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On Bi-complex Emad- Falih Integral Transform

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Abstract:

In this Work, we define the Emad-Falih in Bi-complex space with convergence requirements. Additionally, we determine its fundamental features. We show an example of using the Bi-complex Emad- Falih integral transform to solve differential equations involving Bi- complex – valued functions

Keywords: Emad-Falih transforms, Bi-complex Laplace, Bi-complex EF, Bi-complex numbers..

1. Introduction

Emad-Falih integral transform, which was derived from the Fourier integral and has all of the fundamental properties of the Fourier integral with a deeper connection to Elzaki, Laplace, Mohand and Aboodh transforms, was introduced to make solving ordinary and partial differential equations in the time domain easier. Despite being developed from well-Known techniques, the Emad- Falih integral transform has an advantage over other approaches for solving differential equations since it tends to be easy and simple[1].

Futagawa Michiji introduced the idea of holomorphic functions of a Bi-complex variable in numerous papers between 1928and 1932 [2, 3]. Price [4] and Ro ̈nn [5] established the Bi-complex algebra and function theory, while Dragoni [6] provided some fundamental contributions in the study of Bi-complex holomorphic functions in 1934.

In recent years, authors have expanded the following: the Tauberian theorem of Laplace-Stieltjes transform [7], the inverse Laplace transform and its convolution theorem [8], the polygamma function [9], the Stieltjes transform [10], and the Bochner Theorem of Fourier transform. Regarding the Stieltjes Integral transform [11] , the Mellin integral transform and its using [12], Bicomplex variable is not the same as its complex equivalent. Bicomplex number's idempotent representation is crucial to their operation.

2- Basic concept

Definition 2.1[13][15]

If $z_1, z_2 \in x_o$, then $z_1 + i_2 z_2$ is denoted by Bi-complex number and is said to be £. The set of all Bi-complex numbers is denoted by C_2, we have

$C_2 = \{£: £ = z_1 + i_2 z_2 | z_1, z_2 \in C_1\}$.Equivalently
$C_2 = \{£: £ = x_o + i_1 x_1 + i_2 x_2 + j x_3 | x_o, x_1, x_2, x_3 \in C_o\}$ where $i_1 and\ i_2$ are imaginary numbers such that$i_1{}^2 = i_2{}^2 = -1$, $i_1 i_2 = i_2 i_1 = j, j^2 = 1$.

Definition 2.2[13][15]

There is a unique way to express each Bi-complex number as a complex combination of e_1, e_2 as, $£ = z_1 + i_2 z_2 = (z_1 - i_2 z_2)e_1 + (z_1 + i_2 z_2)e_2$

Where $e_1 = \frac{1+j}{2}, e_2 = \frac{1-j}{2}, e_1 + e_2 = 1, e_1 e_2 = e_2 e_1 = 0$. This terms is idempotent representation of a Bi-complex number. The values $(z_1 - i_2 z_2)$ and $(z_1 + i_2 z_2)$ are said to be idempotent elements of the Bi-complex number $£ = z_1 + i_2 z_2$ and $\{e_1, e_2\}$ is called idempotent basis.

Definition 2.3 [14][15]

If $f(t)$ be a continuous piecewise function in $t \geq 0, m_1 \leq \varphi \leq m_2$and $|f(t)| < Ke^{m^2 j|t|}$ $, if\ t\epsilon(-1)^j X[0, \infty)$, then the Emad- Falih integral transform of $f(t)$, $t \geq 0$ is defined by

$$EF\{f(t)\} = T(\varphi) = \frac{1}{\varphi}\int_0^\infty e^{-\varphi^2 t} f(t)dt \qquad (1)$$

3- Bi-complex Emad – Falih transform

Let $f(t)$ be Bi-complex valued continuous piecewise function of exponential order K. Then Bi-complex Laplace transform of $f(t)$ is

$$L\{f(t), £\} = \int_0^\infty e^{-£t} f(t)dt = K(£), \quad £ \in D$$

Where $D = \{£ \in C_2: Re(£) > K + |Im(£)|\}$

Now, If $f(t)$be Bi-complex – valued continuous piecewise function of exponential order K, then the Bi-complex Emad- Falih transform of $f(t)$ is defined as

$$EF\{f(t), £\} = \frac{1}{£}\int_0^\infty e^{-£^2 t} f(t)dt = \bar{f}(£), \quad £ \in \tau$$

Where

$$\tau = \{£ \in C_2: Re(£^2) > K + \left|Im_j(£^2)\right| and\ £ \notin O_2\} \qquad (2)$$

$$and\ O_2 = \{z_1 + i_2 z_2 \in C_2: z_1^2 + z_2^2 = 0\} \qquad (3)$$

In Table 1,we review the Bi-complex Emad- Falih transform for some elementary functions.

Table1: The Bi-complex Emad- Falih transform of some functions

S.No	$f(t)$	$EF\{f(t), £\}$
1	K	$\frac{K}{£^3}$
2	t^k	$\frac{k!}{£^{2k+3}}$
3	e^{at}	$\frac{1}{£(£^2 - a)}$
4	$\sin(at)$	$\frac{a}{£(£^4 + a^2)}$
5	$\cos(at)$	$\frac{£}{(£^4 + a^2)}$
6	$\sinh(at)$	$\frac{a}{£(£^4 - a^2)}$

7	$\cosh(at)$	$\frac{£}{(£^4 - a^2)}$
8	$f'(t)$	$-\frac{f(o)}{£} + £^2\bar{f}(£)$
9	$f''(t)$	$\frac{1}{£}[£^5\bar{f}(£) - f'(0) - £^2 f(0)]$
10	$f^{(n)}(t)$	$\frac{1}{£}[-f^{(n-1)}(0) - £^2 f^{(n-2)}(0) - £^4 f^{(n-3)}(0) + £^7 EF\{f^{(n-3)}(t);£\}$

4- Properties of Bi-complex Emad- Falih Transform

Theorem4.1

Let $f(t) and\ g(t)$ are two functions whose Bi-complex Emad –Falih transform exist for $|£| < k$ and let n be a constant, then

1- $EF[f(t) + g(t);£] = EF[f(t),£] + EF[g(t),£]$

2- $EF[nf(t),£] = nEF[f(t),£]$

Proof

$$EF[f(t) + g(t);£] = \frac{1}{£}\int_0^\infty e^{-£^2 t}[f(t) + g(t);£]dt$$
$$= \frac{1}{£}\int_0^\infty e^{-£^2 t}[f(t);£]dt + \frac{1}{£}\int_0^\infty e^{-£^2 t}[g(t);£]dt$$
$$= EF[f(t),£] + EF[g(t),£]$$

Now, we prove the second part of theorem by use definition as

EF [n f(t), £] $= \frac{1}{£}\int_0^\infty e^{-£^2 t}[nf(t);£]dt$ =n $[\frac{1}{£}\int_0^\infty e^{-£^2 t}[f(t);£]dt]$
$$= nEF[f(t),£]$$

Theorem 4.2

Let $\bar{f}(£) = EF[f(t);£]$ be the Bi-complex Emad- Falih transform of Bi-complex – valued function$f(t)$, then

i) $EF[f'(t);£] = \frac{1}{£}[£^3\bar{f}(£) - f(0)]$

ii)$EF[f''(t);£] = \frac{1}{£}[£^5\bar{f}(£) - f'(0) - £^2 f(0)]$

iii)$EF\left[f^{(n)}(t);£\right] = \frac{1}{£}[-f^{(n-1)}(0) - £^2 f^{(n-2)}(0) - £^4 f^{(n-3)}(0) + £^7 EF\left\{f^{(n-3)}(t);£\right\}$, where $£ \in \tau$.

Proof

i) By applying the definition of Bi-complex Emad – Falih transform, we get

$$EF[f'(t);£] = \frac{1}{£}\int_0^\infty e^{-£^2 t}f'(t)dt = \left(\frac{1}{s_1}\int_0^\infty e^{-{s_1}^2 t}f_1'(t)dt\right)e_1 + (\frac{1}{s_2}\int_0^\infty e^{-{s_2}^2 t}f_2'(t)dt)e_2$$

Using integrating by parts, we get

$$EF[f'(t);£] = \frac{1}{s_1}\left[-f_1(0) + {s_1}^3\bar{f_1}(s_1)\right]e_1 + \frac{1}{s_2}\left[-f_2(0) + {s_2}^3\bar{f_2}(s_2)\right]e_2$$
$$= \frac{1}{s_1 e_1 + s_2 e_2}\left[{s_1}^3\bar{f_1}(s_1)e_1 + {s_2}^3\bar{f_2}(s_2)e_2 - f_1(0)e_1 - f_2(0)e_2\right]$$
$$= \frac{1}{s_1 e_1 + s_2 e_2}[-f_1(0)e_1 - f_2(0)e_2] + (s_1 e_1 + s_2 e_2)^2[\bar{f_1}(s_1)e_1 + \bar{f_2}(s_2)e_2]$$
$$= -\frac{f(o)}{£} + £^2\bar{f}(£)$$

$$= \frac{1}{£}\left[£^3\bar{f}(£) - f(0)\right] \qquad (4)$$

ii- If we replace $f'(t) = h(t)$, Using (4) then

$EF[f''(t);£] = EF[h'(t);£] = \frac{1}{£}\left[£^3\bar{h}(£) - h(0)\right] = \frac{1}{£}[£^5\bar{f}(£) - f'(0) - £^2 f(0)]$

iii)

$$EF\left[f^{(n)}(t);£\right] = -\frac{f^{(n-1)}(0)}{£} + £^2 EF\left\{f^{(n-1)}(t);£\right\}$$
$$= \frac{1}{£}[-f^{(n-1)}(0) - £^2 f^{(n-2)}(0) - £^4 f^{(n-3)}(0) + £^7 EF\left\{f^{(n-3)}(t);£\right\}$$

Theorem4.3 (Duality)

Let $K(£)$ be Bi-complex Laplace transform of $f(t)$, then the Bi-complex Emad- Falih transform $\bar{f}(£)$ of $f(t)$ is giving by

$\bar{f}(£) = \frac{1}{£^2} L\left[f\left(\frac{u}{£}\right);£\right]$

Proof

$$\bar{f}(£) = \frac{1}{£}\int_0^\infty e^{-£^2 t} f(t)dt$$

Let $t = \frac{u}{£}$, then we have

$$\bar{f}(£) = \frac{1}{£}\int_0^\infty e^{-£u} f\left(\frac{u}{£}\right)\left(\frac{du}{£}\right) \quad = \frac{1}{£^2}\int_0^\infty e^{-£u} f\left(\frac{u}{£}\right)du$$
$$= \frac{1}{£^2} L[f\left(\frac{u}{£}\right);£]$$

Theorem4.4 (Convolution)

Let $\bar{f}(£) = EF[f(t);£]$ and $\bar{g}(£) = EF[g(t);£]$ be the Bi-complex of Emad- Falih transforms of the Bi-complex valued function $f(t) and\ g(t)$ of exponential orders $S_1\ and\ S_2$ respectively, then

$$EF[(f*g)(t);£] = £\bar{f}(£)\bar{g}(£)$$

Proof

$$EF[(f*g)(t);£] = \frac{1}{£}\int_0^\infty e^{-£^2 t}(f*g)(t)dt = \frac{1}{£}\int_0^\infty e^{-£^2 t}\int_0^t f(u)g(t-u)du\,dt$$

By putting $t - u = z$, we get

$$EF[(f*g)(t);£] = \frac{1}{£}\int_0^\infty f(u)du\int_u^\infty e^{-£^2 t} g(t-u)dt$$
$$= \frac{1}{£}\int_0^\infty e^{-£^2 u} f(u)du\int_0^\infty e^{-£^2 z} g(z)dz$$
$$= £\bar{f}(£)\bar{g}(£)$$

5- Inverse Bi- complex Emad- Falih transform

In this section, we denoted the inverse for Bi-complex Emad – Falih as follows:

Theorem 5.1

Let $\bar{f}(£)$ be Bi-complex Emad- Falih transform of Bi-complex –valued function $f(t)$ of exponential order S, analytic in τ, then

$f(t) = \frac{1}{2\pi i_1}\int_G e^{£^2 t}£\ \bar{f}(£)d£ = \sum Res[e^{£^2 t}£\bar{f}(£)]$,

where G=(G_1, G_2) is continuous piecewise function differential closed contour in Bi-complex space and

$$\tau = \left\{ £ \in C_2 : Re(£^2) > K + \left| Im_j(£^2) \right| and\ £ \notin O_2 \right\}$$
$$O_2 = \{z_1 + i_2 z_2 \in C_2 : z_1^2 + z_2^2 = 0\}$$

Proof

Since Bi-complex Laplace transform [14] of Bi-complex –valued function $f(t)$ is

$$L[f(t);\ £^2] = K(£^2) = \int_0^\infty e^{-£^2 t} f(t)dt,$$
$$\frac{1}{£}K(£^2) = \frac{1}{£}\int_0^\infty e^{-£^2 t} f(t)dt\ = \bar{f}(£)$$
$$K(£^2) = £\bar{f}(£) \dots\dots\dots (5)$$

Using (5), inverse Bi-complex Laplace transform [14] of (5)is

$$f(t) = \frac{1}{2\pi i_1}\int_G e^{£^2 t}\ K(£^2)d£ = \frac{1}{2\pi i_1}\int_G e^{£^2 t}\ £\bar{f}(£)d£$$

Hence the result (5) holds.

Theorem 5.2

If $\bar{f}(£) = EF[f(t);£]$ be Bi-complex Emad- Falih transform of Bi-complex – valued function$f(t)$, then

$$EF[tf(t);£] = \frac{-1}{2£}\frac{d}{d£}\bar{f}(£) + £\bar{f}(£)$$

Where τ is denoted in (2).

Proof

Since $\frac{d}{d£}\bar{f}(£) = \frac{d}{d£}\int_0^\infty \frac{1}{£} e^{-£^2 t} f(t)dt$

$$= \left(\frac{d}{ds_1}\int_0^\infty \frac{1}{s_1} e^{-s_1{}^2 t} f_1(t)dt\right)e_1 + \left(\frac{d}{ds_2}\int_0^\infty \frac{1}{s_2} e^{-s_2{}^2 t} f_2(t)dt\right)e_2$$

By use Leibniz law for integration of complex functions, we get

$$\frac{d}{d£}\bar{f}(£) = \left(\int_0^\infty \frac{\partial}{\partial s_1}\frac{1}{s_1} e^{-s_1{}^2 t} f_1(t)dt\right)e_1 + \left(\int_0^\infty \frac{\partial}{\partial s_2}\frac{1}{s_2} e^{-s_2{}^2 t} f_2(t)dt\right)e_2$$
$$= \left(\int_0^\infty -2e^{-s_1{}^2 t} t f_1(t)dt - \int_0^\infty \frac{1}{s_1^2} e^{-s_1{}^2 t} f_1(t)dt\right)e_1 + (\int_0^\infty -2e^{-s_2{}^2 t} t f_2(t)dt$$
$$- \int_0^\infty \frac{1}{s_2^2} e^{-s_2{}^2 t} f_2(t)dt)e_2$$
$$= \left(-2s_1 EF[tf_1(t);s_1] - \frac{1}{s_1}EF[f_1(t);s_1]\right)e_1 + (-2s_2 EF[tf_2(t);s_2] - \frac{1}{s_2}EF[f_2(t);s_2])e_2$$
$$= -2(s_1e_1 + s_2e_2)EF[t(f_1(t)e_1 + f_2(t)e_2);s_1e_1 + s_2e_2]$$
$$- \frac{1}{s_1e_1 + s_2e_2)}(EF[(f_1(t)e_1 + f_2(t)e_2);s_1e_1 + s_2e_2]$$
$$= -2£EF[tf(t);£] - \frac{1}{£}EF[f(t);£]$$

Therefore,

$$EF[tf(t);£] = \frac{-1}{2£}\frac{d}{d£}\bar{f}(£) + £\bar{f}(£)$$

Theorem 5.3

If $\bar{f}(£) = EF[f(t);£]$ be Bi-complex Emad- Falih transform of Bi-complex – valued function$f(t)$, then

$$EF\left[\int_0^t f(u)du;£\right] = \frac{1}{£^2}\bar{f}(£)$$

Where τ is defined in (2)

Proof

From definition of Bi-complex Emad- Falih transform , we have

$$EF\left[\int_0^t f(u)du;£\right] = \frac{1}{£}\int_0^\infty e^{-£^2 t}\int_0^t f(u)du\,dt$$

By changing the order of integration, we get

$$EF\left[\int_0^t f(u)du;£\right] = \frac{1}{£}\int_0^\infty f(u)du\int_u^\infty e^{-£^2 t}\,dt = \frac{1}{£}\int_0^\infty \frac{1}{£^2}e^{-£^2 u}f(u)du = \frac{1}{£^2}\bar{f}(£)$$

Theorem 5.4

Let $f(t)$ be Bi-complex –valued function of period$T > 0$, then

$$EF[f(t);£] = \frac{\frac{1}{£}\int_0^T e^{-£^2 t}f(t)dt}{(1 - e^{-£^2 T})}\ , £ \in \tau$$

Proof From the def. of Bi- complex Emad- Falih transform,we have

$EF[f(t);£] = \frac{1}{£}\int_0^\infty e^{-£^2 t}f(t)dt = \frac{1}{£}\sum_{n=0}^\infty \int_{nT}^{(n+1)T} e^{-£^2 t}f(t)dt$ (6)

Take $t = \iota + nT$ in the n^{th}integral ,(6) becomes

$$EF[f(t);£] = \frac{1}{£}\sum_{n=0}^\infty e^{-£^2 nT}\int_0^T e^{-£^2 \iota}f(\iota + nT)d\iota$$

$= \frac{1}{£}\sum_{n=0}^\infty e^{-£^2 nT}\int_0^T e^{-£^2 \iota}f(\iota)d\iota$, [because $f(t + nT) = f(t)$]

$$= \frac{1}{£(1 - e^{-£^2 T})}\int_0^T e^{-£^2 t}f(t)dt$$

6- Application

In this section, we solved an example about an ordinary differential equations with initial conditions by using Bi-complex Emad –Falih transform.

Example1

Solve the differential equation using Bi-complex Emad –Falih transform

$$\frac{dy}{dt} + y = e^t \quad , y(0) = 0 \qquad (7)$$

Solution

Taking the Bi-complex Emad – Falih transform of equation (7), we get

$$\frac{£^3\bar{y}(£) - y(0)}{£} + \bar{y}(£) = \frac{1}{£(£^2 - 1)}$$

$$£^2\bar{y}(£) + \bar{y}(£) = \frac{1}{£(£^2 - 1)}$$

$$(£^2 + 1)\bar{y}(£) = \frac{1}{£(£^2 - 1)(£^2 + 1)}$$

$$\bar{y}(£) = \frac{1}{£(£^4 - 1)}$$

Taking inverse of Bi-complex Emad-Falih transform and using Theorem4 (Convolution)

$$y(t) = sinht$$

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