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Compute a Finite Presentation for the Automorphism Group of Free Group and Free Abelian Group Using GAP System

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Abstract:

I have developed an algorithm and written a GAP functions pautfreegroup:=function(n) (Finite Presentation of automorphism Groups of Free group of Rank n), and autfreeabeliangroup:=function(n) (Finite Presentation of automorphism Groups of Free abelian group of Rank n). Functions using GAP system for computation of a finite presentation for Aut(F_n) and GL(n,z) respectively. In order to do that we have given a description for the presentation of Aut(F_n) and GL(n,z), the automorphism groups of free groups and free abelian groups respectively.

Keywords: Free group, integral general linear group GL(n,z), automorphism of groups, presentation of group, GAP system.

1. Introduction

As we know a very famous theorem in the theory of free groups states that every group G is a homomorphic image of some free group. This means that for every group G, exists a surjective homomorphism $\sigma: F \to G$ of a of free group F onto G. This homomorphism π is called a *presentation* of the group$G$. Let $R \coloneqq \ker(\sigma)$ be the kernel of $\sigma.$ Then R is a normal subgroup of F and $F/R \cong G.$ The elements in R are called the *relators* of the presentation. Now suppose that Y is a set of free generators for F and $S \subseteq R$ with the property that the normal closure of S equals R. Then $X: = \sigma(Y)$ is obviously a set of generators of the group $G.$ Let F_n be the free group on $n \geq 2$ elements and $Aut(F_n)$ its group of automorphisms. Nielsen in [17] and [18] has proved that $Aut(F_n)$ is a finitely presented group. A well-known representation of $Aut(F_n)$ is given by

$$\xi_1: Aut(Fn) \to Aut(F_n \,/\, F_n^{'}) \cong GL(n, \mathbb{z}),$$

Where $F_n^{'}$ is the commutator subgroup of F_n and $\xi_1(\varphi)$ is the automorphism of the abelian group $F_n \,/\, F_n^{'}$ induced by $\varphi \in Aut(F_n)$. Siegmund in [19] has constructed a presentation for $GL(n, \mathbb{z})$ the automorphism groups of free abelian group.In Section 2, we present the definitions of free group, automorphism of groups and presentation of group with some facts and examples. In Section 3, we give a description for the presentation of $Aut(F_n)$, the automorphism group of the free group of rank n following McCool and Newman work see [13] and [16]. We give also a description for the presentation of $GL(n, \mathbb{Z})$, the automorphism group of the free abelian group $\mathbb{Z}^n$ following Siegmund's work see [19]. A presentation for $SL(n, \mathbb{Z})$, is given in Milnor's work see [14]. In Section 4, develop an algorithm and written GAP functions that provides a finite presentation for the automorphism group $Aut(Fn)$ of the free group of rank n, and a finite presentation for the automorphism group $GL(n, \mathbb{Z})$ of the free abelian group of rank n.

2. Background

In this section we will give the definitions of Free group, presentation of group, automorphism of groups and their properties with some examples, further details may be found in [1], [3], [6], [7], [8], [9], [10], [11], [12] and [20].

Definition 2.1 Assuming that A is a set with a related non-negative (n) which is consider as an integer number. The sequence of a form $a_1, \dots, a_n$, where $a_i \in A$ is called a *word* with specific length n. The *empty word* can be obtained when $n = 0$. Let $= a_1, \dots, a_m$ and $\boldsymbol{b} = b_1, \dots, b_n$ might construct a new word which will be $\boldsymbol{ab} = a_1, \dots, a_m, b_1, \dots, b_n$. The new word will be called as the concatenation of $\boldsymbol{a}$ and $\boldsymbol{b}$. In fact, the concatenation is a binary operation on the words which is belong to a specific set.

Let X be a set with define a set $X^{-1} = \{x^{-1} : x \in X\}$, disjoint from X. A word over $X \cup X^{-1}$ is *reduced* if it contains no subsequence x, x^{-1} or x^{-1}, x, where $x \in X$. For notational convenience we write words over $X \cup X^{-1}$ in the form ${x_1}^{\epsilon_1} \cdots {x_n}^{\epsilon_n}$, where $x_i \in X, \epsilon_i = \pm 1$ and x^{+1} denotes the element x of X.

Example 2.2 Let $A = \{a, c, s, t\}$.

1. c, a, t and s, a, t are words; so is $stattc\ cat$ and sat have concatenation $cat\ sat$.
2. $A^{-1} = \{a^{-1}, c^{-1}, s^{-1}, t^{-1}\}$ then aca^{-1} is reduced while $saa^{-1}t$ is not, neither is ss^{-1} or $ass^{-1}t$.
3. $s^{\epsilon_1} a^{\epsilon_2} t^{\epsilon_3} = s^{-1}at^{-1}$ if $\epsilon_1 = \epsilon_3 = -1$, $\epsilon_2 = +1$.

Theorem 2.3 Let X be a set, let G be a group and let $f: X \to G$ be a map. Then there exists a unique homomorphism $\emptyset: f(X) \to G$ extending f: that is, such that $\emptyset(x) = f(x), for\ all\ x \in X$.

Theorem 2.4 Let X_1 and X_2 be sets. Then $F(X_1) \cong F(X_2)$ if and only if $|X_1| = |X_2|$.

Definition 2.5 Let X be a set and let G be a group isomorphic to $F(X)$. Then we say that G is a *free group* of rank |X|

Definition 2.6 Let G be a group and R a subset of G. The *normal closure* $N(R)$ of R is the intersection of all normal subgroup of G which contact R.

Lemma 2.7

1. $N(R)$ is the smallest normal subgroup of G containing R and it is generated by the set of element $g^{-1}rg$, where $r \in R$ and $g \in G$.

Example 2.8 Let $G = S_3$ and $R = \{(1,2,3)\}$.

Conjugates of (123): $(123)\ and\ (123) \Rightarrow N(R) = \langle R^{\wedge} \rangle = \langle (123), (132) \rangle$

$$= \{ e, (1,2,3), (1,3,2)\}.$$

Definition 2.9 Let X be a set and let R be a subset of $F(X)$. The quotient group G equal to $F(X)/ N(R)$, is said have *presentation* $\langle X|R \rangle$.

Example 2.10

1. $G = \langle x, y \mid yx = xy^2,\ xy = yx^2 \rangle$.
   i.e. $\langle x, y \mid yxy^{-2}x^1,\ xyx^{-2}y^{-1} \rangle$,

   In G: $yx = xy^2 = (xy)y = (yx^2)y = (yx)(xy)$

   $\Rightarrow yx = yx(xy) \Rightarrow 1 = xy \Rightarrow y = x^{-1}$. $(*)$

   Hence, $yx = xy^2 \overset{(*)}{\Rightarrow} 1 = x(x^{-2}) = x^{-1} \Rightarrow x = 1 \overset{(*)}{\Rightarrow} y = 1$.

   Thus, $G = \{1\}$.

Theorem 2.11 [12] Every group has a presentation.

Definition 2.12 The isomorphism of a group G to itself is called an *automorphism*.

Example 2.13 If G be an abelian group and $f: G \to G$ be such that $f(x) = x^{-1}$.

Then as $f(xy) = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1} = f(x)f(y)$, so f is a homomorphism. Now, $f(x) = f(y)$ implies that $x^{-1} = y^{-1}$ implies that $x = y$.

So, f is one to one. f is clearly onto. Thus, f is an automorphism.

Let G be a group and let $Aut(G)$ denote the set of all automorphisms of G. Then $Aut(G)$ is a group under the composition of functions, for the proof you can see [7], or [11]. It is called *automorphism of groups*.

3. Presentations for $Aut(F_n)$ and $GL(n, \mathbb{Z})$

The GAP system utilized to find finite presentations for the automorphism groups of $Aut(F_n)$ and $GL(n, \mathbb{Z})$. In order to do this work a description for the presentation of $Aut(F_n)$ will be given, the automorphism groups of the free group of rank n following McCool and Newman work see [13] and [16]. We give also a description for the presentation of $GL(n, \mathbb{Z})$, as given in [19]. A presentation for $SL(n, \mathbb{Z})$, is given in Milnor's work see [14].

3.1 Presentation of $Aut(F_n)$

We give a description for $Aut(F_n)$ by description of the generating sets of $Aut(F_n)$ according to McCool work in [13] and Newman work in [16].
If $[x_1, x_2, \cdots, x_n]$ are the free generators of the group $aut(F_n)$ then the automorphim will maps the free generators $[x_1, x_2, \cdots, x_n]$ of group to:

O : $[{x_1}^{-1}, x_2, \cdots, x_n]$, $(n \geq 1)$
P : $[x_2, x_1, x_3, \cdots, x_n]$, $(n \geq 2)$
U : $[x_1 x_2, x_2, x_3, \cdots, x_n]$, $(n \geq 2)$
S : $[{x_2}^{-1}, {x_3}^{-1}, \cdots, {x_n}^{-1}, {x_1}^{-1}]$, $(n \geq 1)$
T : $[x_2, {x_1}^{-1}, x_3, \cdots, x_n]$, $(n \geq 2)$
Q : $[x_2, x_3, \cdots, x_n, x_1]$, $(n \geq 2)$
R : $[{x_2}^{-1}, x_1, x_3, x_4, \cdots, x_{n-2}, x_n {x_{n-1}}^{-1}, {x_{n-1}}^{-1}]$, $(n \geq 2)$

3.2 Presentation of $GL(n, \mathbb{Z})$

First, we will give some matrices in $SL(n, \mathbb{Z})$ and $GL(n, \mathbb{Z})$ for $n \geq 2$ and $1 \leq i, j \geq n$

Which are as follows:

- Suppose that $E_{ij} \in SL(n, \mathbb{Z})$ is $(n \times n) - identity\, matrix$ with an additional 1 in the $(i,j) - th\, position$ $(i \neq j)$,

$$E_{ij} \coloneqq \begin{bmatrix} 1 & & & & & & & & & \\ & \ddots & & | & & & | & & & \\ & & 1 & & & & & & & \\ & - & & 1 & & - & 1 & & - & \\ & & & & 1 & & & & & \\ & & & | & & \ddots & | & & & \\ & & & & & & 1 & & & \\ & - & & 0 & & - & 1 & & - & \\ & & & & & & & 1 & & \\ & & & | & & & | & & \ddots & \\ & & & & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow i - th\, row \\ \\ \\ \\ \leftarrow j - th\, row \\ \\ \\ \\ \end{matrix}$$

- Suppose that $E_{ij}(a) \in SL(n, \mathbb{Z})$ is $(n \times n) - identity\, matrix$ with an additional $a \in \mathbb{Z}$ in the $(i,j) - th\, position$ $(i \neq j)$,

$$E_{ij}(a) := \begin{bmatrix} 1 & & & & & & & & & & \\ & \ddots & & \vert & & & & \vert & & & \\ & & 1 & & & & & & & & \\ & - & & 1 & & - & & a & & - & \\ & & & & 1 & & & & & & \\ & & & \vert & & \ddots & & \vert & & & \\ & & & & & & 1 & & & & \\ & - & & 0 & & - & & 1 & & - & \\ & & & & & & & & 1 & & \\ & & & \vert & & & & \vert & & \ddots & \\ & & & & & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow i-th\,row \\ \\ \\ \\ \leftarrow j-th\,row \\ \\ \\ \\ \end{matrix}$$

- Suppose that $P_{ij} \in SL(n,\mathbb{Z})$ is the following permutation matrix $(i \neq j)$,

$$P_{ij} := \begin{bmatrix} 1 & & & & & & & & & & \\ & \ddots & & \vert & & & & \vert & & & \\ & & 1 & & & & & & & & \\ & - & & 0 & & - & & 1 & & - & \\ & & & & 1 & & & & & & \\ & & & \vert & & \ddots & & \vert & & & \\ & & & & & & 1 & & & & \\ & - & & 1 & & - & & 0 & & - & \\ & & & & & & & & 1 & & \\ & & & \vert & & & & \vert & & \ddots & \\ & & & & & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow i-th\,row \\ \\ \\ \\ \leftarrow j-th\,row \\ \\ \\ \\ \end{matrix}$$

- Suppose that $O_i \in GL(n,\mathbb{Z})$ be the following diagonal matrix with -1 in the $(i,i)-th$,

$$O_i := \begin{bmatrix} 1 & & & & & & \\ & \ddots & & \vert & & & \\ & & 1 & & & & \\ & - & & (-1) & & - & \\ & & & & 1 & & \\ & & & \vert & & \ddots & \\ & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow i-th\,row \\ \\ \\ \\ \end{matrix}$$

- Suppose that $O_{1i} \in SL(n,\mathbb{Z})$ be the following diagonal matrix with -1 in the $(1,1)-th$ $and\ in\ the\ (i,i)-th\,position$ $(i \neq j)$,

$$O_{1i} := \begin{bmatrix} -1 & & & & \vert & & & \\ & 1 & & & & & & \\ & & \ddots & & \vert & & & \\ & & & 1 & & & & \\ - & & - & & (-1) & & - & \\ & & & & & 1 & & \\ & & & & \vert & & \ddots & \\ & & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \leftarrow i-th\,row \\ \\ \\ \\ \end{matrix}$$

Now, suppose that $E_{ij} \in SL(n,\mathbb{Z})$ is $(n \times n)-identity\,matrix$ with an additional 1 in the $(i,j)-th\,position$ $(i \neq j)$ and $O_i := diag(1, \dots, 1, -1, 1, \dots, 1)$ be the diagonal matrix with a -1 at the $(i,i)-th$ position.

Proposition 3.2.1 (Presentation for $SL(n,\mathbb{Z})$) according to Milnor's work in [14].

(i) $SL(n,\mathbb{Z}) = 1$.
(ii) $SL(n,\mathbb{Z})$ $has\ a\ finite\ presentation\ with\ the\ two\ generators$ E_{12} and E_{21} $subject\ to\ the\ following\ relations$

$$E_{12}\, E_{21}{}^{-1}\, E_{12}\, E_{21}\, E_{12}{}^{-1} E_{21} = 1,$$

$$(E_{12}\, E_{21}{}^{-1}\, E_{12})^4 = 1.$$

(iii) $For\ n \geq 3,\ \ the\ group\ \ SL(n,\mathbb{Z}))\ \ has\ a\ finite\ presentation\ with\ \ n(n-1)\ \ generators$ $E_{ij}\ (i \neq j)\ subject\ to\ the\ following\ relations$

$$[E_{ij}, E_{kl}] = 1\ if\ j \neq k, i \neq l,$$

$$[E_{ij}, E_{jk}] = E_{ik}\ if\ i, j, k\ are\ pairwise\ distinct, \qquad (E_{12}\ {E_{21}}^{-1}\ E_{12})^4 = 1.$$

Proof: (i) is clear and (iii) can be found in [14]. The proof of (ii) can be found in [19].

The target now is to find a finite presentation of $GL(n,\mathbb{Z})$ in terms of the matrices E_{ij} and O_i. In order to do that, we use the following exact sequence (see [5], [15] and [19])

$$1 \to SL(n,\mathbb{Z}) \xrightarrow{det} \{-1,1\} \to 1.$$

More general, let G be an extension of H by N, say

$$1 \to N \overset{i}{\to} G \overset{\pi}{\to} H \to 1.$$

Assume further that N has the following finite presentation

$$N = \langle n_1, \dots, ; n_r \mid R_1\ (n_1, \dots, n_r), \dots, R_k(n_1, \dots, n_r)\rangle$$

And that H has the finite presentation

$H = \langle h_1, \dots, ; h_r \mid W_1\ (h_1, \dots, h_r), \dots, W_k(h_1, \dots, h_r)\rangle.$

We wish to find a finite presentation of G.

Since π is surjective, there are $g_1, \dots, g_s \in G$ with $\pi(g_i) = h_i$ for $1 \leq i \leq s$. By identifying N with the kernel of π and G, it is easy to see that G is generated by $g_1, \dots, g_s$ and $n_1, \dots, n_r$. Thus, we have found generators for G. We start collecting relations in terms of $g_1, \dots, g_s$ and $n_1, \dots, n_r$:

- The relations $R_1(n_1, \dots, n_r), \dots, R_k(n_1, \dots, n_r)$ in N are, of course, also relations in G.
- Let $W_i(g_1, \dots, g_s)$ be the word obtained from $W_i(h_1, \dots, h_s)$ by replacing each h_j by g_j. We have
  $\pi\big(W_i(g_1, \dots, g_s)\big) = W_i\big(\pi(g_1), \dots, \pi(g_s)\big) = W_i(h_1, \dots, h_s).$
  Hence $W_i(g_1, \dots, g_s) \in \ker(\pi), i.e.\ W_i(g_1, \dots, g_s) \in N$. This means that we can write $W_i(g_1, \dots, g_s)$ as a product of the n_i, say
  $W_i(g_1, \dots, g_s) = \widetilde{W_i}(n_1, \dots, n_r).$
  This gives us more relations in G.
- Finally, we mention that, since N is a normal subgroup in G. Each conjugate $g_j n_j g_i^{-1}$ and $g_i^{-1}\ n_j\ g_j$ is in N. Thus, we get relations
  $g_j n_j g_i^{-1} = V_{ij}(n_1, \dots, n_r) \qquad g_i^{-1} g_j n_j = U_{ij}(n_1, \dots, n_r).$

The next proposition tells us that the above relations are sufficient for a presentation of G.

Proposition 3.2.2 (P. Hall) see [11]. Let G be an extension of H by N

$$1 \to N \overset{i}{\to} G \overset{\pi}{\to} H \to 1.$$

If N has the finite presentation

$$N = \langle n_1, \dots, n_r \mid R_1\ (n_1, \dots, n_r), \dots, R_k(n_1, \dots, n_r)\rangle$$

And H has the finite presentation

$$H = \langle h_1, \dots, h_s \mid W_1\ (h_1, \dots, h_s), \dots, W_k(h_1, \dots, h_s)\rangle.$$

Then G has the following finite presentation

$$G = \langle n_1, \dots, n_r,\ g_1, \dots, g_s \mid R_1\ (n_1, \dots, n_r), \dots, R_k(n_1, \dots, n_r),$$

$$g_j n_j g_i^{-1} = V_{ij}(n_1, \dots, n_r)$$

$$g_i^{-1} g_j n_j = U_{ij}(n_1, \dots, n_r),$$

$$W_i(g_1, \dots, g_s) = \widetilde{W_i}(n_1, \dots, n_r)\rangle.$$

Where $V_{ij}, (n_1, \dots, n_r), U_{ij}, (n_1, \dots, n_r)$ and $\widetilde{W_i}(n_1, \dots, n_r)$ are as above.

By using Proposition 3.2.2 and the Presentation of $SL(n, \mathbb{Z})$ given in Proposition 3.2.1 can compute a finite presentation for $GL(n, \mathbb{Z})$. To make this define $O_1 \coloneqq diag\ (-1, 1, \dots, 1)$ to be the diagonal matrix with an entry -1 at the first position.

Now we will give a presentation for $GL(n, \mathbb{Z})$ according to Siegmund's work in [19] as follows: for more information about $GL(n, \mathbb{Z})$ see [4], [5] and [15].

Proposition 3.2.3 (Presentation of $GL(n, \mathbb{Z})$).

a) $GL(1, \mathbb{Z}) = \langle O_1 \mid O_{1=1}^2 \rangle = C_2$.
b) $GL(2, \mathbb{Z})$ has a finite presentation with the three generators E_{12}, E_{21} and O_1 subject to the following relations

$$E_{12} E_{21}^{-1} E_{12} E_{21} E_{12}^{-1} E_{21} = 1,$$
$$(E_{12} E_{21}^{-1} E_{12})^4 = 1,$$
$$(O_1 E_{12})^2 = 1,$$
$$(O_1 E_{21})^2 = 1,$$
$$O_1^2 = 1.$$

c) For $n \geq 3$, the group $GL(n, \mathbb{Z})$ has a finite presentation with $n(n-1) = 1$ generators E_{ij} and O_1 subject to the following relations

$$[E_{ij} E_{kl}] = 1\ if\ j \neq k, i \neq l$$
$$[E_{ij} E_{jk}] = E_{ik}\ if i,\ j, k\ are\ pairwise\ distinct,$$
$$(E_{12} E_{21}^{-1} E_{12})^4 = 1,$$
$$O_1 E_{ij} O_1 E_{ij}^{-1} = 1\ if\ i, j, j \neq 1,$$
$$(O_1 E_{12})^2 = 1\ if\ j\ \neq 1,$$
$$(O_1 E_{12})^2 = 1\ if\ i \neq 1,$$
$$O_1^2 = 1.$$

Proof: The proof can be found in [19].

4. An algorithm for $\boldsymbol{Aut(F_n)}$ and $\boldsymbol{GL(n, \mathbb{Z})}$ using GAP system

We have developed and written an algorithm using GAP system to produce finite presentation for $Aut(F_n)$ the automorphism groups of the free group of rank n, which are described in Section 3.1. Also, we have developed an algorithm using GAP to compute a finite presentation for $GL(n, \mathbb{Z})$ the automorphism groups of the free abelian group of rank n, according to the presentation in Proposition 3.2.3. For more information about GAP presentations see [1] and [2].

4.1 Function $\boldsymbol{pautfreegroup} := \boldsymbol{function(n)}$

We have developed $pautfreegroup := function(n)$ (**Finite Presentation of automorphism Groups of Free group of Rank $\boldsymbol{n}$**). A function using GAP system for computation of a finite presentation for $Aut(F_n)$. The input of the function $pautfreegroup := function(n)$ is an integer n that provides a finite presentation for $Aut(F_n)$ the automorphism groups of the free group of rank n according to the presentation in Section 3.1. The output of this function is consists of three sets FF, $gens$ and $rels$, where FF is fp group on the generators of the free group of rank n, $gens$ is the list of the generators of the $Aut(F_n)$ and $rels$ is the list of relators subject to the generators $gens$. Appropriate GAP package is "gap> LoadPackage("fga");"

```
pautfreegroup:=function(n)
local F,FF,A,iso,gens,rels;
F:=FreeGroup(n);
A:=AutomorphismGroup(F);
```

```
iso:=IsomorphismFpGroup(A);
FF:=Range(iso);
gens:=GeneratorsOfGroup(FF);
rels:=RelatorsOfFpGroup(FF);
return[FF,gens,rels];
end;
```

For example:

```
gap> pautfreegroup(2);
[ <fp group on the generators[O, P, U ]>, [O, P, U ], [P^2, O^2, (O*P)^4, (U^-1*O^-1)^2*(U*O)^2,
(P*O*P*U)^2, (U*P*O)^3]]
```

##

```
gap> pautfreegroup(3);
[ <fp group on the generators [S, T, U ]>,[S, T, U ],
[ (S^5*T^-1)^2,T^-1*S*T^2*S^8*T^-1*S*T^2*S^-4,(S^3*(S*T^-1)^2)^2, T^4,U^-1*S^-1*(S^-1*T)^2*S^-
2*U*S*(S*T^-1)^2*S^2, (U^-1*T*S^-1*T*S^2)^2*(U*S^-2*T^-1*S*T^-1)^2,U^-1*S^-2*T^-1*S*T*S*T^-
1*U^-1*T*S^-1*T^-1*S^-1*T*S^2*U*S^-2*T^-1*S*T*S*T^-1*U*T*S^-1*T^-1*S^-1*T*S^2,(U^-1*T^-
2*S^-1*T)^2*(U*T^-1*S*
T^2)^2, S^-2*T^-1*S*T^2*S^2*U*S^2*T^-1*S*T^2*S^2*U*S^-2*U*
S^2*U^-1*S^-2*U^-1, (S^-2*T^-1*S*T*U)^2,(U*T)^3 ]]
```

4.2 Function $\boldsymbol{autfreeabeliangroup := function(n)}$

We have developed $autfreeabeliangroup := function(n)$ (**Finite Presentation of automorphism Groups of Free abelian group of Rank $\boldsymbol{n}$**). The input of the function $autfreeabeliangroup := function(n)$ that provides a finite presentation for $GL(n,\mathbb{Z})$ the automorpism groups of free abelian group, is an integer n. The output of $autfreeabeliangroup := function(n)$ consists of three sets Aut, $gens$ and $rels$ where, Aut is the automorphism group $GL(n,\mathbb{Z})$, $gens$ is the list of the generators of $GL(n,\mathbb{Z})$ and $rels$ is the list of the relators subject to the generators $gens$. In addition the output of this function consists of the values of the generators E_{ij} and O_1. Appropriate GAP package is “polycyclic package”.

```
autfreeabeliangroup:=function(n)
local
i,j,S,T,Aut,gens,rels,G,newthings,O1,invEij,k,l,Ekl,invEkl,R1,R2,R3,R4,R5,R6,R7,W,N,M1,M2,M3,M4,M5,M
6,M7;
G:=AbelianPcpGroup(n);
Aut:=GL(n,Integers);
gens:=[];
rels:=[];
S:=[];
T:=[];
O1:=IdentityMat(n);
O1[1][1]:=-1;
Print(" ","\n");
Print("O1=");Print(O1);
W:=(Concatenation(["O",String(1)]));
Print(" ","\n");
Add(gens,W);
Print(" ","\n");
if n=1 then
  return([Aut, ["O1"],["O1*O1"]] );
fi;
for i in [1..n] do
   S[i]:=[]; T[i]:=[];
   for j in [1..n] do
   S[i][j]:=[];T[i][j]:=[];
     if i<>j then
        S[i][j]:=IdentityMat(n);
```

```
          S[i][j][i][j]:=1;
          T[i][j]:= Inverse(S[i][j]);
                N:=Concatenation(["E",String(i),String(j)]);
                Add(gens,N);
          Print(" ","\n");
          Print(N);Print("=");Print(S[i][j]);Print(",");
          Print(N);Print("^-1");Print("=");Print(T[i][j]);
          Print(",");
Print(" ","\n");
fi;
od;
od;
Print(" ","\n");
if n=2 then
   return([Aut,   ["O1","E12","E21"],[   "E12*E21^-1*E12*E21*E12^-1*E21",   "E12*E21^-1*E12*E12*E21^-
1*E12*E12*E21^-1*E12*E12*E21^-1*E12", "O1*E12*O1*E12", "O1*E21*O1*E21", "O1*O1"]]);
fi;

for i in [1..n] do
    for j in [1..n]do
        for k in [1..n] do
            for l in [1..n]do

                if i<>j and i<>l and j<>k and l<>k then

                  Print("                                                                                          ","\n");
M1:=Concatenation(["E",String(i),String(j),"*","E",String(k),String(l),"*","E",String(i),String(j),"^-
1","*","E",String(k),String(l),"^-1"]);
                  Add(rels,M1);
                fi;
            od;
         od;
     od;
od;

for i in [1..n] do
    for j in [1..n]do
        for k in [1..n] do
            if i<>j and j<>k and i<>k then
            Print("                                                                                          ","\n");
M2:=Concatenation(["E",String(i),String(j),"*","E",String(j),String(k),"*","E",String(i),String(j),"^-
1","*","E",String(j),String(k),"^-1","*","E",String(i),String(k),"^-1"]);
            Add(rels,M2);

            fi;
         od;
     od;
od;

for i in [1..n] do
    for j in [1..n]do

        if i<>j and i<>1 and j<>1 then

        Print(" ","\n");

M3:=(Concatenation(["O",String(1),"*","E",String(i),String(j),"*","O",String(1),"*","E",String(i),String(j),"^-
1"]));
        Add(rels,M3);
        fi;
```

```
    od;
od;

for j in [2..n] do

        Print(" ","\n");

M4:=(Concatenation(["O",String(1),"*","E",String(1),String(j),"*","O",String(1),"*","E",String(1),String(j)]));
        Add(rels,M4);
od;

for i in [2..n] do
        Print(" ","\n");

M5:=(Concatenation(["O",String(1),"*","E",String(i),String(1),"*","O",String(1),"*","E",String(i),String(1)]));
        Add(rels,M5);
od;

        Print(" ","\n");
        M6:=(Concatenation(["O",String(1),"*","O",String(1)]));
        Add(rels,M6);

        Print(" ","\n");
        M7:=Concatenation(["E",String(1),String(2),"*","E",String(2),String(1),"^-
1","*","E",String(1),String(2),"*","E",String(1),String(2),"*","E",String(2),String(1),"^-
1","*","E",String(1),String(2),"*","E",String(1),String(2),"*","E",String(2),String(1),"^-
1","*","E",String(1),String(2),"*","E",String(1),String(2),"*","E",String(2),String(1),"^-
1","*","E",String(1),String(2)]);
        Add(rels,M7);

Print(" ","\n");

return[Aut,gens,rels];

end;
```

For example:

```
gap> autfreeabeliangroup(1);
O1=[ [ -1 ] ]
[ GL(1,Integers), [ "O1" ], [ "O1*O1" ] ]

#################################################################

gap> autfreeabeliangroup(2);
O1=[ [ -1, 0 ], [ 0, 1 ] ]
E12=[ [ 1, 1 ], [ 0, 1 ] ],E12^-1=[ [ 1, -1 ], [ 0, 1 ] ],
E21=[ [ 1, 0 ], [ 1, 1 ] ],E21^-1=[ [ 1, 0 ], [ -1, 1 ] ],

[ GL(2,Integers), ["O1", "E12", "E21"],["E12*E21^-1*E12*E21*
E12^-1*E21", "E12*E21^-1*E12*E12*E21^1*E12*E12*E21^-1*E12*
E12*E21^-1*E12", "O1*E12*O1*E12", "O1*E21*O1*E21","O1*O1"] ]

#################################################################

gap> autfreeabeliangroup(3);

O1=[ [ -1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ]
E12=[ [ 1, 1, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ],E12^-1=[ [ 1, -1, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ],E13=[ [ 1, 0, 1 ], [ 0, 1, 0 ], [
0, 0, 1 ] ],E13^-1=[ [ 1, 0, -1 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ],E21=[ [ 1, 0, 0 ], [ 1, 1, 0 ], [ 0, 0, 1 ] ],E21^-1=[ [ 1, 0, 0 ],
[ -1, 1, 0 ], [ 0, 0, 1 ] ],E23=[ [ 1, 0, 0 ], [ 0, 1, 1 ], [ 0, 0, 1 ] ],E23^-1=[ [ 1, 0, 0 ], [ 0, 1, -1 ], [ 0, 0, 1 ] ],E31=[ [
```

1, 0, 0], [0, 1, 0], [1, 0, 1]],E31^-1=[[1, 0, 0], [0, 1, 0], [-1, 0, 1]],E32=[[1, 0, 0], [0, 1, 0], [0, 1, 1]],E32^-1=[[1, 0, 0], [0, 1, 0], [0, -1, 1]],

[GL(3,Integers), ["O1", "E12", "E13", "E21", "E23", "E31", "E32"],["E12*E12*E12^-1*E12^-1", "E12*E13*E12^-1*E13^-1", "E12*E32*E12^-1*E32^-1", "E13*E12*E13^-1*E12^-1", "E13*E13* E13^-1*E13^-1","E13*E23*E13^-1*E23^-1", "E21*E21*E21^-1*E21^-1", "E21*E23*E21^-1*E23^-1", "E21*E31*E21^-1*E31^-1", "E23* E13*E23^-1*E13^-1","E23*E21*E23^-1*E21^-1", "E23*E23*E23^-1*E23^-1", "E31*E21*E31^-1*E21^-1", "E31*E31*E31^-1*E31^-1", "E31*E32*E31^-1*E32^-1","E32*E12*E32^-1*E12^-1","E32*E31*E32^-1*E31^-1", "E32*E32*E32^-1*E32^-1", "E12*E23*E12^-1*E23^-1*E13^-1","E13*E32*E13^-1*E32^-1*E12^-1", "E21*E13*E21^-1* E13^-1*E23^-1", "E23*E31*E23^-1*E31^-1*E21^-1", "E31*E12* E31^-1*E12^-1*E32^-1","E32*E21*E32^-1*E21^-1*E31^-1", "O1*E23*O1*E23^-1", "O1*E32*O1*E32^-1", "O1*E12*O1*E12", "O1* E13*O1*E13", "O1*E21*O1*E21","O1*E31*O1*E31", "O1*O1", "E12* E21^-1*E12*E12*E21^-1*E12*E12*E21^-1*E12*E12*E21^-1*E12"]]

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