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Applications of Fermatean Uncertainty weighted Average Accumulation operator to Multiple attribute group Decision Making

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Abstract:

In this article, fermatean uncertainty weighted Averaging Aggregation operator has been studied along their several properties namely impotency, boundedness and monotonicity. Secondly we applied this proposed operator to deal with multiple attribute group decision making problem under fermatean uncertainty information. Finally, we constructed an algorithm for multiple attribute group decision making problems with suitable example.

Introduction: Xu demonstrated several operators such as intuitionist uncertainty weighted averaging (IFWA), intuitionist uncertainty ordered weighted averaging (IFOWA) and intuitionist uncertainty hybrid Averaging (IFHA) operators. Xu and Yager explored geometric addition operators such as intuitionistic uncertainty weighted geometric (IFWG) operator, intuitionist uncertainty hybrid geometric (IFWG) operators. They also applied them to multiple alternate group decision making (MAGDH) based on intuitionist fuzzy set (IFS). The advantage of the aggregation operators in this work, we familiarize the noted of (3, 2) Uncertainty weighted averaging aggregation of operator and also discuss some of their basic properties.

Conclusions: The main advantage of using the proposed method and operator is that this method provides more general, accurate and precise results. Therefore, the suggested methodology can be used for any type of selection problem involving any number of selection attributes. This method plays a vital role in real world situations. We ended the paper with an application of fermatean uncertainty decision making problem. In future, some author may develop this gives operators in various fuzzy Environment.

Key words: Fuzzy set, intuitionist fuzzy set, Pythagorean fuzzy set, fermatean fuzzy set, boundaries, weighted average, Decision making, impotency algorithm.

1. Introduction

The concept of uncertainty set was first introduced by Zadeh in 1965 [24]. In 1986, Atanassov [2] presented the concept of intuitionist fuzzy set (IFS), which is a general form of the uncertainty set [2]. Bustince and Burillo [9] Chen and Tan [10] discussed multi criteria paper fuzzy decision making based on Vague set. Xu [21] demonstrated several operators such as intuitionist uncertainty weighted averaging (IFWA), intuitionist uncertainty ordered weighted averaging (IFOWA) and intuitionist uncertainty hybrid Averaging (IFHA) operators. Xu and Yager [22] explored geometric addition operators such as intuitionistic uncertainty weighted geometric (IFWG) operator, intuitionist uncertainty hybrid geometric (IFWG) operators. They also applied them to multiple alternate group decision making (MAGDH) based on intuitionist fuzzy set (IFS). Xu [21] was developed technique for order of preference by simultaneously to ideal solution (TOPSIS) method for multiple attribute group decision making. The advantage of the aggregation operators in this work, we familiarize the noted of (3, 2) Uncertainty weighted averaging aggregation of operator and also discuss some of their basic properties. The idea of (3,2)-uncertainty set proposed by [13]. Pythagorean uncertainty subsets was discussed by [23].

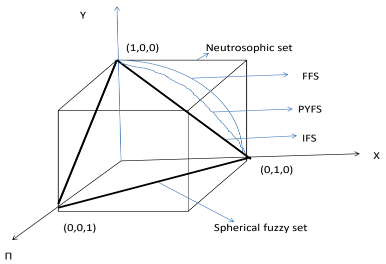
To illustrate the importance of Pythagorean uncertainty collection to extend the grade of membership and non-membership degrees, assume that $\alpha_D(x) = 0.9$ and $\beta_D(x) = 0.8$ for $X = \{x\}$. We obtain $0.9 + 0.8 = 1.7 > 1$, $(0.9)^2 + (0.8)^2 = 1.45 > 1$ and $(0.9)^3 + (0.8)^3 = 1.241 > 1$, which means that $D = (0.9,0.8)$ neither following the condition of Fermatean uncertainty set nor follows the condition of fermatean uncertainty set.

2. Preliminaries

Definition 2.1: (Fuzzy set) Let U be a non-empty set. Then by a fuzzy set on U is meant a function $A : U \to [0,1]$ .A is called the membership function, A(x) is called the membership grade of x in A. We also write A= {(x, A(x)): x∈U}.

Example 2.2: Consider U = { a, b, c ,d } and $A : U \to [0,1]$ defined by A(a)=0, A(b)=0.7, A(c)=0.4, A(d)=1**.**

Definition 2.3: (Pythagorean Fuzzy Set (PFS)) A Pythagorean uncertainty set D on a set X is defined by $D = \{(x, (\alpha_D(x), \beta_D(x))/x \in X\}$, where$\alpha_D: X \to [0,1]$ is the degree of membership and $\beta_D: X \to [0,1]$ is the degree of non – membership of $x \in X$, respectively which fulfill the condition $0 \le {\alpha_D}^2(x) + {\beta_D}^2(x) \le 1$, for all $x \in X$. The degree of indeterminacy $\pi_D(x) = \sqrt[4]{1 - (\alpha_D(x))^2 - (\beta_D(x))^2}$.



Definition 2.4: (Fermatean fuzzy set) [**Senapati and Yager, 2019a**] [18] Let ‘X’ be a universe of discourse A. Fermatean uncertainty set “F” in X is an object having the form $F = \{x,\ m_F(x),\ n_F(x)/x \in X\}$, where $m_F(x) : X \rightarrow 0{,}1$ and $n_F(x) : X \rightarrow 0{,}1$ including the condition $0 \le m_F(x^3) + n_F(x^3) \le 1$, for all $x \in X$. The numbers $m_F(x)$ signifies the level (degree) of membership and $n_F(x)$ indicate the non-membership of the element ‘x’ in the set F.

Definition 2.5 : Let X be a universal set. Then the fermatean uncertainty set (briefly, fermatean uncertainty is defined by the following; $D = \{\langle x,\ \alpha_D(x), \beta_D(x)\rangle / x \in X\}$, where$\alpha_D : X \rightarrow [0,1]$ is the degree of membership and $\beta_D : X \rightarrow [0,1]$ is the degree of non–membership of $x \in X$ to D, with the condition

$$0 \le \left(\alpha_D(x)\right)^3 + \left(\beta_D(x)\right)^3 \le 1$$

the degree of indeterminacy of $x \in X$ to D is defined by

$$\pi_D(x) = \sqrt[6]{1 - \left[\left(\alpha_D(x)\right)^3 + \left(\beta_D(x)\right)^3\right]}.$$

It is clear that, $\left(\alpha_D(x)\right)^3 + \left(\beta_D(x)\right)^3 + \left(\pi_D(x)\right)^6 = 1$ and $\pi_D(x) = 0$

Whenever $\left(\alpha_D(x)\right)^3 + \left(\beta_D(x)\right)^3 = 1$. In the case of simplicity, we shall mention the symbol $\mathrm{D} = (\alpha_{\mathrm{D}}, \beta_{\mathrm{D}})$ for the fermatean uncertainty set $\mathrm{D} = \left\{\left(x, (\alpha_{\mathrm{D}}(x), \beta_{\mathrm{D}}(x)\right)/x \in X\right\}$.

Here, ${\alpha_{\mathrm{D}}}^3(x) = \left(\alpha_D(x)\right)^3$ and ${\beta_{\mathrm{D}}}^3(x) = \left(\beta_D(x)\right)^3$, for all $x \in X$.

Example 2.6: Let D be fermatean fuzzy set and $x \in X$ such that $\beta_D(x) = 0.82$ and $\pi_D(x) = 0$. Then,

$$\begin{aligned} |\alpha_D(x)| &= \sqrt[3]{|(\beta_D(x) - 1)(\beta_D(x) + 1)|} \\ &= \sqrt[3]{|(-0.18)(1.82)|} \\ &= \sqrt[3]{0.3276} \end{aligned}$$

In 2013, Yager defined fermatean uncertainty subset (PUS) as a generalization of intuitionistic uncertainty set (IUS).

Definition 2.7: Let $\sigma = (\alpha_\sigma, \beta_\sigma)$, $\sigma_1 = (\alpha_{\sigma_1}, \beta_{\sigma_1})$, $\sigma_2 = (\alpha_{\sigma_2}, \beta_{\sigma_2})$, are three fermatean uncertainty numbers and $\gamma > 0$. Then

(i) $\sigma^c = (\beta_\sigma, \alpha_\sigma)$

(ii) $\sigma_1 \oplus \sigma_2 = \sqrt{\alpha\sigma_1^3 + \alpha\sigma_2^3 - \alpha\sigma_3^3,\ \beta_{\sigma_1}, \beta_{\sigma_2}}$

(iii) $\sigma_1 \otimes \sigma_2 = (\alpha\sigma_1, \alpha\sigma_2, \sqrt{(\beta\sigma_1^3 + \beta\sigma_2^3 - \beta\sigma_1^3\beta\sigma_2^3)}$

(iv) $\gamma_\sigma = \sqrt{(1-(1-\alpha_\sigma^3), \beta_\sigma^\gamma)})$

(v) $\alpha_\sigma^\gamma = \sqrt{(1-(1-\beta_\sigma^3)^\gamma)}$

Definition 2.8: Let σ = $(\alpha_\sigma, \beta_\sigma)$ be a fermatean uncertainty value. Then we can find the score of 'σ' as the following,

S (σ) = $\alpha_\sigma^3 - \beta_\sigma^3$ where S (σ) ϵ [-1, 1] → (1)

Definition 2.9: Let σ = (α_σ, β_σ) be a fermatean uncertainty number. Then the accuracy degree 'σ' can be defined as follows:

H ((σ) = $\alpha_\sigma^3 + \beta_\sigma^3$ where H (σ)ϵ [0, 1] → (2)

Definition 2.10: Let σ_1 = ($\alpha_{\sigma1}$, $\beta_{\sigma1}$) and σ_2 = ($\alpha_{\sigma2}$, $\beta_{\sigma2}$) be the two fermatean uncertainty numbers. Then

S (σ_1) = $\alpha_{\sigma_1}^3 - \beta_{\sigma_1}^3$ S (σ_2) = $\alpha_{\sigma_2}^3 + \beta_{\sigma_2}^3$

H (σ_1) = $\alpha_{\sigma_1}^3 + \beta_{\sigma_1}^3$ H (σ_2) = $\alpha_{\sigma_2}^3 + \beta_{\sigma_2}^3$ are the score and accuracy of σ_1 and σ_2

respectively. The following are the holds:

(i) If S (σ_2) > S (σ_1), then σ_2 is greater than σ_1 represented by $\sigma_1 < \sigma_1$

(ii) If S (σ_1) = S (σ_2), then,

(a) If H (σ_1) = H (σ_2), then σ_1 and σ_2 have the same information

(ie). $\alpha_{\sigma_1} = \alpha_{\sigma_2}$ and $\beta_{\sigma_1} = \beta_{\sigma_2}$ represented by $\sigma_1 = \sigma_2$.

(iii) If H (σ_1) < H (σ_2), then $\sigma_2$is greater than σ_1.

3. Fermatean uncertainty Weighed averaging aggregation operators

Fermatean **f**uzzy sets is introduced by [23] but in this paper, we familiarize uncertainty weighted averaging operator with their properties.

Definition 3.1: Let $\sigma_j = \left(\alpha_{\sigma j}, \beta_{\sigma i}\right) (j = 1,2, \dots n)$ be fermatean uncertainty variables and let fermatean uncertainty weighed average is a mapping from $\Delta^n \to \Delta$. Then the fermatean uncertainty weighed averaging aggregation operator can be defined as,

fermatean FWA ($\sigma_1, \sigma_2, \dots \sigma_n$) $= r_1\sigma_1 \oplus r_2\sigma_2 \oplus \dots \dots \oplus r_n\sigma_n$

Where $r = (r_1, r_2, r_3 \dots . r_n)$ is the weighted vector of σ_j with condition, r_j ϵ [0,1] and $\sum_{j=1}^n r_j = 1$.

If $r = (\frac{1}{n}, \frac{1}{n}, \frac{1}{n} \dots . \frac{1}{n})$, then the fermatean FWA is converted to fermatean uncertainty average which is defined as,

fermatean FA ($S_1, S_2 \dots S_n$) = $\frac{1}{n}(\sigma_1 \oplus \sigma_2 \oplus \dots \oplus \sigma_n)$ → (3)

Example: 3.2: Let σ_1 = (0.4, 0.3), σ_2 = (0.6, 0.4), σ_3 = (0.7, 0.5), σ_4 = (0.8, 0.4) and r = $(0.1,0.2,0.3, 0.4)^T$, be the weighted vector of σ_1 (j=1,2…..n).

Then fermatean FWAr ($\sigma_1, \sigma_2, \sigma_3, \sigma_4$)

$$= \sqrt{1 - \prod_{j=1}^{4}\left(1 - \alpha_{\sigma_J}^3\right)^{r_j}}, \prod_{j=1}^{4}\left(\beta_{\sigma_j}\right)^{r_J}$$

$$= \left(\sqrt{1 - (1 - \alpha_{\sigma_1}^3}\right)^{r_1}, \left(\beta_{\sigma_1}\right)^{r_1} + \left(\sqrt{1 - (1 - \alpha_{\sigma_2}^3}\right)^{r_2}, \left(\beta_{\sigma_2}\right)^{r_2} + \left(\sqrt{1 - (1 - \alpha_{\sigma_3}^3}\right)^{r_3}, \left(\beta_{\sigma_3}\right)^{r_3} + \left(\sqrt{1 - (1 - \alpha_{\sigma_4}^3}\right)^{r_4}, \left(\beta_{\sigma_4}\right)^{r_4}$$

= (0.5247, 0.4275)

Theorem 3.3: Let $\sigma_j = (\alpha_{\sigma j}, \beta_{\sigma j})$ (j=1, 2….n) be fermatean uncertainty variables, Then their aggregated value by applying fermatean uncertainty weighted average operator is also a fermatean uncertainty value Pythagorean FWA ($\sigma_1, \sigma_2, \dots \sigma_n$) = $\sqrt{1 - \prod_{j=1}^{4}\left(1 - \alpha_{\sigma_J}^3\right)^{r_j}}$, $\prod_{j=1}^{4}\left(\beta_{\sigma_j}^2\right)^{r_J}$ and also the weighted vector of σ_j (j=1, 2….n) is $r_n = (r_1, r_2 \dots r_n)^T$ with some conditions $r_j \in [0,1]$and $\sum_{j=1}^{n} r_j = 1$.

Proof: By mathematical induction, we can prove that equation (3) holds for all n.

First we can show that equation (3) holds for n = 2, Since,

$$r_1\sigma_1 = \left(\left(\sqrt{1 - (1 - \alpha_{\sigma_1}^3}\right)^{r_1}, \left(\beta_{\sigma_2}^3\right)^{r_1}\right)$$

$$r_2\sigma_2 = \left(\sqrt{1 - (1 - \alpha_{\sigma_2}^3}\right)^{r_2}, \left(\beta_{\sigma_2}^3\right)^{r_2}$$

So $r_1\sigma_1 \oplus r_2\sigma_2$

$$= \left(\left(\sqrt{1 - (1 - \alpha_{\sigma_1}^3}\right)^{r_1}, \left(\beta_{\sigma_2}^2\right)^{r_1}\right) \oplus \left(\left(\sqrt{1 - (1 - \alpha_{\sigma_2}^3}\right)^{r_2}, \left(\beta_{\sigma_2}^3\right)^{r_2}\right)$$

$$= \left(1 - (1 - \alpha_{\sigma_1}^3\right)^{r_1} + \left(1 - (1 - \alpha_{\sigma_2}^3\right)^{r_2} - \left(1 - (1 - \alpha_{\sigma_1}^3\right)^{r_1}\left(1 - (1 - \alpha_{\sigma_2}^3\right)^{r_2}, \left(\beta_{\sigma_2}^3\right)^{r_1} \left(\beta_{\sigma_2}^3\right)^{r_3}$$

$$= \left(\sqrt{1 - \prod_{j=1}^{2}\left(1 - \alpha_{\sigma_J}^3\right)^{r_j}}, \prod_{j=1}^{2}\left(\beta_{\sigma_2}^3\right)^{r_J}\right)$$

Thus equation (3) is true for n=2. Let us suppose that equation (3) is true for n=k. Then we have fermatean FW$A_r(\sigma_1, \sigma_2, \sigma_3 \dots . \sigma_n)$

$= \left(\sqrt{1 - \prod_{i=1}^{k}\left(1 - \alpha_{\sigma_1}^{3}\right)^{r_j}\ \prod_{i=1}^{k} \beta^3 \sigma_j^{r_j}}\right)$

Now we show that equation (3) is true for n = k+1.

fermatean FW$A_r(\sigma_1, \sigma_2, \sigma_3 \ldots . \sigma_{k+1})$

$= \left(\sqrt{1 - \prod_{i=1}^{k}\left(1 - \alpha_{\sigma_1}^{3}\right)^{r_j}\ \prod_{i=1}^{k} \beta^3 \sigma_j^{r_j}}\right) \oplus \left(\sqrt{1 - \left(1 - \alpha\sigma^3{}_{k+1}\right)^{k+1}}\right)$

$= \left(\sqrt{1 - \prod_{i=1}^{k+1}\ \left(1 - \alpha_{\sigma_J}^{3}\right)^{r_j}}\ , \prod_{i=1}^{k+1}\left(\beta^3 \sigma_j\right)^{r_j}\right)$

Hence equation (3) holds for n = k+1. Thus equation (3) holds for all n.

Theorem 3.4: Let $\sigma_i = \left(\alpha\sigma_i, \beta\sigma_j\right)(j = 1,2, \ldots n)$ be the fermatean uncertainty variables and the weight vector of $\sigma_j(j = 1,2,3 \ldots . n)$ is $r = (r_1, r_2, r_3, \ldots . r_n)$ with some conditions $r_j \in [0,1]$ and $\sum_{j=1}^{n} r_j = 1$. If $\sigma_j(j = 1,2,3 \ldots . n)$ are mathematically equal, then fermatean uncertainty $FWA_r(\sigma_1, \sigma_2, \ldots \sigma_n) = \sigma$ → (4)

Proof: As we know that, fermatean $FWA_r(\sigma_1, \sigma_2, \ldots \sigma_n) = r_1\sigma_1 \oplus r_2\sigma_2 \oplus \ldots \oplus r_n\sigma_n$.

Let $\sigma_j(j = 1,2, \ldots n) = \sigma$ then (3,2) uncertainty $FWA_r(\sigma_1, \sigma_2, \ldots \sigma_n) = r_1\sigma_1 \oplus r_2\sigma_2 \oplus \ldots \oplus r_n\sigma_n = \sigma \sum_{j=1}^{n} r_j = \sigma$

Theorem 3.5: Let be fermatean uncertainty variable and let the weighted vector of σ_j *be* $r = (r_1, r_2, \ldots r_n)^T$ such that $r_j \in [0,1]$ *and* $\sum_{j=1}^{n} r_j = 1$.

$$\sigma^- = (\min_j\left(\alpha\sigma_j\right), \max_j(\beta\sigma_J))$$

$$\sigma^+ = (\max_j\left(\alpha\sigma_j\right), \min_j(\beta\sigma_j))$$

Then $\sigma^- \leq$ fermatean $FWA_r(\sigma_1, \sigma_2, \ldots \sigma_n) \leq \sigma^+$. → (5)

Proof: we know that

$\min_j(\alpha\sigma_j) \leq \alpha\sigma_j \leq \max_j(\alpha\sigma_j)$ → (6)

$\min_j(\beta\sigma_j) \leq \beta\sigma_j \leq \max_j(\beta\sigma_j)$ → (7)

From equation (6), we have

$\Leftrightarrow \sqrt{\min_j\left(\alpha\sigma_j\right)^3} \leq \sqrt{\left(\alpha\sigma_j\right)^3} \leq \sqrt{\max_j\left(\alpha\sigma_j\right)^3}$

$\Leftrightarrow \sqrt{\left(1 - \max\left(\alpha\sigma_j\right)^3\right)^{r_j}} \leq \sqrt{\left(1 - \alpha^3\sigma_j\right)^{r_j}} \leq \sqrt{\left(1 - \min\left(\alpha\sigma_j^3\right)\right)^{r_j}}$

$\Leftrightarrow \sqrt{(-1 + \min\left(\alpha\sigma_j^3\right))} \leq \sqrt{-\prod_{i=1}^{n}\left(1 - \alpha^3\sigma_j\right)^{r_j}} \leq \sqrt{(-1 + \min\left(\alpha\sigma_j^3\right))}$

$\Leftrightarrow \min_j(\alpha\sigma_j) \leq \sqrt{1-\prod_{i=1}^{n}\left(1-\alpha^3\sigma_j\right)^{r_j}} \leq \max_j\left(\alpha\sigma_j\right)$

Now from equation (7), we have

$$\Leftrightarrow \min_j\left(\beta^3\sigma_j^{r_j}\right)^{r_j} \leq \prod_{j=1}^{n}\left(\beta^3\sigma_j\right)^{r_j} \leq \max_j\left(\beta^3\sigma_j\right)^{r_j}$$

$$\Leftrightarrow \min_j\left(\beta^3\sigma_j\right)^{\sum_{j=1}^{n} r_j} \leq \prod_{j=1}^{n}\left(\beta^3\sigma_j\right)^{r_j} \leq \max_j\left(\beta^3\sigma_j\right)^{\sum_{j=1}^{n} r_j}$$

$$\Leftrightarrow \min_j\left(\beta^3\sigma_j\right) \leq \prod_{j=1}^{n}\left(\beta^3\sigma_j\right)^{r_j} \leq \max_j\left(\beta^3\sigma_j\right) \qquad \rightarrow \quad (8)$$

Let fermatean $FWA_r(\sigma_1,\sigma_2,\sigma_3,..\sigma_n)$, then $S(\sigma) = \alpha_\sigma^3 - \beta_\sigma^3$

$$\leq \max_j(\alpha_\sigma)^3 - \min_j(\beta_\sigma)^3 = S(\sigma^+)$$

Thus, $S(\sigma) \leq S(\sigma^+)$.

Again, $S(\sigma) = \alpha_\sigma^3 - \beta_\sigma^3$

$$\geq \min_j(\alpha\sigma)^3 - \max(\beta\sigma)^3 = S(\sigma^-).$$

Thus, $S(\sigma) \geq S(\sigma^-)$.

If $S(\sigma) < S(\sigma^+)$ *and* $S(\sigma) > S(\sigma^+)$.Then

$$\sigma^- < (3,3)FWG_r(\sigma_1,\sigma_2,\sigma_3,..\sigma_n) < \sigma^+ \qquad \rightarrow \quad (9)$$

If $S(\sigma) = S(\sigma^+)$, then

$$\Leftrightarrow \alpha_\sigma^3 - \beta_\sigma^3 = \max_j\left(\alpha_{\sigma j}\right)^3 - \min_j\left(\beta\sigma_j\right)^3$$

$$\Leftrightarrow \alpha_\sigma^3 = \max_j\left(\alpha\sigma_j\right)^3 , \beta_\sigma^2 = \min_j\left(\beta\sigma_j\right)^3$$

$$\Leftrightarrow \alpha_\sigma = \max_j(\alpha\sigma_j) , \beta_\sigma = \min_j(\beta\sigma_j)$$

Since, $H(\sigma) = \alpha_\sigma^3 + \beta_\sigma^3$

$$= \max_j\left(\alpha\sigma_j\right)^3 + \min_j\left(\beta\sigma_j\right)^3 = H(\sigma^+)$$

Thus, Pythagorean $FWG_r(\sigma_1,\sigma_2,\sigma_3,..\sigma_n) = \sigma^+$ $\qquad \rightarrow \quad (10)$

$$\Leftrightarrow \alpha_\sigma^3 - \beta_\sigma^3 = \min_j\left(\beta_{\sigma j}\right)^3 - \min_j\left(\alpha\sigma_j\right)^3$$

$$\Leftrightarrow \alpha_\sigma^2 = \min_j\left(\beta\sigma_j\right)^3 , \beta_\sigma^2 = \min_j\left(\alpha\sigma_j\right)^3$$

$$\Leftrightarrow \alpha_\sigma = \min_j(\beta\sigma_j) , \beta_\sigma = \max_j(\alpha\sigma_j)$$

Since, $H(\sigma) = \alpha_{\sigma}^{3} + \beta_{\sigma}^{3} = \min_{j}\left(\beta\sigma_j\right)^3 + \min_{j}\left(\alpha\sigma_j\right)^3 = H(\sigma^-)$

Thus, fermatean $FWG_r(\sigma_1, \sigma_2, \sigma_3, .. \sigma_n) = \sigma^-$ → (11)

Thus, from the equation (10) to (11), we have

$\sigma^- \leq$ fermatean $FWA_r(\sigma_1, \sigma_2, \sigma_3, .. \sigma_n) \leq \sigma^+$.

Hence the proof.

Theorem 3.6: Let $\sigma_j (j = 1,2, \dots n)$ and $\sigma_j^* (j = 1,2, \dots n)$ be the two collection of fermatean uncertainty variables. If $\alpha\sigma_j \leq \alpha\sigma_j^*$ and $\beta\sigma_j \geq \beta\sigma_j^*$. Then

Pythagorean $FWA_r(\sigma_1, \sigma_2, \sigma_3, .. \sigma_n) \leq$ fermatean $FWA_r(\sigma_1^*, \sigma_2^*, \sigma_3^*, .. \sigma_n^*)$

Proof: Since $\alpha\sigma_j \leq \alpha\sigma_j^*$

Then $\Leftrightarrow \alpha^3\alpha_j \leq \alpha^3\sigma_j^*$

$$\Leftrightarrow \sqrt{1 - \alpha^3\sigma_j^*} \leq \sqrt{1 - \alpha^3\alpha_j}$$

$$\Leftrightarrow \sqrt{\left(1 - \alpha^3\sigma_j^*\right)^{r_j}} \leq \sqrt{\left(1 - \alpha^3\sigma_j\right)^{r_j}}$$

$$\Leftrightarrow \sqrt{\left(1 - \prod_{i=1}^{n}(1 - \alpha^3\sigma_j\right)^{r_j}} \leq \sqrt{\left(1 - \prod_{j=1}^{n}(1 - \alpha^3\sigma_j^*\right)^{r_j}} \quad \rightarrow \quad (12)$$

Now $\beta\sigma_j \geq \beta\sigma_j^*$

$\beta^{3^{r_j}}\sigma_j \geq \beta^3\beta^{r_j}\sigma_j^*$

$$\Leftrightarrow \prod_{i=1}^{n}\beta^{3^{r_j}}\sigma_j \geq \prod_{i=1}^{n}\beta^{3^{r_j}}\sigma_j^* \quad \rightarrow \quad (13)$$

Let, fermatean $FWA_w(\sigma_1, \sigma_2, \sigma_3, .. \sigma_n) = \sigma$ → (14)

fermatean n$FWA_w(\sigma_1^*, \sigma_2^*, \sigma_3^*, .. \sigma_n^*) = \sigma^*$ → (15)

Then form equation (14) and (15), we have

$S(\sigma) < S(\sigma^*)$, then

fermatean $FWA_w(\sigma_1, \sigma_2, \sigma_3, .. \sigma_n) <$ fermatean $FWA_w(\sigma_1^*, \sigma_2^*, \sigma_3^*, .. \sigma_n^*)$ → (16)

$S(\sigma) = S(\sigma^*)$

Then $\Leftrightarrow \alpha\sigma_j^3 - \beta\sigma_j^3 = \alpha\sigma_j^3 - \beta\sigma_{j^*}^3$

$\Leftrightarrow \alpha\sigma_j^3 = \alpha^3\sigma_j^*, \beta\sigma_j^3 = \beta^3\sigma_j^*$

$\Leftrightarrow \alpha\sigma_j = \alpha\sigma_j^*, \ \beta\sigma_j = \beta\sigma_j^*$

Since, $H(\sigma) = \alpha\sigma_j^3 + \beta\sigma_j^3$

$= \alpha^3\sigma_j^* + \beta^3\sigma_j^*$

$= H(\sigma^*)$

Thus, fermatean $FWA_r(\sigma_1, \sigma_2, \sigma_3, .. \sigma_n) = (3,3)FWA_r(\sigma_1^*, \sigma_2^*, \sigma_3^*, .. \sigma_n^*)$ $\rightarrow$ (17)

Thus from equation (16) and (17),

we have $FWA_r(\sigma_1, \sigma_2, \sigma_3, .. \sigma_n) \leq$ fermatean $FWA_r(\sigma_1^*, \sigma_2^*, \sigma_3^*, .. \sigma_n^*)$

Example 3.7: $\sigma_1 = (0.4,0.6),\ \sigma_2 = (0.4,0.7)$

$\sigma_3 = (0.5,0.7),\ \sigma_4 = (0.6,0.6)$

and $\sigma_1^* = (0.7,0.6),\ \sigma_2^* = (0.8,0.6),\ \sigma_3^* = (0.9,0.6)\ ,\ \sigma_4^* = (0.8,0.3)$

Where $r = (0.1, 0.2, 0.3, 0.4)$.

Now using the fermatean FWA_r operator, we get the following results

fermatean $FWA_r(\sigma_1, \sigma_2, \sigma_3, .. \sigma_n) = \left(\sqrt{1 - \prod_{j=1}^{4}\left(1 - \alpha_{\sigma_J}^3\right)^{r_j}}\ , \prod_{j=1}^{4}\left(\beta^3{}_{\sigma_j}\right)^{r_J}\right)$

= (0.527,0.5210)

Again fermatean $FWA_r(\sigma_1^*, \sigma_2^*, \sigma_3^*, .. \sigma_n^*) = \left(\sqrt{1 - \prod_{j=1}^{4}\left(1 - \alpha^3\sigma_j^*\right)^{r_j}}\ , \prod_{j=1}^{4}\left(\beta^3{}_{\sigma_j^*}\right)^{r_J}\right)$

= (0.7267, 0.1297)

Theorem 3.8: **(Commutative law)** If $\sigma_{ij} = \left(\alpha_{\sigma_{ij}}, \beta_{\sigma_{ij}}\right)(j = 1,2,3..)$ be two fermatean uncertainty number, then

(i) $\sigma_{11} \oplus \sigma_{12} = \sigma_{12} \oplus \sigma_{11}$

(ii) $\sigma_{11} \otimes \sigma_{12} = \sigma_{12} \otimes \sigma_{11}$

Proof: It is obvious

Theorem 3.9: (Associative law) If $\sigma_{ij} = \left(\alpha_{\sigma_{ij}}, \beta_{\sigma_{ij}}\right)(j = 1,2,3..)$ be three fermatean uncertainty numbers, then

(i) $(\sigma_{11} \oplus \sigma_{12}) \oplus \sigma_{13} = \sigma_{11} \oplus (\sigma_{12} \oplus \sigma_{13})$

(ii) $(\sigma_{11} \otimes \sigma_{12}) \otimes \sigma_{13} = \sigma_{11} \otimes (\sigma_{12} \otimes \sigma_{13})$

Proof: It is obvious

Theorem 3.10: Let $\sigma = (\alpha, \beta)$ and $\sigma_{ij} = \left(\alpha_{\sigma_{ij}}, \beta_{\sigma_{ij}}\right)(j = 1,2,3..)$ be three uncertainty number and a real number $\lambda > o$, we have

(i) $\lambda(\sigma_{11} \oplus \sigma_{12}) = \lambda\sigma_{11} \oplus \lambda\sigma_{12}$

(ii) $(a_{11} \otimes a_{12})^\lambda = a_{11}^\lambda \otimes a_{12}^\lambda$

(iii) $\lambda_1\sigma \oplus \lambda_2\sigma = (\lambda_1 + \lambda_2)\sigma$

(iv) $\sigma^{\lambda_1} \otimes \sigma^{\lambda_2} = \sigma^{\lambda_1+\lambda_2}$.

Proof: Here, we prove the parts (i) and (iii) only and the proof of others are similar.

$$\lambda\sigma_{11} = \left(1-(1-\alpha_{11})^{\lambda}, (1-\alpha_{11})^{\lambda}-(1-\alpha_{11}-\beta_{11})^{\lambda})\right)$$

and $$\lambda\sigma_{12} = \left(1-(1-\alpha_{12})^{\lambda}, (1-\alpha_{12})^{\lambda}-(1-\alpha_{12}-\beta_{12})^{\lambda})\right)$$

Thus, we have

$$\begin{aligned}\lambda\sigma_{11} + \lambda\sigma_{12} = &\big(1-(1-\alpha_{11})^{\lambda}, (1-\alpha_{11})^{\lambda}, (1-\alpha_{12})^{\lambda} \\ &-\{(1-1+(1-\alpha_{11})^{\lambda}-(1-\alpha_{11})^{\lambda}+(1-\alpha_{11}-\beta_{11})^{\lambda})) \times (1-1 \\ &+(1-\alpha_{12})^{\lambda}, (1-\alpha_{12})^{\lambda}-(1-\alpha_{12}-\beta_{12})^{\lambda})\},\big)\end{aligned}$$

$$= \left(1-(1-\alpha_{11})^{\lambda}(1-\alpha_{12})^{\lambda},\ (1-\alpha_{11})^{\lambda}(1-\alpha_{12})^{\lambda}-(1-\alpha_{11}-\beta_{11})^{\lambda})(1-\alpha_{12}-\beta_{12})^{\lambda}\right)$$

$$= \lambda(\sigma_{11} \oplus \sigma_{12}).$$

For $\lambda_1, \lambda_2 > 0$ and the fermatean uncertainty numbers $\sigma = (\alpha, \beta)$, we have

$$\lambda_1\sigma = \left(1-(1-\alpha)^{\lambda_1,} (1-\alpha)^{\lambda_1}-(1-\alpha-\beta)^{\lambda_1}\right)$$

and $$\lambda_2\sigma = \left(1-(1-\alpha)^{\lambda_2,} (1-\alpha)^{\lambda_2}-(1-\alpha-\beta)^{\lambda_2}\right)$$

$$\begin{aligned}\lambda_1\sigma + \lambda_2\sigma = &\big(1-(1-\alpha)^{\lambda_1,} (1-\alpha)^{\lambda_2}, (1-\alpha)^{\lambda_1}\ (1-\alpha)^{\lambda_2} \\ &-\ (1-\alpha-\beta)^{\lambda_1} \times (1-\alpha-\beta)^{\lambda_2}\big)\end{aligned}$$

$$= \left(1-(1-\alpha)^{\lambda_1+\lambda_2}, (1-\alpha)^{\lambda_1+\lambda_2}-(1-\alpha-\beta)^{\lambda_1+\lambda_2}\right)$$

$$=(\lambda_1+\lambda_2)\sigma$$

4. Fermatean uncertainty weighted averaging Aggregation operator to multiple attribute group decision making

Let $P = \{p_1, p_2, p_3 \ldots\ldots p_4 \ldots\ldots p_n\}$ *be a set of n alternatives and* $Q = \{q_1, q_2, \ldots\ldots q_m\}$ be a set of m alternatives and $r = (r_1, r_2 \ldots\ldots r_m)^T$ be the weighted vector of the attributes $Q_i(i = 1,2 \ldots m)$ such that $w_i \epsilon [0,1]$ *and* $\sum_{i=1}^{m} w_i = 1$.

4.1 Algorithm:

Step-1: The decision makers provide the information in the form of a matrix.

Step-2: Compute $\sigma_j(j = 1,2 \ldots n)$ using fermatean fuzzy weighted averaging fermatean FWA aggregation operator.

Step-3: Compute the score of $\sigma_j(j = 1,2 \ldots n)$ .If there is no difference between two or more than two scores, then we must have to calculate the degree of accurancy.

Step-4: Arrange the score function of all alternatives in the form of descending order and select the alternatives, which has the highest score function value

4.2 Numerical Example: we consider an example for selecting a watch form different cell phones. Support a customer wants to buy a cell phone from different cell phones. Let

p_1, p_2, p_3, p_4, p_5 represent the five cellphones of different companies. Let Q_1, Q_2, Q_3 be the criteria of these cellphones. In the process of choosing one of the cellphones; three factors are considered.

Q_1: Price of each cellphone

Q_2: Model of each cellphone

Q_3: Design of each cell phone

Support the weight vector of $Q_j(j = 1,2,3)$ is $r = (0.3,0.4,0.5)^T$ and the fermatean uncertainty values of the alternative $P_j(j = 1,2,3,4,5)$ are represented by the following decision matrix.

Step-1: The decision maker gives the decision in Table

Table: fermatean uncertainty decision matrix

	P_1	P_2	P_3	P_4	P_5
Q_1	(0, 0)	(0.2, 0.1)	(0.4, 0.2)	(0.5, 0.4)	(0.6, 0.3)
Q_2	(0.3, 0.1)	(0.4, 0.2)	(0.4, 0.3)	(0.3, 0.2)	(0.7, 0.5)
Q_3	(0.6, 0.4)	(0.4, 0.3)	(0.5, 0.2)	(0.6, 0.3)	(0.4, 0.2)

Step-2: Compute $\sigma_j(j = 1,2,3,4,5)$ by applying fermatean uncertainty weighted average operator

$\sigma_1 = (0.5220, 0.3020)$ $\sigma_2 = (0.5000, 0.3267)$

$\sigma_3 = (0.5348, 0.2147)$ $\sigma_4 = (0.7592, 0.2016)$

$\sigma_3 = (0.5201, 0.3020)$

Step-3: We can find the scores of $\sigma_j(j = 1,2,3,4,5)$

$S(\sigma_1) = (0.5220)^3 - (0.3020)^3 = 0.1147$

$S(\sigma_2) = (0.5000)^3 - (0.3267)^3 = 0.0901$

$S(\sigma_3) = (0.5348)^3 - (0.2147)^3 = 0.1431$

$S(\sigma_4) = (0.7592)^3 - (0.2016)^3 = 0.4294$

$S(\sigma_5) = (0.5201)^3 - (0.3020)^3 = 0.1131$

and the accuracy function,

$H(\sigma_1) = (0.5220)^3 + (0.3020)^3 = 0.1698$

$H(\sigma_2) = (0.5000)^3 + (0.3267)^3 = 0.1599$

$H(\sigma_3) = (0.5348)^3 + (0.2147)^3 = 0.1629$

$H(\sigma_4) = (0.7592)^3 + (0.2016)^3 = 0.4458$

$H(\sigma_5) = (0.5201)^3 + (0.3020)^3 = 0.1682$

Step-4: Arrange the scores of the alternatives in the form of descending order and select the alternatives, which has the highest score function. Since $\sigma_4 > \sigma_3 > \sigma_1 > \sigma_5 > \sigma_2$.

Hence $P_1 > P_5 > P_3 > P_4 > P_2$. Thus the type of cellphone is the best option for the customer.

5. Conclusion

An aggregation operator based on fermatean fuzzy number and applied them to the multivariable decision making problem, where a values are fermatean uncertainty numbers is to be presented. Firstly, we have developed fermatean uncertainty weighted averaging aggregation operator along with their properties namely impotency, boundedness and monotonically. Finally, we have developed a method for multi criteria decision making based on the proposed operator and the operational process have illustrated in details. The main advantage of using the proposed method and operator is that this method provides more general, accurate and precise results. Therefore, the suggested methodology can be used for any type of selection problem involving any number of selection attributes. This method plays a vital role in real world situations. We ended the paper with an application of fermatean uncertainty decision making problem

6. Future Work

In future, some author may develop this gives operators in various fuzzy Environment.

Refrences

[1] G.M. Addis, Fuzzy homomorphism theorems on groups, Korean J. Math., 26, (3) (2018), 373-385.Ipsum dolor sit amet consectetur adipiscing elit pellentesque. Orci eu lobortis elementum nibh. Faucibus a pellentesque sit amet porttitor.

[2] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Set. Syst., 20, (1) (1986), 87-96. Augue interdum velit euismod in pellentesque massa placerat duis ultricies. Metus aliquam eleifend mi in nulla posuere sollicitudin aliquam ultrices.

[3] J.M. Anthony and H. Sherwood, Fuzzy groups redefined, J. Math. Anal. Appl., 69, (1) (1979), 124-130.

[4] J.M. Anthony and H. Sherwood, A characterization of fuzzy subgroups Fuzzy Set.Syst.7, (3) (1982), 297-305.

[5] K. Balamurugan and R. Nagarajan, Fermatean fuzzy implicative P-ideal structure in KU-algebra's, Neuro Quantology, Aug-2022, Vol.20, (10), 2587-2597.

[6] Bhattacharya, Fuzzy Subgroup: Some characterizations, II. Inform. Sciences, 38, (3) (1986), 293-297.

[7] S. Bhunia, G. Ghorai and Q. Xin, On characterization Pythagorean fuzzy subgroups, AIMS Mathematics, 6, (1) (2021), 962-978.

[8] R. Biswas, Fuzzy subgroups and anti-fuzzy subgroups, Fuzzy Set. Syst., 35, (1) (1990), 121-131.

[9] H. Bustinse and P. Burillo, Vague sets are intuitionist fuzzy sets system, Vol.79, pp.403-405, 1996.

[10] Chen, S.M., Tan, J.M.: Handling multi-criteria fuzzy decision making problems based on vague set theory. Fuzzy Sets and Systems 67, (2), 163–172 (1994), 24.

[11] P.S. Das, Fuzzy groups and level subgroups, J.Math.Anal.Appl.,84, (1) (1981), 264-269.

[12] P.A. Ejewa, Pythagorean fuzzy set and its applications in career placements based on academics' performance using max-min-max composition, Complex Intell. Syst., 5, (2) (2019), 165-175.

[13] Ibrahim, Tareq M. Al-shami and O.G. Elbarbary, (3,2)-Fuzzy Sets and Their applications to topology and Optimal Choices, Computational Intelligence and Neuroscience, Volume 2021, Article ID 1272266, 14 pages.

[14] S.V. Manemaran, R. Nagarajan, Bipolar Fermatean fuzzy soft sub algebra's over R-ideals, AIP Conference Proceedings, 2850, 050005 (2004), https://doi.org/10.1063/5.0209145

[15] N.P. Mukherjee and P. Bhattacharya, Fuzzy normal subgroups and Fuzzy cossets, Inform. Sciences, 34, (3) (1984), 225-239.
[16] N.P. Mukherjee and P. Bhattacharya, Fuzzy groups: some group-theoretic analogs, Inform. Sciences, 39, (3) (1986), 247-267.
[17] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl., 35, (3) (1971), 512-517.
[18] T. Senapti and R.R. Yager, Fermatean fuzzy sets, Journal of Ambient Intelligence and Humanized computing, 11, (2) (2020), 663-674.
[19] I. Silambarasan, New operators for Fermatean fuzzy sets, Annals of Communications in Mathematics, 3, (2) (2020), 116-131.
[20] M. Tarnauceanu, Classifying fuzzy normal subgroups of finite groups, Iran. J. Fuzzy Syst., 12, (2) (2015), 107-115.
[21] Z.S. Xu and R.R. Yager, “Some geometric aggregation operators based on intuitionistic fuzzy sets,” Int. J. Gen. syst. Vol.65. pp.417-433, 2006.
[22] Z.S. Xu and R.R. Yager, Dynamic intuitionistic fuzzy multi-attributes decision making, “Int. J. Approx. Reason, Vol.48, pp.246-262, 2008.
[23] R.R. Yager, Pythagorean fuzzy subsets. In:2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), 2013, 36286152.
[24] L.A. Zadeh, Fuzzy sets, Inform. Contr., 8(3) (1965), 338-353.