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# Numerical Analysis of Integrals with the Quadrature Based Haar Wavelet Technique

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**Abstract:**

This paper presents a quadrature rule based on uniform Haar wavelets to approximate the values of definite integrals. The key advantages of this method lie in its effectiveness and ease of application. Numerical examples and error estimates are provided to demonstrate the convergence and accuracy of the proposed method, validating its effectiveness for practical use.

**AMS subject classification: 65D30, 65D32, 65Gxx**

**Key words:** Numerical integration, Quadrature method and Haar wavelets, Mat lab.

## 1. Introduction

We learned a variety of approaches to solving integral problems in calculus courses, such as the change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, and others. Numerous technical applications, including Fourier transform, signal processing and transmission, image recognition, fluid dynamics, and electrodynamics, usually need the numerical integration of these types of functions. Even though integrals cannot be evaluated analytically, the numerical solution of these types of integrals is a difficult task that does not produce a solution using standard numerical techniques like the Simpson, trapezoidal rule, and Gauss Legendre quadrature rule, very pulsating integrals.

The four types of integrals listed below are the focus of this study because it is difficult to

Solve them using the methods mentioned before.

$$I_1 = \int \frac{1 - e^{ax}\cos(bx+c) - e^{a(n+1)x}\cos(n+1)(bx+c) + e^{a(n+2)x}\cos n(bx+c)}{1 - 2e^{ax}\cos(bx+c) + e^{2ax}}\,dx$$

$$I_2 = \int \frac{e^{ax}\sin(bx+c) - e^{a(n+1)x}\sin(n+1)(bx+c) + e^{a(n+2)x} sinn(bx+c)}{1 - 2e^{ax}\cos(bx+c) + e^{2ax}}\,dx$$

Where $n$ is any non-negative integer, $a, b, c$ are real numbers and

$$a^2 + b^2 \neq 0, \qquad \text{and} \qquad (ax)^2 + (bx+c)^2 \neq 0.$$

$$I_3 = \int_0^{\pi} \frac{cos^m x}{(a + bcosx)^{n+m+1}}\,dx$$

$$I_4 = \int_0^{\pi/2} \frac{cos^{2p}x \, . \, sin^{2q}x}{(r + \lambda cos^2 x + \beta sin^2 x)^{n+p+q+1}} \, dx$$

Where $m, n, p, q$ are non-negative integers, $a, b, r, \lambda, \beta$ are real numbers, and $a > |b|$, $2r + \lambda + \beta > |\lambda - \beta|$. By applying differentiation with respect to a parameter and the Leibniz differential rule, Euler's formula, De Moivre's formula, and finite geometric series Chii-Huei Yu is able to obtain the closed forms of these four types of definite integrals in [1-2].

Here we solving these integrals by using a numerical method for solving integral functions is the Haar wavelet method. It is built on using Haar wavelets, a family of orthogonal functions that can be used to represent a variety of signals and functions.

The fundamental idea behind the Haar wavelet method is to use a number of Haar wavelets to approximate the integral function. The coefficients of the wavelets are obtained by solving a series of linear equations, and the Haar wavelets serve as the foundation for the approximation. The approximation can be used to evaluate the integral function at any point after the coefficients have been established.

Compared to alternative numerical approaches for calculating integral functions, the Haar wavelet method has a number of benefits. Its ease of implementation and computational efficiency are two benefits. The ability to approximate functions with discontinuities and other irregularities, which can be challenging to manage using other techniques, is another benefit.

The Haar wavelet approach does, however, have significant drawbacks. One drawback is that it could take a lot of wavelets to attain a high level of precision, which can raise the cost of computing. Another drawback is that, for some functions, the approach might not be as accurate as others. In general, the Haar wavelet method strikes a fair mix between simplicity and precision when solving integral functions.

With the help of this method, K.T. Shivaram et al. in [3] were able to solve numerical integration of the forms $\int_0^1 f(x) \sin\left(\frac{w}{x^r}\right)$ and $\int_0^1 f(x) \cos\left(\frac{w}{x^r}\right)$ and evaluate them numerically while using various values of $\omega$ and $r$. To assess the usual integration of highly oscillating functions controlled by the suggested technique, they used Haar wavelet and Hybrid function. The obtained findings show superb agreement with the precise values, and the current methodology can be used to multi-dimensional integrals.

In [11-13] the numerical integration of double and triple integrals with variable limits is accomplished in this study using a novel method based on Haar wavelets and hybrid functions. This approach is a generalization and enhancement of our prior approach [10]. Compared to the standard quadrature rule, (Evans G.A., in [8] and in [16-17] K.T. Shivaram et al. also solved some integrals using quadrature method) this technique has a number of benefits. Several benchmark problems are used to evaluate the new methodology. Comparing the two approaches reveals that the hybrid functions approach outperforms the Haar wavelets approach in terms of results. It has been demonstrated that the current method is more accurate and simpler to use when compared to the symmetric Gauss Legendre quadrature and the hybrid functions method. In [18] K.T. Shivaram et al. extended his work with Chebyshev and Block Pulse Wavelet method for solving Non-Linear Quasi-Singular Integral Equation.

The ability of computers to produce complex activities like abstract art like to Picasso's paintings and musical compositions akin to Beethoven's is a natural question as information technology develops. This seems to be impossible right now. It also seems improbable that computers will ever be able to create abstract mathematical theories similar to those human mathematicians or solve abstract and challenging mathematical issues. However, we may research the help that mathematical software can offer as we look for alternatives. This study demonstrates how to use the mathematical programme Mat lab to carry out mathematical research.

The key benefits of adopting Matlab in this study are its user-friendly interface and straightforward instructions, which make it easy for newcomers to pick up operating principles quickly. Difficult problems can be easily solved by using Matlab's robust computational capabilities.

## 2. Objectives

- Develop a Quadrature Rule Using Haar Wavelets: To propose a quadrature rule based on uniform Haar wavelets for approximating the values of definite integrals.
- Analyze Effectiveness and Applicability: To demonstrate the effectiveness, simplicity, and ease of implementation of the Haar wavelet-based quadrature method.
- Provide Numerical Illustrations: To validate the convergence and correctness of the proposed method through numerical examples and error analysis.
- Enable Comparative Analysis: To perform a comparative study of the Haar wavelet-based quadrature technique with existing methods for approximating integrals.
- Explore Application Potential: To investigate the applicability of Haar wavelets in solving integral equations, partial differential equations, and ordinary differential equations efficiently.
- Ensure Quick Convergence: To emphasize the rapid convergence properties of Haar wavelets for numerical approximation tasks.

## 3. Numerical Integration using Haar wavelets

### a. Haar wavelets

For the family of haar wavelets specified in the range $[a, b]$, the scaling function is

$$h_1(x) = \begin{cases} 1 \; for \; x \in [a, b) \\ 0 \quad elsewhere. \end{cases} \tag{1}$$

Additionally, defined on the interval $[a, b)$ and provided by, the mother wavelet for the family of Haar wavelets.

$$h_2(x) = \begin{cases} 1 \; for \; x \in a, \frac{a+b}{2}) \\ -1 \;\; for \; x \in [\frac{a+b}{2}, b) \\ 0 \quad elsewhere. \end{cases} \tag{2}$$

The dilation and translation processes used to create $h_2(x)$ produce all the other functions in the Haar wavelet family, which are specified on subintervals of $[a, b)$. With the exception of the scaling function, all functions defined for $x \in [a, b)$ in the Haar wavelets family can be represented as

$$h_i(x) = \begin{cases} 1 \; for \; x \in \; [\alpha, \beta) \\ -1 \; for \; x \in [\beta, \gamma) \\ 0 \qquad elsewhere. \end{cases} \tag{3}$$

where

$$\alpha = a + (b - a)\frac{k}{m}, \quad \beta = a + (b - a)\frac{k + 0.5}{m}, \gamma = a + (b - a)\frac{k + 1}{m},$$

$$\text{for } \; i = 3, 4, .., 2M$$

The integer $m = 2^j$, where $j = 0,1, ..J$ with $J = 2^M$ and integer and the integer $j$ represents the wavelet's level. For each level $j$, m takes on the values $m = 2^j$, where $j = 0,1, .., m - 1$.

The **translation parameter** is denoted by the number $k$, while the integer $j$ corresponds to the wavelet's resolution level. The integer $J$ represents the **highest level of resolution**.

The relationship between $I, m,$ and $k$ is given by the formula:

$$I = m + k + 1$$

This relationship ensures that the Haar wavelet functions are **orthogonal** to one another.

$$\int_a^b h_j(x)\, h_k(x)\, dx = \begin{cases} (b-a)2^{-j} & when\ j = k \\ 0 & when\ j \neq k \end{cases} \tag{4}$$

Consequently, any function $f(x)$ that can be squarely integrated in the interval $[a, b)$ can be represented as an infinite sum of Haar wavelets.

$$f(x) = \sum_{i=1}^{\propto} a_i\, h_i(x) \tag{5}$$

If $f(x)$ is piecewise constant or can be approximated as piecewise constant throughout each subinterval, then the aforementioned series ends at finite terms.

**b. Method of numerical integration based on Haar wavelets**

The numerical integration of single integrals using Haar wavelets is discussed in this section.

We consider the integral

$$\int_a^b f(x)dx \tag{6}$$

Over the stretch $[a, b]$ Using Haar wavelets, the function $f(x)$ can be approximated as

$$f(x) = \sum_{i=1}^{2M} a_i\, h_i(x) \tag{7}$$

By raising the value of M, the Haar wavelets approximation quickly converges to the precise function.

**Lemma 3.1. [11]** The approximate value of the integral is

$$\int_a^b f(x)dx \approx a_1(b-a).$$

It is clear from that Haar approximation involves only one coefficient in the evaluation of the definite integral. To calculate the Haar coefficient $a_1$ we consider the nodal points

$$x_k = a + \frac{(b-a)(k-0.5)}{2M}, k = 1,2,..,2M \tag{8}$$

(7) Can be expressed as the discretized form, which is

$$f(x_k) = \sum_{i=1}^{2M} a_i\, h_i(x_k) \tag{9}$$

**Lemma 3.2. [11]** The solution of the system (9) for $a_1$ is

$$a_1 = \frac{1}{2M}\sum_{i=1}^{2M} f(x_k)$$

Consequently, the following formula is derived for numerical integration using the quadrature method with Haar wavelets:

$$\int_a^b f(x)dx \approx \frac{(b-a)}{2M}\sum_{i=1}^{2M} f\left(a+\frac{(b-a)(k-0.5)}{2M}\right) \qquad (10)$$

## 4. Numerical Examples

**Example 4.1.**

$$I_1 = \int_{-10}^{-3} \frac{1-e^{2x}\cos(4x+3)-e^{12x}\cos 6(4x+3)+e^{14x}\cos 5(4x+3)}{1-2e^{2x}\cos(4x+3)+e^{4x}}\,dx$$

$$= 6.99957050820771214882333 \quad \text{(Chii-Huei Yu., [1])}$$

**Example 4.2.**

$$I_2 = \int_4^9 \frac{e^{-x}sinx - e^{-5x}sin5x + e^{-6x}sin4x}{1-2e^{-x}cosx+e^{-2x}}\,dx$$

$$= -0.012814751181223 \quad \text{(Chii-Huei Yu., [1])}$$

**Example 4.3.**

$$I_3 = \int_0^{\pi} \frac{cos^4x}{(5+3cosx)^7}\,dx$$

$$= 0.0026408801975462 \quad \text{(Chii-Huei Yu., [2])}$$

**Example 4.4.**

$$I_4 = \int_0^{\pi} \frac{cos^5x}{(7-2cosx)^9}\,dx$$

$$= 2.1552539566445 * 10^{-7} \quad \text{(Chii-Huei Yu., [2])}$$

**Example 4.5.**

$$I_5 = \int_0^{\pi/2} \frac{cos^4x\,.\,sin^2x}{(2+3cos^2x+2sin^2x)^8}\,dx$$

$$= 5.1015015791996 * 10^{-7} \quad \text{[(Chii-Huei Yu., [2])}$$

**Example 4.6.**

$$I_6 = \int_0^{\pi} \frac{cos^4x\,.\,sin^8x}{(5-2cos^2x+4sin^2x)^{10}}\,dx$$

$$= 2.7089130065827 * 10^{-10} \quad \text{(Chii-Huei Yu., [2])}$$

### 4.7 Numerical results

We have compared the numerical results obtained using the proposed method with the exact values for various orders of $n$ and $m$, which are tabulated in the table.

| **Exact value** | ***N*** | **Haar wavelet** | **Error** |
|---|---|---|---|
| $I_1 = 6.99957050820771214882$3 | 20 | 6.999570508207168 | 5.435651928564766e-13 |
| $I_2 = -0.012814751181223$ | 20 | □0.012814751181222 | □9.992007221626409e-16 |
| $I_3 = 0.0026408801975462$ | 10 | 0.002640880197546 | 1.999268806063270e-16 |
| $I_4 = 2.1552539566445e\text{-}07$ | 10 | 2.155253956644444e-07 | 5.585123495958043e-21 |
| $I_5 = 5.1015015791996e\text{-}07$ | 12 | 5.101501579199564e-07 | 3.599890025830776e-21 |
| $I_6 = 2.7089130065827 * 10^{-10}$ | 11 | 2.708913006582699e-10 | 1.033975765691285e-25 |

## 5. Conclusion

A comparative analysis of quadrature-based Haar wavelets is conducted to discover numerical approximations for various types of integrals. The straightforward applicability and rapid convergence of Haar wavelets support their use in the numerical approximation of integral equations, partial differential equations, and ordinary differential equations.

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