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Results for a Novel Family of Vector Hemivariational-Like Inequalities in Hausdorff Topological Linear Spaces via ζ-η-Monotone Operators

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Abstract:

This article's goal is to present a novel family of vector hemivariational-like inequalities in Hausdorff topological linear spaces (HTLS) using a formulation of the ζ-η -monotone operator. We get existence results for this novel class of vector hemivariational-like inequalities by applying the FKKM method and making appropriate assumptions about the nonlinear mappings that are being investigated. Enhancing both theoretical and practical comprehension of certain forms of inequalities, the solutions given in this work contribute to the development and enhancement of several corresponding solutions supplied by other writers.

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Introduction

Vector hemivariational-like inequalities encompass non-coercive and multivalued relations from mechanics, optimization, and mathematical physics. Vector hemivariational inequalities enable more problems with non-unique solutions than traditional variational inequalities, which need monotonicity and single-valued mappings. Recent advances in hemivariational inequality theory have demonstrated the close relationship between optimization, non-smooth analysis, and variational formulations. These links aid in the resolution of complex problems in nonlinear elasticity, contact mechanics, and other engineering subjects requiring the examination of many methodologies or states. Therefore, vector hemivariational inequalities enrich mathematics while also assisting scientists and engineers in understanding complex systems with several criteria and interactions (see [1-10]).

With the help of an h-generalized convex map, Ahmad and Khan [11] showed weak linear variational-like inequalities and provided new solutions of existence. Furthermore, in response to an unanswered question put forward by Hou and Chen [12]. Fang and Huang [13] introduced novel existence results for solutions pertaining to a category of robust linear variational inequalities in Banach spaces. Xie and Gong in 2019 presented results based on the problems of variational variance of fuzzy mappings and their applications to the theory of Zadeh's decomposition and formally described the exact sets that represent the cut sets of fuzzy mappings directly [14]. Khisbag and al. et., developed a novel definition of invexity, particularly $T{-}\zeta{-}\mu{-}$invex. As part of their research, they also looked into some ideas about variational inequality and complementarity problems in the context of topologically ordered vector spaces [15]. In 2024, Noor proposed and evaluated several iterative techniques for higher-order hemivariational inequalities using the auxiliary principal technique [16].

This work aims to prove the existence of a novel family of vector inequalities akin to hemivariational ones. This study aims to improve these findings. In three sections, the paper stays focused. Along with the introduction, we examine the concepts and key theorems needed to grasp the second section's major discoveries. We use FKKM–Mapping to analyze many fresh discoveries in the third segment. This paper examines compact and noncompact convex set assumptions. Several authors say this study strengthens and generalizes previous findings (see [17-20]).

1. Preliminaries

Assume that E and M are two separate sets that are not part of the HTLS. Additionally, assume that $P \subset E$ is a closed, convex, pointed cone by its tip located at the origin. A nonempty K is a convex and closed set of M. The set E is an ordered HTLS, which is represented by the notation (E, P), as you may recall. As shown in E, the following is an illustration of an example of an ordering connection with the following:

$$\forall\, \sigma, v \in E, \sigma \leq v \text{ iff } v - \sigma \in P;$$

When int P is non-zero, the weak ordering relations in E are more often called

$$\forall\, \sigma, \tau \in E, \sigma \nless \tau \text{ iff } \tau - \sigma \notin \text{int } P.$$

Considering that $T\colon M \to \Pi(M, E)$ and that $\Pi(M, E)$ is the space of all continuous vector maps from M to E, we may discuss the value of $\imath \in \Pi(M, E)$ on $\sigma \in M$ using $(\imath, \sigma)$.

In this work, we will refer to a set of E cones that are closed, convex, and pointed as $P(\sigma)\colon \sigma \in K$: Suppose that $\Theta\colon K \times K \to M$ and $g\colon K \times K \to E$ be functions, where int $P(\sigma) \neq \emptyset$ for all $\sigma \in K$.

Here, we mention that a function $J\colon B \to \mathbb{R}$ is said Locally Lipschitz if for each $u \in B$ then, there exists U is a neighborhood of u and a constant $L_u \in (0, \infty)$in which

$$|J(w) - J(\tau)| \leq L_u \parallel w - \tau \parallel_B, \text{ for all } \tau, w \in U.$$

Definition 1.1. [21] We will assume that $J\colon B \to \mathbb{R}$ is a locally Lipschitz function. In the direction of J at $u \in B$, the generalized derivative of $\tau \in B$ is represented by $J^0(u;\tau)$.

$$J^0(u;\tau) = \lim_{\substack{w \to u \\ \alpha \downarrow 0}} sup \frac{J(w+\alpha\tau) - J(w)}{\alpha}.$$

Lemma 1.2. [21] Assume that $J\colon B \to \mathbb{R}$ is locally Lipschitz function of rank L_u near the point $u \in B$. Thus,

i. $\tau \to J^0(u;\tau)$ is a subadditive function that is finite, positively homogenous, and satisfies that
$$|J^0(u;\tau)| \leq L_u \|\tau\|_B.$$
ii. $J^0(u;\tau)$,is upper as a function of (u, τ).
iii. $J^0(u;-\tau), = (-J)^0(u;\tau)$.

Lemma 2.1 [22]. Assuming a nonemty set K is a convex, and closed set of an H.T.S, and $\Lambda: K \to 2^M$ a multi-valued mapping, if every finite set $\{u_1, u_2, \dots, u_n\} \subset K$, one have $conv\,\{u_1, u_2, \dots, u_n\} \subset \bigcup_{i=1}^{n} \Lambda(u_i)$.

Specifically, it indicates that Λ is a KKM-mapping, and $\Lambda(u)$ is closed for all u in M, while it is compact for a few u in M. The term " $conv$ " refers to the convex hull operator. Therefore, the value of , $\bigcap_{u\in M} \Lambda(u) \neq \emptyset$.

Theorem 2.2[23]. Let K be a nonempty set within an HTLS M. Let the set-valued mapping $\theta : K \to 2^M$ be defined as a KKM map, where

i. $\theta(\varrho)$ is closed set, for any $\varrho \in K$.
ii. $\theta(\varrho)$ is a compact set for at least one $\varrho \in K$.

Therefore, $\cap \{\theta(\varrho) : \varrho \in K\} \neq \emptyset$.

In order to demonstrate our findings, we will make use of the following notations for research findings.

Remark 2.3[24] . For cleanliness, we use these terms:

i. $\tau \notin -intP$ iff $\tau \geq_P 0$.
ii. $\tau \in -intP$ iff $\tau <_P 0$.
iii. $\tau - w \notin -intP$ iff$\tau - w \geq_P 0, i.e(\tau \geq_P w)$.
iv. $\tau \notin -intP$ and $w \notin -intP$ imply $\tau + w \notin -intP$.

Definition 2.4 [25] Suppose that $T: M \to L(M, Y)$, $\zeta: M \times M \to Y$ and $\Theta: K \times K \to K$ are three mappings in which $P = \bigcap_{u\in K} P(u)$. Then, T is said to be $\zeta - \Theta$ −monotone in P iff

$$< T(\tau) - T(u), \Theta(\tau, u) > +\zeta(\tau, u) \in P \text{ for all } u, \tau \in K.$$

Definition 2.5 [21] Consider the two functions $T: K \to \Pi\,(M, E)$ and $\Theta: K \times K \to K$ We assert that T is Θ-hemi continuous if the mapping $t \longmapsto \langle T(\tau + t(u - \tau)), \Theta(\tau, u)\rangle$ is continuous at 0^+. for any given all $u, \tau \in K$, and $t \in (0,1)$, which is denoted by Θ −H.C.

We will examine the following the category of linear variational inequalities that is called Generalized vector hemivariational-like inequalities (for brief, GVHLI): For all $\tau \in K$, find $\sigma \in K$, such that

$$< T(\sigma), \Theta(\tau, \sigma) > +J^0(\sigma, \tau - \sigma) + g(\tau, \sigma) \notin -\operatorname{int} P(\sigma)\ . \qquad (2.1)$$

We assume that the following hypotheses have been satisfied in addition to the resolution of problem (2.1):

H_1: $\dfrac{\zeta(t\tau + (1 - t)\sigma, \sigma)}{t} = 0$;

H_2: $T: K \to \Pi\,(M, E)$ is an Θ −H.C mapping;

H_3: $P = \bigcap_{\sigma\in K} P(\sigma) \neq \emptyset$ and *the operator* T is $\zeta - \Theta$ −monotone in P.

2. Main Results

Utilizing the be $\zeta - \Theta -$monotone operator, this section suggests a novel set of vector hemivariational-like inequalities in Hausdorff topological linear spaces (HTLS). The existence of this new class of vector hemivariational-like inequalities is proven by using the FKKM method under suitable nonlinear mapping assumptions. By building on the work of other authors, the answers presented in this book deepen readers' understanding of inequality both theoretically and in practice.

Theorem 3.1. Assume M to be (HTLS) and let K be a closed, nonempty, convex subset of M. For every σ in K let $(Y, P(u))$ be an OTLS with $\mathrm{int}P(\mu) \neq \emptyset$. Let $\zeta: M \times M \to E$ be a bifunction that fulfills the conditions $(H_1 - H_3)$. Considering that $\Theta: K \times K \to M$, $g: K \times K \to M$. Thus, the issues listed below are all interchangeable.

(A) Considering $\sigma \in K$, $\forall\, \tau \in K$ in which

$$\langle T(\sigma), \Theta(\tau, \sigma)\rangle + J^0(\sigma, \tau - \sigma) + g(\tau, \sigma) \notin -\mathrm{int}\ P(\sigma).$$

(B) Considering $\sigma \in K$, $\forall\, \tau \in K$ in which

$$\langle T(\sigma), \Theta(\tau, \sigma)\rangle + \zeta(\tau, \sigma) + J^0(\sigma, \tau - \sigma) + g(\tau, \sigma) \notin -\mathrm{int}\, P(\sigma).$$

Proof. Assuming that (A) is correct, One can have $\sigma \in K$,

$$\langle T(\sigma), \Theta(\tau, \sigma)\rangle + J^0(\sigma, \tau - \sigma) + g(\tau, \sigma) \notin -\mathrm{int}\, P(\sigma), \forall\, \tau \in K$$

One can obtain the result by virtue of the fact that T is $\zeta - \Theta$-monotone.

$$\langle T(\tau) - T(\sigma), \Theta(\tau, \sigma)\rangle + \zeta(\tau, \sigma) \in P \quad \forall\, \sigma, \tau \in K.$$

Indeed, it is accurate.

$$\langle T(\sigma), \Theta(\tau, \sigma)\rangle - \langle T(\tau), \Theta(\tau, \sigma)\rangle - \zeta(\tau, \sigma) \in -P\, , \forall\, \tau \in K.$$

From, $P = \bigcap_{\sigma \in K} P(\sigma)$, then $\forall\, \tau \in K$

$$[\langle T(\sigma), \Theta(\tau, \sigma)\rangle + J^0(\sigma, \tau - \sigma) + g(\tau, \sigma)] - [\langle T(\tau), \Theta(\tau, \sigma)\rangle + J^0(\sigma, \tau - \sigma) + \zeta(\tau, \sigma) + g(\tau, \sigma)] \in -P \subset -P(\sigma)\, .$$

After applying Remark (2.3), then $\forall\, \tau \in K$,

$$\langle T(\tau) + \Theta(\tau, \sigma)\rangle + J^0(\sigma, \tau - \sigma) + g(\tau, \sigma) + \zeta(\tau, \sigma) \notin -\mathrm{int}\ P(\sigma).$$

Due to the fact that this is the case, σ is a solution to formula (B).

Conversely, should (B) prove to be valid, then exists. $\sigma_0 \in K$, $\forall\, \tau \in K$. Then

$$\langle T(\tau), \Theta(\tau, \sigma_0)\rangle + J^0(\sigma_0, \tau - \sigma_0) + \zeta(\tau, \sigma_0) + g(\tau, \sigma_0) \notin -\mathrm{int}\, P(\sigma_0),$$

for all $\tau \in K$, $t \in (0,1)$, let $\tau_t = t\tau + (1-t)\sigma_o$. Surely, it seems evident that $\tau_t \in K$, then

$$\langle T(\tau_t), \Theta(\tau_t, \sigma_0)\rangle + J^0(\sigma_0, \tau_t - \sigma_0) + \zeta(\tau_t, \sigma_0) + g(\tau_t, \sigma_0) \notin -\mathrm{int}\, P(\sigma_0).$$

In consideration of the fact that g and Θ are affine and H_1, one can obtain

$$\langle T(t\tau + (1-t)\sigma_0), t\Theta(\tau, \sigma_0)\rangle + J^0(\sigma_0, t\tau + (1-t)\sigma_o - \sigma_0)$$

$$+\zeta(t\tau+(1-t)\sigma_0,\sigma_0)+tg(\tau,\sigma_0)$$

$$=t\langle T(t\tau+(1-t)\sigma_0),\Theta(\tau,\sigma_0)\rangle+tJ^0(\sigma_0,\tau-\sigma_0)+\zeta(t\tau+(1-t)\sigma_0,\sigma_0)+tg(\tau,\sigma_0)\notin -\text{int}\,P(\sigma_0).$$

$$=\langle T(t\tau+(1-t)\sigma_0),\Theta(\tau,\sigma_0)\rangle+J^0(\sigma_0,\tau-\sigma_0)+\frac{\zeta(t\tau+(1-t)\sigma_0,\sigma_0)}{t}+g(\tau,\sigma_0)\notin -\text{int}\,P(\sigma_0).$$

In the context of the Θ −H.C of T, H_1, and the assumption that $t\to 0^+$, we obtain the following:

$$\langle T(\sigma_0),\Theta(\tau,\sigma_0)\rangle+J^0(\sigma_0,\tau-\sigma_0)+g(\tau,\sigma_0)\notin -\text{int}\,P(\sigma_0),\forall\,\tau\in K.$$

The proof is completed.

Let us consider K as a convex closed subset of the topological vector space M. Furthermore, we define the set $\{P(\sigma):\sigma\in K\}$ as a collection of convex closed and pointed cones within a topological space E, ensuring that the $\text{int}\,P(\sigma)\neq\emptyset$ for any $\sigma\in K$. In the course of this assignment, we will employ a set-valued map.

$$\overline{P}:K\to 2^E \text{ as shown: } \overline{P}(\sigma)=E\setminus\{-\text{int}\,P(\sigma)\}, \text{for each } \sigma\in K\}.$$

Theorem 3.2. Take that M is an HTLS, and a nonempty subset K is compact, closed, and convex in M. Then, and $(E,P(\sigma))$ is an OTLS, as $\text{int}\,P(\sigma)\neq\emptyset$ for any $\sigma\in K$. Let both $\Theta:K\times K\to M$ and $g:K\times K\to M$ be affine bifunctions such that $\Theta(\sigma,\sigma)=0=g(\sigma,\sigma)$, for any $\sigma\in K$ and $\zeta:M\times M\to E$ be continuous bifunctions and the conditions (H_1-H_3) are satisfied. İf, $\overline{P}:K\to 2^E$ is an upper semi-continuous. Then, there is $\sigma_0\in K$, where

$$<T(\sigma_0),\Theta(\tau,\sigma_0)>+J^0(\sigma,\tau-\sigma)+g(\tau,\sigma_0)\notin -\text{int}\,P(\sigma_0)\quad\forall\,\tau\in K.$$

Proof. For each $\tau\in K$, we define

$$\Lambda_1(\tau)=\{\sigma\in K: <T(\sigma),\Theta(\tau,\sigma)>+J^0(\sigma,\tau-\sigma)+g(\tau,\sigma)\notin -\text{int}\,P(\sigma)\};$$

$$\Lambda_2(\tau)=\{\sigma\in K: <T(\tau),\Theta(\tau,\sigma)>+J^0(\sigma,\tau-\sigma)+\zeta(\tau,\sigma)+g(\tau,\sigma)\notin -\text{int}\,P(\sigma)\}.$$

Assuming this, Λ_1 and Λ_2 are not empty, as $\in\Lambda_1$ and $\tau\in\Lambda_2$, respectively. Our proof has been divided into three distinct sections for your convenience.

Step one. First, we will present Λ_1 is a KKM mapping. İf Λ_1 is not a KKM , so there is $\nu_1,\nu_2,\dots,\nu_n\in K$ and $\Omega_1\geq 0$, $\Omega_2\geq 0,\dots,\Omega_m\geq 0$ together with $\sum_{i=1}^m\Omega_i=1$ and $\mu=\sum_{i=1}^m\Omega_i\,\nu_i$ where $\mu\notin\bigcup_{i=1}^m\Lambda_1(\nu_i)$. It means that, for all $i=1,2,\dots,m$

$$<T(\mu),\Theta(\nu_i,\mu)>+J^0(\mu,\nu_i-\mu)+g(\nu_i,\mu)\in -\text{int}\,P(\mu),$$

Because Θ and g are affine and $\mathscr{H}$ is semicontinuous, we get for every $\mu\in K$.

$$<T(\mu),\Theta(\mu,\mu)>+J^0(\mu,\mu-\mu)+g(\mu,\mu)$$

$$=<T(\mu),\Theta(\textstyle\sum_{i=1}^m\Omega_i\,\nu_i,\mu)>+J^0(\mu,\sum_{i=1}^m\Omega_i\,\nu_i-\mu)+g(\sum_{i=1}^m\Omega_i\,\nu_i,\mu)$$

$$=\sum_{i=1}^m\Omega_i\,[<T(\mu),\Theta(\nu_i,\mu)>+J^0(\mu,\nu_i-\mu)+g(\nu_i,\mu)]\in -\text{int}\,P(\mu).$$

Considered from the opposite direction, we see that $\Theta(\mu,\mu)=g(\mu,\mu)=0$, we get

$$0 =< \mathrm{T}(\mu), \Theta(\mu\,, \mu) > +J^0(\mu, \mu - \mu) + \mathrm{g}(\mu, \mu) \in -\mathrm{int}\,\mathrm{P}(\mu),$$

Because it is impossible to do so the mapping $\Lambda_1 : \mathrm{K} \to 2^{\mathrm{K}}$ is a KMM.

In the second step, we assert that

$$\bigcap_{\tau \in \mathrm{K}} \Lambda_1(\tau) = \bigcap_{\tau \in \mathrm{K}} \Lambda_2(\tau).$$

Actually, if $\sigma \in \Lambda_1(\tau)$ then

$< \mathrm{T}(\sigma), \Theta(\tau, \sigma) > +J^0(\sigma, \tau - \sigma) + \mathrm{g}(\tau, \sigma) \notin -\mathrm{int}\,\mathrm{P}(\sigma)$, from H_3 we get

$$< \mathrm{T}(\tau) - \mathrm{T}(\sigma), \Theta(\tau, \sigma) > +\zeta(\tau, \sigma) \in \mathrm{P},$$

So,

$$[< \mathrm{T}(\sigma), \Theta(\tau, \sigma) > +J^0(\sigma, \tau - \sigma) + \mathrm{g}(\tau, \sigma)]$$
$$-[< \mathrm{T}(\tau), \Theta(\tau, \sigma) > -J^0(\sigma, \tau - \sigma) + \zeta(\tau, \sigma) + \mathrm{g}(\tau, \sigma)] \in -\mathrm{P} \subset -\mathrm{P}(\sigma).$$

The usage of Remark (2.3) makes this feasible.

$$< \mathrm{T}(\tau), \Theta(\tau, \sigma) > +J^0(\sigma, \tau - \sigma) + \zeta(\tau, \sigma) + \mathrm{g}(\tau, \sigma) \notin -\mathrm{int}\ \mathrm{P}(\sigma)$$

At the same time, we get $\sigma \in \Lambda_2(\tau)$ for each $\tau \in \mathrm{K}$. That means,

$$\Lambda_1(\tau) \subset \Lambda_2(\tau).$$

Thus,

$$\bigcap_{\tau \in \mathrm{K}} \Lambda_1(\tau) \subset \bigcap_{\tau \in \mathrm{K}} \Lambda_2(\tau).$$

In contrast, it is reasonable to infer that $\sigma \in \bigcap_{\tau \in \mathrm{K}} \Lambda_2(\tau)$. Therefore,

$$< \mathrm{T}(\tau), \Theta(\tau, \sigma) > +J^0(\sigma, \tau - \sigma) + \zeta(\tau, \sigma) + \mathrm{g}(\tau, \sigma) \notin -\mathrm{int}\,\mathrm{P}(\sigma)\ , \forall\, \tau \in \mathrm{K}$$

The process of the following from Theorem 3.1:

$$< \mathrm{T}(\sigma), \Theta(\tau, \sigma) > +J^0(\sigma, \tau - \sigma) + \mathrm{g}(\tau, \sigma) \notin -\mathrm{int}\,\mathrm{P}(\sigma)\ , \forall\, \tau \in \mathrm{K}$$

So, $\sigma \in \bigcap_{\tau \in \mathrm{K}} \Lambda_1(\tau)$ and $\bigcap_{\tau \in \mathrm{K}} \Lambda_2(\tau) \subset \bigcap_{\tau \in \mathrm{K}} \Lambda_1(\tau)$, It shows that

$$\bigcap_{\tau \in \mathrm{K}} \Lambda_1(\tau) = \bigcap_{\tau \in \mathrm{K}} \Lambda_2(\tau).$$

The third step is of our assertion that

$$\bigcap_{\tau \in \mathrm{K}} \Lambda_2(\tau) \neq \emptyset$$

In light of the fact that Λ_1 is a KKM mapping, we are cognizant of the fact that, for every finite set $\{\tau_1, \tau_2, \dots, \tau_n\} \in \mathrm{K}$, It is imperative that one possess

$$\mathrm{conv}\{\tau_1, \tau_2, \dots, \tau_n\} \subset \bigcup_{i=1}^{n} \Lambda_1(\tau_i) \subset \bigcup_{i=1}^{n} \Lambda_2(\tau_i).$$

Here, the fact that Λ_2 is a KKM map is obvious. We now claim to show $\Lambda_2(\tau)$ is closed for each $\tau \in$ K.

Consider that there exists a net $\{\sigma_\alpha\} \subset \Lambda_2(\tau)$ by $\sigma_\alpha \to \sigma \in K$. So,

$$< T(\tau), \Theta(\tau, \sigma_\alpha) > + J^0(\sigma_\alpha, \tau - \sigma_\alpha) + \zeta(\tau, \sigma_\alpha) + g(\tau, \sigma_\alpha) \notin -\mathrm{int}\, P(\sigma_\alpha).$$

By employing the definition of $\overline{P}$, one can obtain

$$< T(\tau), \Theta(\tau, \sigma_\alpha) > + J^0(\sigma_\alpha, \tau - \sigma_\alpha) + \zeta(\tau, \sigma_\alpha) + g(\tau, \sigma_\alpha) \notin \overline{P}(\sigma_\alpha).$$

According to the continuous property applicable to both Θ and g, it can be concluded that

$< T(\tau), \Theta(\tau, \sigma_\alpha) > + J^0(\sigma_\alpha, \tau - \sigma_\alpha) + \zeta(\tau, \sigma_\alpha) + g(\tau, \sigma_\alpha)$ is approach to

$$< T(\tau), \Theta(\tau, \sigma) > + J^0(\sigma, \tau - \sigma) + \zeta(\tau, \sigma) + g(\tau, \sigma).$$

Due to the fact that $\overline{P}$ is a lsc with values that are in close proximity. The utilization of Lemma 2.1 enables us to ascertain that P is closed; consequently,

$$< T(\tau), \Theta(\tau, \sigma) > + J^0(\sigma, \tau - \sigma) + \zeta(\tau, \sigma) + g(\tau, \sigma) \in \overline{P}(\sigma).$$

This demonstrates that

$$< T(\tau), \Theta(\tau, \sigma) > + J^0(\sigma, \tau - \sigma) + \zeta(\tau, \sigma) + g(\tau, \sigma) \notin \mathrm{int}\, P(\sigma),$$

Therefore, $\Lambda_2(\tau)$ is closed. As a result of the compactness of K and the closeness of $\Lambda_2(\tau) \subset K$. One can observe that $\Lambda_2(\tau)$ is compact. By employing Theorem 2.2, it is feasible to acquire

$$\bigcap_{\tau \in K} \Lambda_2(\tau) \neq \emptyset.$$

As a result, it implies that

$$\bigcap_{\tau \in K} \Lambda_1(\tau) \neq \emptyset.$$

This implies, there is $\sigma_0 \in K$ in which

$$< T(\sigma_0), \Theta(\tau, \sigma_0) > + J^0(\sigma_0, \tau - \sigma_0) + g(\tau, \sigma_0) \notin -\mathrm{int}\, P(\sigma_0), \forall \tau \in K.$$

Consequently, (GVHLI) is solves. The proof is done.

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