a local regular Dirichlet form on the SC (similar to that on PCF), was of great use for us,especially, for the proof of the Sobolev-type inequality (4). That one which allowed us to show others result easily.

One general conclusion is that calculus on the Sierpiński carpet is slightly regular than calculus on the Sierpiński gasket and other PCF fractals, but we are hopeful that by utilizing the Sobolev-type inequality (4), we will be able to examine many more elliptic equation and systems in futur works.

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