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Integral Solutions for the Diophantine Equation of Higher Degree with Six Unknowns $x^6 - y^6 - 3456z^3$=800($p^2 - q^2)R^8$

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Abstract:

Our focus in this paper has been on solving high-power Diophantine equations - those with variables raised to a high degree. These types of equations can be particularly challenging as they involve finding integer solutions that satisfy the given polynomial equation. We have utilized various techniques such as brute force methods and substitution strategies to solve these high power Diophantine equations, successfully deriving their solutions. Furthermore, our investigation has led us to uncover intriguing relationships among these solutions, which manifest in four distinct patterns. The equation x^6-y^6-3456z^3=800(p^2-q^2)R^8 is analysed with properties. Some of the special numbers are discussed in properties. Special numbers are unique and have special qualities that set them apart from other numbers. Learning about these special qualities helps us understand how numbers work and their significance in different areas of math and science.

Keywords: Diophantine Equations, Higher degree, Transformation, Real, Imaginary, Factorising, Properties.

Notations:

Tet_n	-Tetrahedral number of rank n
S_n	-Star number of rank n
J_n	- Jacobsthal Integer of rank n
So_n	- Stella Octangular Number of rank n
j_n	- Jacobsthal Lucas Integer of rank n
Co_n	- Centered octagonal number of rank n
$F_{4,n,3}$	-Four dimensional figurative number of rank n
Pt_n	-Pentatope number of rank n
OH_n	-Octahedral number of rank n
Obl_n	-Oblong number of rank n
SqP_n	-Square pyramidal number of rank n
Ky_n	-Keynea number of rank n

1. Introduction:

Number Theory, acknowledged as foundational knowledge, occupies a pivotal role in the realm of mathematics.

- It distinguishes itself uniquely and surpasses any doubts, establishing its position as the purest domain within the field of pure mathematics.

- The integration of modern technology has brought a new dimension to the application of Number Theory.
- This mathematical branch plays a crucial role in deciphering the complexities of numerical patterns and possesses the potential to astound unsuspecting individuals with its remarkable tricks.
- Focusing primarily on the properties of integers and rational numbers, Number Theory actively addresses real-world issues through various substantive applications. Its swift integration into fields such as coding theory, cryptology, and computer science technologies emphasizes its importance and versatility.
- Collectively, mathematicians have diligently crafted a strong and fundamental body of knowledge in Number Theory.

Exploring Number Theory yields the following rewards:

(i) It possesses a wealth of historical significance.

(ii) Special recognition is attributed to integers, particularly positive integers, as the foundational elements in constructing the real number system.

(iii) Furthermore, Number Theory distinguishes itself through its intrinsic beauty, offering both enjoyment and excitement.

(iv)It pushes the boundaries of unresolved mathematical problems, presenting challenges to mathematicians over the course of centuries.

(v) Recognized by the Greek mathematician Pythagoras as a distinct discipline, Number Theory owes much of its development to his followers, known as the Pythagoreans.

Diophantine Equations:

Diophantine equations, named after the ancient Greek mathematician Diophantus who extensively researched them, are algebraic equations specifically dealing with solutions that must be integers.

The general representation of a Diophantine equation takes the form:

$$a_1x_1^n + a_2x_2^n + a_3x_3^n + \cdots + a_mx_m^n = c$$

Here $a_1, a_2, a_3, \ldots, a_m, c, x_1, x_2, x_3, \ldots, x_m$ are integers and the variables $x_1, x_2, x_3, \ldots, x_m$ represent the unknowns to be determined.

Problem:

The Higher Degree Diophantine Equation is

$$x^6 - y^6 - 3456z^3 = 800(p^2 - q^2)R^8 \quad (1)$$

Execute the transformation

$$x=3s+2t, y=3s-2t, z=st, p=6t+6s, q=6t-6s \quad (2)$$

Equation (1) simplifies to

$$9s^2+4t^2 = 40R^4 \quad (3)$$

Method of investigation the equation

1. **Assume R =$9u^2+4v^2$** (4)

The number 40 can be written as

$$(6+2i)(6-2i) \quad (5)$$

Using the values of R and 40 in equation (3).Then

$$3s+i2t = (6+2i)(3u + i2v)^4 \quad (6)$$

Equating the real and imaginary parts in the above equation, we get

$s=162u^4+32v^4-432u^2v^2-144u^3v+64uv^3$

$t=81u^4 + 16v^4 - 216u^2v^2 +648u^3v-288uv^3$

The Integral solutions of (1) are

$x(u,v)=648u^4+128v^4-1728u^2v^2+864u^3v-384uv^3$

$y(u,v)=324u^4+64v^4-864u^2v^2-1728u^3v+768uv^3$

$z(u,v)=[162u^4+32v^4-432u^2v^2-144u^3v+64uv^3][81u^4 + 16v^4 - 216u^2v^2 +648u^3v-288uv^3]$

$p(u,v)= 1458u^4+288v^4 - 3888u^2v^2 +3024u^3v-1344uv^3$

$q(u,v)=-486u^4-96v^4+1296u^2v^2+4752u^3v-2112uv^3$

R(u,v)=9u^2+4v^2

Properties:

1. 3x(1,1)-y(1,1)=Tet_4
2. q(1,1)+7[p(1,1)]+j_1=S_5
3. R(4,5)+J_1 = So_5

2.The equation (3) can also be written as

$$9s^2+4t^2 = 40R^4 *1 \quad (7)$$

The number 1 can be written as

$$\frac{[12+9i][12-9i]}{225} \quad (8)$$

Using (4),(5) and (8) in (7), we get

$$3s+i2t = (6+2i)(3u+i2v)^4 \ \frac{(12+9i)}{15}$$

Equating the real and imaginary parts in the above equation, we get

$$3s = \frac{1}{15}[4374u^4+864v^4-11664u^2v^2-16848u^3v+7488uv^3]$$

$$2t = \frac{1}{15}[6318u^4+1248v^4-16848u^2v^2+11664u^3v-5184uv^3]$$

Replacing u=15U, v=15V in the above two equations , then we get

$$s=4920750U^4+972000V^4-13122000U^2V^2-18954000U^3V+8424000UV^3$$

$$t=10661625U^4+2106000V^4-28431000U^2V^2+19683000U^3V-8748000UV^3$$

Using the above two equations in (2), then the integral solutions of (1) are

$$x(U,V)=36085500U^4+7128000V^4-96228000U^2V^2-17496000U^3V+7776000UV^3$$

$$y(U,V)=-6561000U^4-1296000B^4+17496000U^2V^2-96228000U^3V+42768000UV^3$$

$$z(U,V)=[4920750U^4+972000V^4-13122000U^2V^2-18954000U^3V+8424000UV^3]$$
$$[10661625U^4+2106000V^4-28431000U^2V^2+19683000U^3V-8748000UV^3]$$

$$p(U,V)=93494250U^4+18468000V^4-249318000U^2V^2+4374000U^3V-1944000UV^3$$

$$q(U,V)=34445250U^4+6804000V^4-91854000U^2V^2+231822000U^3V-103032000UV^3$$

$$R(U,V)=2025U^2+900V^2$$

Properties:

1. $\frac{1}{100}$ {$R(76,90)-[y$(1,1)-x(1,1)]} =Co_{14}
2. {312 R(1,1) - [p(2,1)-7q(1,1)]+Pt_3} = $F_{4,12,3}$
3. R(1,2)-4OH_{12}=Pt_{11}
4. R(1,2)-2$F_{4,n,4}$-Obl_1=OH_{11}

3.The equation (3) can also be written as

$$9s^2+4t^2 = 40R^4 *1 \quad (9)$$

The number 1 can be written as

$$\frac{[3+4i][3-4i]}{25} \quad (10)$$

Using (4),(5) and (10) in (9), we have

$$3s+i2t = (6+2i)(3u+i2v)^4 \ \frac{(3+4i)}{5}$$

Equating the real and imaginary parts in the above equation, we get

$$3s = \frac{1}{5}[810u^4+160v^4-2160u^2v^2-6480u^3v+2880uv^3]$$

$$2t = \frac{1}{5}[2430u^4+480v^4-6480u^2v^2+2160u^3v-960uv^3]$$

Replacing u=15U,v=15V in the above two equations , then we get

$$s=2733750U^4+540000V^4-7290000U^2V^2-21870000U^3V+9720000UV^3$$

$$t=12301875U^4+2430000V^4-32805000U^2V^2+10935000U^3V-4860000UV^3$$

Using the above two equations in (2), then the integral solutions of (1) are

$$x(U,V)=32805000U^4+6480000V^4-87480000U^2V^2-43740000U^3V+19440000UV^3$$

y(U,V)=-16402500U^4-3240000B^4 $+$ 43740000U^2V^2-87480000U^3V +38880000UV^3
z(U,V)=[2733750U^4+540000V^4-7290000U^2V^2 $-$ 21870000U^3V+9720000UV^3] [12301875U^4+2430000V^4-32805000U^2V^2+10935000U^3V-4860000UV^3]
p(U,V)= 90213750U^4+17820000V^4-240570000U^2V^2-65610000U^3V $+$ 29160000UV^3
q(U,V)=57408750U^4+11340000V^4 $-$ 153090000U^2V^2 $+$ 196830000U^3V $-$ 87480000UV^3
R(U,V)=2025U^2+900V^2

Properties:

1. R(215,65) – [x(1,2)+y(1,2)]- 11M_9= Tet_2
2. 2077 R(1,1) – [7q(1,1)+p(1,1)] = Co_8
3. R(3,5)-37 Ky_5 = SqP_{11}

4.The equation (3) can also be written as

$9s^2+4t^2 = 36R^4+4R^4$

Factorising the above equation and we get

$4(t^2-R^4)=9(4R^4-s^2)$

The above equation can be written in the ratio form as

$\frac{4(t+R^2)}{(2R^2+s)} = \frac{9(2R^2-s)}{(t-R^2)} = \frac{u}{v}$, v≠0

It is equivalent to the form

-us+4vt+R^2[4v-2u]=0

-4vs-ut+R^2[18v+u]=0

By the method of cross multiplication, we have

s = -2u^2+72v^2+8uv
t = u^2-36v^2+36uv (11)
R^2=u^2+36v^2 (12)

Here $R^2(u,v)$ is of the form $z^2= Dx^2+y^2$ ($D>0$ *and square free*).
Then the solutions for the above equations are
u = 3$m^2$$-$$n^2$, v = 2mn, R = 3m^2+n^2
The above three values substitute in (11), then

s(m,n) = -18m^4-2n^4+300m^2n^2+48m^3n-16mn^3
t(m,n) = 9m^4+n^4-150m^2n^2+216m^3n-72mn^3

Using the values of s and t in (2), then the integral solutions of (1) are
x(m,n)= -36m^4-4n^4+600m^2n^2+576m^3n-192mn^3
y(m,n)= -72m^4-8n^4+1200m^2n^2 $-$ 288m^3n+96mn^3
z(m,n)= [-18m^4-2n^4+300m^2n^2+48m^3n-16mn^3][9m^4+n^4-150m^2n^2+216m^3n-72mn^3]
p(m,n)= -54m^4-6n^4+900m^2n^2+1584m^3n-528mn^3
q(m,n)= 162m^4+18n^4 $-$ 2700m^2n^2+1008m^3n-336mn^3
R(m,n)= 3m^2+n^2

Properties:

1. [x(1,1) – y(1,1)] is a transcendental perfect number.
2. p(1,1) –x(1,1)-y(1,1) is a nasty number.
3. y(2,2) – 2[x(2,1)] – p(1,1) – x(1,1)+R(6,4) = OH_4
4. R(2,3) = J_6

2. Conclusion:

This work aims to demonstrate the number of potential outcomes for Diophantine equations. Comprehending and resolving Diophantine equations equips individuals with valuable tools to address a wide array of challenges spanning various fields of study. Understanding Diophantine equations goes beyond just math; it gives us a glimpse into how smart we are and our endless curiosity about how things work in the universe. As we explore and learn more about these equations, let's take on the challenges they bring and celebrate the valuable insights they offer to those who try to solve them.

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