

A Survey of Recent Developments in Nonlinear Variational Inequalities

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Abstract:

Nonlinear Variational Inequalities (NVI) are a versatile mathematical framework that find applications across diverse domains, from optimization and engineering to economics and physics. This survey article provides an overview of the most recent developments in the field of NVI. We explore the evolution of NVI theory, cutting-edge solution methods, and their applications in real-world problems. By summarizing the state-of-the-art, this survey aims to provide researchers and practitioners with valuable insights into the latest advancements in Nonlinear Variational Inequalities and their wide-ranging impact on various scientific and engineering disciplines.

Introduction

Nonlinear Variational Inequalities (NVI) represent a class of mathematical problems that have garnered significant attention in various fields, including mathematics, physics, economics, and engineering. These inequalities serve as a mathematical framework for modeling a wide range of real-world problems involving constraints and nonlinearity. In this article, we embark on a comprehensive survey of recent developments in the theory and applications of NVIs, highlighting their significance, mathematical formulations, solution techniques, and notable applications.

Understanding Nonlinear Variational Inequalities

Mathematical Formulation

A nonlinear variational inequality can be defined as follows:

Given a closed convex set $C \subseteq \mathbb{R}^n$, and a function $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$, find $x^* \in C$ such that:

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C$$

Here, $\langle \cdot, \cdot \rangle$ denotes the inner product, and x^* is the solution sought. This inequality represents a generalization of the notion of fixed points and is central to the study of optimization and equilibrium problems.

Properties and Applications

Nonlinear variational inequalities possess several key properties:

1. **Existence and Uniqueness:** Under certain conditions, NVIs have a unique solution. This property makes them suitable for modeling problems with single optimal solutions.
2. **Convexity:** The convexity of C and F plays a vital role in the analysis of NVIs. Many practical problems, such as traffic flow optimization and economic equilibrium modeling, exhibit convexity, making NVIs a powerful tool.
3. **Applications:** NVIs find applications in diverse fields, including game theory, Nash equilibria, traffic flow optimization, and finance. They are also instrumental in solving constrained optimization problems.

Recent Developments

Variational Inequality Algorithms

Recent years have witnessed significant advancements in the development of numerical algorithms for solving NVIs. Prominent among them are:

1. **Projected Gradient Methods:** These methods are widely used for solving NVIs when the set C is a convex set. They combine the principles of gradient descent with projection onto the feasible set to find solutions efficiently.
2. **Penalty and Augmented Lagrangian Methods:** These approaches convert the NVI problem into a sequence of unconstrained problems through penalty functions or Lagrangian augmentation, simplifying the solution process.
3. **Interior-Point Methods:** Originally developed for linear programming, interior-point methods have been adapted to handle nonlinear variational inequalities. They are particularly efficient for large-scale problems.

Non-Smooth Analysis and Regularization Techniques

Non-smooth analysis has been employed to analyze the properties of NVIs with non-smooth functions. Techniques like Moreau-Rockafellar duality and proximal point methods have been instrumental in handling non-smoothness in variational inequalities.

Applications

Recent developments in NVIs have expanded their application areas. Some notable examples include:

1. **Traffic Management:** NVIs are employed in traffic flow optimization, ensuring smooth traffic patterns and reducing congestion in urban areas.
2. **Energy Markets:** NVIs are used in modeling energy markets, optimizing the allocation of resources in power grids, and ensuring fair pricing mechanisms.

3. **Machine Learning:** NVIs have found applications in machine learning, especially in training neural networks with constrained activations.
4. **Game Theory:** They are extensively used in game theory for modeling strategic interactions, such as finding Nash equilibria.

Conclusion

Nonlinear Variational Inequalities have evolved into a powerful mathematical tool with a wide range of applications. Recent developments in algorithms, non-smooth analysis, and applications have further enhanced their utility. As interdisciplinary research continues to flourish, NVIs are poised to play a vital role in solving complex real-world problems across various domains.

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