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On $\widetilde{sw}$-D Sets and Associated Separation Axioms in Topological Spaces

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Abstract:

In this article, we introduced the concept of semi-weakly-D ($\widetilde{sw}$-D) sets based on weakly-D open sets to obtain some semi-weakly-D separation axioms. From these new concepts of separation axioms, we obtained axioms of semi-weakly-D_0, semi-weakly-D_1, and semi-weakly-D_2 spaces. We also derived some characterizations.

Keywords: Topological space, D-set, semi-open set, weakly-D open set.

1. Introduction

A set $\tilde{O}$ in a topological space X is semi-open if and only if there exists an open set ω such that $\omega \subset \tilde{O} \subset \overline{\omega}$, which was first formulated by Norman Levine [14], and it generalized into semi continuity for a single-valued function. The α-open set notion was introduced by Njastad [15]. In this direction, numerous researchers expanded general open sets into pre-open sets [2], β-open sets [13], semi-preopen sets [5], supra-semi open sets [21] and other categories. Maheshwari and R. Prasad [20] established the idea of new separation axioms, which are the axioms semi-T_0, semi-T_1 and semi-T_2, who also examined the relationships among them. Semi-$T_{1/2}$ space was presented by Caldas [12], who also derived new separation axioms. Wali et al. [18] constructed semi-regular-weakly-open sets and studied irresolute and srw-continuous functions in topological spaces.

In this paper, we introduced more general form of open sets like semi-open sets, preopen sets, regular-open sets, supra-semi open sets etc. and verify their properties in topological spaces. In order to define a new type of set, open sets are very important. The difference set (D-sets) are one of the extensions of a general open set, which was first formulated by Tong [8] by using open sets and used these new sets to define and investigate a new type of separation axioms, particularly $D_e(e = 0, 1, 2)$ spaces. To enlarge the application and scope of D-sets theory, researchers have proposed many kinds of D-sets by replacing a new class of open sets. Many useful concepts have been proposed and generalized in the study of D-sets. For example, Caldas [10] proposed the concept of semi-D sets and he established some new separation axioms like semi-D_0, semi-D_1 and semi-D_2. Jafari [19] used the ideas of pre-open sets to develop pD-sets and establish some new separation axioms $pD_e(e = 0, 1, 2)$. Through the use of α-open sets, Caldas et al. [11] established a new kind of open sets termed αD-sets and new separation axioms referred to αD_e spaces, where $e = 0, 1, 2.$ In order to achieve decomposition of continuous functions, Ekici and Jafari [4] proposed the theories of D-sets, DS-sets, D-continuity, and DS-continuity. They also explored various characteristics of the classes of D-sets and DS-sets. Keskin and Noiri [1] defined the ideas of bD-sets using b-open sets, derived a few weak separation axioms, and provided various examples relevant to the digital line.

The concept Q^*D-open sets was established by Padma et al. [17], who also used this idea to establish some new separation axioms and highlight the relationship between new types of D-sets and existing D-sets in topological spaces. Mustafa et al. [7] determined $supra$-D sets using supra-open sets and also presented with novel separation axioms. Jardo [9] identified new separation axioms $sD_e(e = 0, 1, 2)$ by defining the iD-set based on the i-open sets. Afterward, by using this new concept several researchers worked in this direction to establish certain important results and obtain some new properties of topological spaces.

In this paper, we propose an alternative approach to general forms of weakly-D open sets [16], semi-D sets, and more generalized forms of open sets. The purpose of this study is to provide a novel kind of open set called $\widetilde{sw}$-D sets and to use this notion to propose several new separation axioms: semi-weakly-D_0, semi-weakly-D_1, and semi-weakly-D_2. Furthermore, we describe how $\widetilde{sw}$-D sets relate to more general D-sets. Finally, we draw a diagram that highlights the relationships among each D-set that has been considered in this study.

2. Preliminaries

In this section, we introduce some definitions of topological spaces that will be used in this paper to prove our main results.

Definition 2.1 [8] Let (Y, τ) be a topological space. A subset H of Y is D-set, if there are two open sets F_1 and F_2 such that $F_1 \neq Y$ and $H = F_1 - F_2$.

Definition 2.2 [6] A subset H of a topological space (Y, τ) is closed if its complement $Y - H$ is open.

Definition 2.3 [14] Let (Y, τ) be a topological space and $K \subset Y$. Then K is said to be semi-open if, $\exists$ an open set $H \in \tau$ such that $H \subset K \subset cl(H)$. The family of all semi-open sets is denoted by $S_H(Y, \tau)$.

Definition 2.4 [6] A point p is a limit point of a subset K of a topological space (Y, τ) if every neighbourhood of p contains a point of K other than p.

Definition 2.5 [16] Let (Y, τ) be a topological space and let K be a subset of Y. Then K is called weakly-D open set if $K - K'$ is τ-open where K' is the set of all limit points of Y.

Definition 2.6 [6] Let Y be any set and τ be the family given by $\tau = \{ \phi \} \cup \{H \subset Y : H^c$ is finite$\}$. Then τ is a topology on Y called the co-finite topology.

Definition 2.7 [20] Let (Y, τ) be a topological space. Then (Y, τ) is said to be semi-T_0, if for each pair y_1, y_2 of members of Y, $y_1 \neq y_2$, $\exists$ semi-open set H of Y containing one of the points but not the other.

Definition 2.8 [20] Let (Y, τ) be a topological space. Then (Y, τ) is said to be semi-T_1, if for each pair y_1, y_2 of members of Y, $y_1 \neq y_2$, $\exists$ semi-open sets H_1 and H_2 in Y such that $y_1 \in H_1$ but $y_2 \notin H_1$ and $y_1 \notin H_2$ but $y_2 \in H_2$.

Definition 2.9 [20] Let ((Y, τ) be a topological space, then (Y, τ) is said to be semi-T_2, if for each pair y_1, y_2 of members of Y, $y_1 \neq y_2$ $\exists$ disjoint semi-open sets H_1 and H_2 in Y such that $y_1 \in H_1$ and $y_2 \in H_2$.

Definition 2.10 [3] A topological space (Y, τ) is weakly-D_0 space, if for any pair p, q of members of Y, $p \neq q$, $\exists$ weakly-D open set D_w of Y containing one of the points but not the other.

Definition 2.11 [3] A topological space (Y, τ) is a weakly-D_1 space if for each pair p, q of members of Y, $p \neq q$, $\exists$ weakly-D open sets G_w and H_w in Y such that $p \in G_w$ but $q \notin G_w$ and $p \notin H_w$ but q $\in H_w$.

Definition 2.12 [3] A topological space (Y, τ) is a weakly-D_2 space if for each pair p, q of members of Y, $p \neq q$, $\exists$ two disjoint weakly-D open sets G_w and H_w in Y such that $p \in G_w$ and q $\in H_w$.

Definition 2.13 [6] Let (Y, τ) be a topological space, then (Y, τ) is said to be T_0-space, if for each pair p, q of members of Y, $p \neq q$, $\exists$ τ-open subset G of Y containing one of the points but not the other.

Definition 2.14 [6] Let (Y, τ) be a topological space. Then (Y, τ) is said to be T_1-space if for each pair p, q of members of Y, $p \neq q$ $\exists$ τ-open subsets G and H of Y such that $p \in G$ but $q \notin G$ and $p \notin H$ but $q \in H$.

Definition 2.15 [6] Let (Y, τ) be a topological space, then (Y, τ) is said to be T_2-space if for each pair p, q of members of Y, $p \neq q$ $\exists$ disjoint τ-open subsets G and H of Y such that $p \in G$ and $q \in H$.

Theorem 2.1 [16] Every D-set is weakly-D open.

Theorem 2.2 [16] Every open set is D-set and so is weakly-D open.

The converse of the above results need not be true.

3. $\widetilde{sw}$-*D* Sets and Associated Separation Axioms

Definition 3.1 A subset K of a topological space (Y, τ) is said to be $\widetilde{sw}$-D open sets if $K - K'$ is semi-open where K' is the set of all limit points of K.

Example 3.1 Let $K \subset Y$, $Y = \{p, q, r, s, t\}$, $\tau = \{\phi, Y, \{p\}, \{p, q\}, \{p, r, s\}, \{p, q, t\}, \{p, q, r, s\}\}$ and $K = \{p, r, s, t\}$. So $K' = \{q, r, s, t\}$ and $K - K' = \{p\}$ which is $\widetilde{sw}$-D open.

Corollary 3.1 Every D-set is $\widetilde{sw}$-D sets in a topological space (Y, τ).

Proof. Let U be a D-set generated by semi-open sets F_1 and F_2, where $F_1 \neq Y$ and $U = F_1 - F_2$. Let $a \in U'$. Then either $a \notin U$ or $a \in U$. If $a \in U$, F_1 must contain a point b of U such that $a \neq b$. Thus $b \in U'$ and therefore $U - U' = \phi$. If $a \notin U$ implies $a \notin F_1$ but $a \in F_2$, then

$\exists\ b \in U$ such that $b \in F_2$ which is a contradiction. Thus $U - U' = \phi$ is semi-open and by Theorem 2.1, it follows that (Y, τ) is $\widetilde{sw}$-D open sets.

Remark 3.1 Every open set is $\widetilde{sw}$-D sets.

Remark 3.2 Every weakly-D open set is $\widetilde{sw}$-D open but not conversely.

Example 3.2 $X = \{s, t, u, v, w\}$ with topology $\tau = \{\phi, X, \{s, t, u, v\}, \{w\}\}$, $A = \{w\}$ is $\widetilde{sw}$-D open but not D-set. So, it is not weakly-D open sets.

Proposition 3.1 Every $\widetilde{sw}$-D is semi-D set.

Proposition 3.2 Every semi-open set is $\widetilde{sw}$-D set.

Definition 3.2 A topological space (Y, τ) is semi-weakly-D_0 ($\widetilde{sw}$-D_0) space, if for any pair p, q of members of Y, $p \neq q$, $\exists$ $\widetilde{sw}$-D open set of Y containing one of the points but not the other.

Example 3.3 Assume that (Y, τ) is a topological space such that $Y = \{p, q\}$ and $\tau = \{\phi, \{p\}, \{q\}, \{p, q\}\}$, For every pair of distinct points of Y, (Y, τ) be a $\widetilde{sw}$-D_0-space.

Definition 3.3 A topological space (Y, τ) is semi-weakly-D_1 ($\widetilde{sw}$-D_1) space if for each pair p, q of members of Y, $p \neq q$ $\exists$ $\widetilde{sw}$-D open sets $\widetilde{sw}$-H_1 and $\widetilde{sw}$-H_2 in Y such that $p \in \widetilde{sw}$-H_1 but $q \notin \widetilde{sw}$-H_1 and $p \notin \widetilde{sw}$-H_2 but $q \in \widetilde{sw}$-H_2.

Definition 3.4 A topological space (Y, τ) is semi-weakly-D_2 ($\widetilde{sw}$-D_2) space if for each pair p, q of members of Y, $p \neq q$, $\exists$ two disjoint $\widetilde{sw}$-D open sets $\widetilde{sw}$-H_1 and $\widetilde{sw}$-H_2 in Y such that $p \in \widetilde{sw}$-H_1 and $q \in \widetilde{sw}$-H_2.

Theorem 3.1 (a) [3] A topological space (Y, τ) is semi-D_i then semi-D_{i-1}, $i = 1,2$.

(b) [10] A topological space (Y, τ) is weakly-D_i, then it is also weakly-D_{i-1}, $i = 1,2$.

(c) [10] A topological space (Y, τ) is T_i, then (Y, τ) is also weakly-D_i, $i = 0,1,2$.

Theorem 3.2 A topological space (Y, τ) is $\widetilde{sw}$-D_0 space, then (Y, τ) is semi-D_0 .

Proof. Let (Y, τ) is $\widetilde{sw}$-D_0 space. Then for each distinct pair $p, q \in Y$ such that $p \in \widetilde{sw}$-D but $q \notin \widetilde{sw}$-D. Let $H_1 \neq Y$ and $S = H_1 - H_2$ where H_1 and H_2 are semi-D-open sets, then $p \in H_1$ and $q \notin S$. Thus $q \notin H_1$. This implies that $p \in H_1$ but $q \notin H_1$. If p belongs to H_1, $q \notin S$, so q belongs to H_2. Clearly, Y is semi-D_0.

Corollary 3.2 Let (Y, τ) be a topological space. Then

1. If (Y, τ) is $\widetilde{sw}$-D_1 space, then (Y, τ) is semi-D_1.
2. If (Y, τ) is $\widetilde{sw}$-D_2 space, then (Y, τ) is semi-D_2.

Theorem 3.3 [3] Let (Y, τ) be a topological space. Then,

(i) (Y, τ) is semi-D_0 iff (Y, τ) is semi-T_1.

(ii) (Y, τ) is semi-D_1 iff (Y, τ) is semi-T_2.

Theorem 3.4 A topological space (Y, τ) is semi-T_ρ space, then (Y, τ) is $\widetilde{sw}$-D_ρ space respectively for $\rho = 0,1,2$.

Proof. Assume that (Y, τ) is semi-T_0. Then for each pair p, q of distinct points of Y, there exists a semi-open set D_s such that $p \in D_s$ but $q \notin D_s$. Since every semi-open set is semi-D set and so semi-weakly-D open [Proposition 3.2], there exists a semi-weakly-D open set $\widetilde{sw}$-D such that $p \in \widetilde{sw}$-D but $q \notin \widetilde{sw}$-D. Thus (Y, τ) is $\widetilde{sw}$-D_0 space. If (Y, τ) is semi-T_1 and semi-T_2 it follows from Remark 3.2, (Y, τ) is $\widetilde{sw}$-D_1 and $\widetilde{sw}$-D_2 respectively.

Theorem 3.5 If a topological space (Y, τ) is $\widetilde{sw}$-D_1, then (Y, τ) is $\widetilde{sw}$-D_0.

Proof. Assume that (Y, τ) is $\widetilde{sw}$-D_1. Then (Y, τ) is $\widetilde{sw}$-D_0.

It follows from Theorem 3.1 and Theorem 3.2.

The converse need not be true.

Example 3.4 Let (Y, τ) be a topological space, where $Y = \{p, q, r\}$ and $\tau = \{ \phi , Y, \{p\}, \{p, q\}\}$. Thus (Y, τ) is semi-weakly-D_0 [Theorem 3.4.] but not semi-weakly-D_1 because p and q can not be separated by disjoint semi-weakly-D open sets. Thus (Y, τ) is semi-weakly-D_0 but not semi-weakly-D_1.

Theorem 3.6 If a topological space (Y, τ) is $\widetilde{sw}$-D_2, then (Y, τ) is $\widetilde{sw}$-D_1.

Proof. It follows from Theorem 3.3. and Theorem 3.1.

The converse need not be true.

Example 3.5. Let (Y, τ) be a topological space,

where $Y = \{p, q, r\}$ and $\tau = \{ \phi, Y, \{p\}, \{q\}, \{r\}, \{p, q\}, \{q, r\}, \{p, r\}\}$ be the semi-T_1.

Thus (Y, τ) is $\widetilde{sw}$-D_1 [Theorem 3.2 et al.] but not $\widetilde{sw}$-D_2.

Definition 3.5 Assume that (Y, τ) is topological space. Then a subset H of Y is said to be semi-weakly-D closed set if $(H - H')^c$ is semi-weakly-D open, where H' is the set of all limit points.

Theorem 3.7 Let (Y, τ) be a topological space. Then (Y, τ) is $\widetilde{sw}$-D_1 space if and only if each singleton of Y is semi-weakly-D closed.

Proof. Assume that the topological space (Y, τ) is $\widetilde{sw}$-D_1 and for any $p \in Y$, suppose that $q \in Y^c$ such that $p \neq q$, and so there exists a semi-weakly-D open set $\widetilde{sw}$-D such that $q \in \widetilde{sw}$-D but $p \notin \widetilde{sw}$-D. Then $q \in \widetilde{sw}$-$D \subset \{p\}^c$, implies $\{p\}^c = \cup \{ \widetilde{sw} - D : q \in \{p\}^c\}$ which is $\widetilde{sw}$-D open and hence $\{p\}$ is weakly-D closed.

Conversely, assume that every singleton set $\{p\}$ of Y is semi-weakly-D closed. Let $s, t \in Y$ such that $s \neq t$. Then $\{t\}^c$ is semi-weakly-D open set such that $s \in \{t\}^c$ and $t \notin \{t\}^c$. Similarly, $\{s\}^c$ is semi-weakly-D open such that $t \in \{s\}^c$ and $s \notin \{s\}^c$. Thus Y is $\widetilde{sw}$-D_1.

Theorem 3.8 Assume that (Y, τ) is a topological space. Then (Y, τ) is $\widetilde{sw}$-D_1 space iff (Y, τ) consists of co-finite topology on Y.

Proof. Assume that (Y,τ) is a $\widetilde{sw}$-D_1 space. Then each singleton subset $\{p\}$ of Y is semi-weakly-D closed [Theorem 3.7.]. Thus $Y-\{p\}$ is semi-weakly-D open. Thus $(Y-\{p\})^c = \{p\}$ is finite for all $p \in Y$ and hence it contains co-finite topology on Y.

Conversely, suppose (Y,τ) contains co-finite topology on τ and let $\{p\}$ be any subset of Y. So, $Y-\{p\}$ is open in co-finite topology on Y. Consequently, $Y-\{p\}$ is τ -open. Therefore $Y-\{p\}$ weakly-D open and so semi-weakly-D open [Remark 3.2. et al.]. Therefore $Y-\{p\}$ is semi-weakly-D open so that $\{p\}$ is closed for all $p \in Y$. Hence there exists $q \in Y$ which is different from p such that q is not the limit point of $\{p\}$. Thus (Y,τ) is $\widetilde{sw}$-D_1 space.

Theorem 3.9 A topological space (Y,τ) is $\widetilde{sw}$-D_0 space, then (Y,τ) is weakly-D_0 space.

Proof. Assume that (Y,τ) is $\widetilde{sw}$-D_0 space. Then for each pair p, q of distinct points of Y, there exists a $\widetilde{sw}$-D open set F_{sw} such that $p \in F_{sw}$ but $q \notin F_{sw}$. Since every $\widetilde{sw}$-D open set is semi-D-set and so is weakly-D open [Proposition 3.1. & Theorem 2.1. et al.], there exists a weakly-D open set F_w such that $p \in F_w$ but $q \notin F_w$. Thus (Y,τ) is weakly-D_0 space.

Theorem 3.10 A topological space (Y, τ) is $\widetilde{sw}$-D_ρ space, then (Y, τ) is weakly-D_ρ space respectively, ρ = 1, 2.

Proof. It follows from proposition 3.1. & Theorem 2.1 et al.

4. Conclusions

We show the concept of "$\widetilde{sw}$-D open sets", which is the main idea of this article. It shows that the concept of a family of $\widetilde{sw}$-D open sets extend the structure of weakly-D open sets. This notion has been constructed by applying "semi-open sets". With this idea of a new class of $\widetilde{sw}$-D open sets, we study the separation axioms of $\widetilde{sw}$-D open sets in topological spaces and investigate the new results, we explore the relationship between $\widetilde{sw}$-D_i, semi-D_i, weakly-D_i and T_i, where $i = 0, 1, 2$. We further discussed some initial properties of semi-weakly-D separation axioms. From the above discussion and results we have the following implications as shown in the diagram:

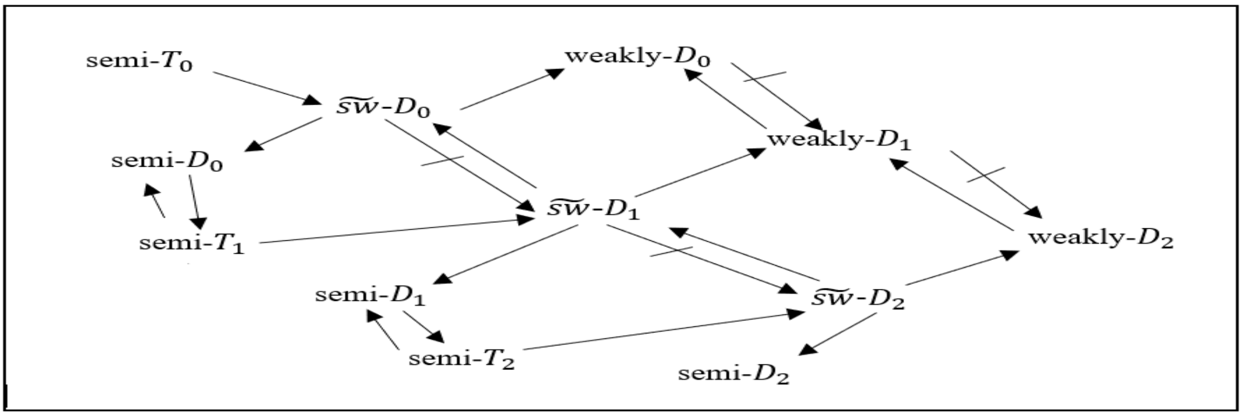

Figure 1: Implications of a family of $\widetilde{sw}$-D open sets extend the various structure of semi-open sets.

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