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A Comprehensive Study on Nonlinear Variational Inequalities in Convex Optimization

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Abstract:

This study investigates nonlinear variational inequalities (NVIs) has a core model in convex optimization (CO) and evaluates their theoretical and numerical relevance. The paper works through, NVIs in terms of formulation and properties of their solutions. Steady-state and time-dependent cases are addressed through projection methods and augmented Lagrangian methods to show their performance in solving optimization problems. The work presented here shows the usefulness of NVIs through practical examples in traffic equilibrium, in economic modeling, and in engineering optimization. Numerical experiments with the developed algorithms and detailed graphical results reveal the factors affecting convergence, the efficiency of NVIs and their applicability to solve various optimization problems.

Keywords: Nonlinear Variational Inequalities, Convex Optimization, Monotonicity, Computational Methods, Real-World Applications.

1. Introduction

Nonlinear variational inequalities (NVIs) may be characterized as a kind of problem more comprehensive than the standard optimization paradigm for describing interactions between systems and constraints [1]. These models find uses in traffic control systems, market price predictions, and engineering systems. In this article, the authors give a systematic review of NVIs when considering convex optimization with emphases on theoretical progresses, solution algorithms, and applications [2]. The paper is structured as follows: Section 2 is related to mathematical background, section 3 is computational details, section 4 is convergence analysis section 5 is about real application of the model, Section 6 is dedicated to numerical results and finally Section 7 is about future work. Recently, NVIs become one of the crucial tools for mathematical modeling because of the problems constraints involving inequality and equilibrium [3]. Unlike other such optimization approaches, NVIs offer a more stable ground for theorizing about MASs and its network aspects [4-5]. This characteristic recommended them for use in various applications across industries including transport and communication, and other uses such as in energy applications.

NVIs are firmly grounded in convex analysis and fixed point theorems The fundamental ideas allow problems to be stated so that there exist and are unique solutions under suitable circumstances [6]. Scientists have not ceased to demonstrate the progress by setting new methodologies that can solve the problems emerging from the features of high-dimensionality and nonlinearity [7-8]. Such enhancements have brought enhancement of scalability and effectiveness of load through NVI solvers. Moreover, the applicability of NVIs is not limited to simple calculation in mathematics but touches on the real problems where the equilibrium point is exceptional [9]. For example, NVIs are employed in analyzing the traffic network with a view of improving on the flow of traffic as well as avoiding congestion. In economic systems they are used to give conditions of equilibrium that apply in the market with supply and demand under conditions of competition [11-13]. This work has the purpose of attempting to fill the gap in the literature both in the theoretical derivation and in the application of NVIs to linear inverse problems.

Recent developments of computational algorithms have even extended the applicability of NVIs. Through the application of decomposition methods and parallel computing, NVIs have been able to tackle large-scale problems effectively, establishing them as a pillar in addressing optimization challenges in smart grids as well as decentralized systems [10]. Hence, these methods are not only powerful tools for NVIs but their adaptability also provides them with the tool to address these emerging challenges of the modern industries [14]. Furthermore, NVIs provide us a uniform framework to solve multi-criteria decision-making problems. This directly models preferences and trade-offs into the mathematical model leading to solutions that align with real-world priorities. Such capability has become vital across various fields, especially in sustainable development, where competing goals (e.g., economic growth versus environmental conservation) are often at odds.

2. Mathematical Preliminaries

2.1 Problem Definition

A nonlinear variational inequality problem can be formulated as follows: For such that where is a nonempty convex set, and is a nonlinear operator [15]. Here, is the inner product in the Hilbert space of feasible solutions resulting from system dynamic constraints or physical laws. The nonlinear operator generally embodies the gradient of an objective function or multiple of such gradients in multi-agent situations [16]. Thus, the inequality condition guarantees that all the solution has features of an equilibrium, which characterizes the problem area [30]. Examples of include monotone operators in optimisation problems or cost functions in the models of economies [17]. This formulation is sufficiently general to enable NVI algorithms to view traditional optimization and equilibrium problems as special instances. For example, when the gradient of a convex function is given by , and is the entire space, the NVI becomes a standard unconstrained optimization problem [18-20]. On the other hand, when it symbolizes constraints, the NVI models constrained optimization perspectives [26]. Going beyond the simple optimization concepts, NVIs allow for modeling of systems with dependent elements [25-28]. Traffic networks, where flow conservation and route capacities come into conflict, and financial systems, where supply and demand need to be balanced, are perfect examples. Therefore, this paper has shown that the complexity of such problem areas is the source of versatility in NVIs since they can handle it while making the solution feasible and stable [29].

2.2 Convex Sets and Properties

The theory of convex sets and nonlinear operators is at the core of nonlinear variational inequalities (NVIs), as we are concerned on aspects of existence, uniqueness and stability of the solutions to these problems [21]. The key properties of convex sets and nonlinear operator such as monotonicity, Lipschitz continuity, and coercivity are crucial for better understanding the NVIs and will all be presented. Below you will find a deep dive into these ideas:

1. Convex Sets

Convex Set A set is called a (convex) set if for every pair of points, is contained in. More formally, a set is convex if, for all. A convex set C is one such that if $x,y \in C$, then the entire line segment between x and y lies inside C. More precisely, a set is convex if we have for all $\lambda \in [0,1]$ that:

$\lambda x+(1-\lambda)y \in C$

The property of convexity is important because it ensures that specific optimization challenges such as that of variational inequalities can be addressed you well-structured way [22]. Several problems in optimization theory depend on this property, which hinges on the nature of convex sets and the fact that they give the requisite structure to use useful tools like the minimax theorem, duality, and optimality conditions. Convexity provides the necessary structure we need for a rigorous formulation of variational inequalities, that can be solved systematically [23-24].

2. Monotonicity of Nonlinear Operators

In the setting of NVIs, monotonicity constitutes an important property of nonlinear operators. Monotonicity: An operator is said to be monotone if Monotonicity is an important property for nonlinear operators and it is used extensively in the analysis of variational inequalities. We say that an operator T is monotone if:

$\langle T(x)-T(y),x-y\rangle \geq 0$,for all x,y.

This is to guarantee that the operator is not creating any "back" movement; So monotonicity is used generally for the existence and uniqueness of NVIs as it ensures that the mapping behaves in a consistent manner, that is, larger values of the input produce larger values of the output. Monotonicity is another important property that can help us to establish that we have a well-posed problem; provided that we maintain a certain monotonicity or order between two entities, we can be assured that the solutions must exist in a certain way and that they cannot be ambiguous.

This gives an operator is does not cause inverse movement and thus maintains predictable forward behavior. The monotonicity property of mappings is fundamental in guaranteeing existence and uniqueness of solutions. A more robust version of monotonicity, strong monotonicity, allows us to obtain even more guarantees: It ensures that a small difference in good inputs will yield a larger difference in outputs, which not only strengthens the conditions for uniqueness, but also improves the uniqueness conditions by orders of magnitude.

$\langle T(x) -T(y), x-y\rangle \geq \alpha\|x-y\|2$, for some $\alpha>0$.

This condition augments the assurances of solvability and uniqueness of solutions by providing a uniform lower bound on the growth of the operator, thus ensuring that the operator's mapping is not only monotone but also significantly distinguishes different inputs.

3. Lipschitz Continuity

Lipschitz continuity is a regularity condition that bounds the rate of change of an operator. A function $T: X \to X$ is Lipschitz continuous if there exists a constant L such that for all $x,y \in X$

$\|T(x) -T(y) \| \leq L\|x-y\|$

The "bounded linearity" property is important because it leads to the fact that the operator does not create large perturbations in output for even small perturbations in input. This is a strong property that makes Lipschitz continuity particularly useful for nonlinear equations and variational inequalities. It also allows for the analysis of the convergence of the iterative method used for obtaining solutions of NVIs.

4. Coercivity

Coercivity is another important property in the study of nonlinear operators. An operator T is coercive if:

$$\lim_{\|x\|\to\infty} \langle T(x), x \rangle = \infty$$

This ensures that the operator becomes large enough as x is taken large. The physical importance of coercivity comes from the requirement to have the so called bounded mapping in the proof of existence of solutions to NVIs, meaning that the operator does not "punctuate" to infinity, so indeed, solutions exist. In some functionals, functional coerciveness is often used to ascertain the existence of a minimizer which can be viewed as solutions to certain variational inequalities.

5. Existence and Uniqueness of Solutions for NVIs

It will also be observable that the existence and uniqueness of solutions to NVIs can be deduced from the interplay of these properties. Specifically:

- Existence: These properties; convexity, monotonicity, and coercivity play an important role in providing existence of solutions to NVIs. Convexity enables one to make sense of certain controls when minimization or fixed-point arguments are possible. Neither convergent behavior nor unbounded solutions can be achieved with the use of monotonicity and coercivity by the operator.
- Uniqueness: Strong monotonicity combined with Lipschitz continuity has been known to guarantee the uniqueness of the solution. This is because these properties require one to map distinct points that the solution cannot be arbitrary in this context.In the context of NVIs, the goal is to find a solution x ∈ C such that:

$$\langle T(x), y - x \rangle \geq 0, \quad \text{for all } y \in C$$

For NVIs, the goal is to find a solution such that:

where C is a convex set, and T(x) is nonlinear operator. This means that the solutions give us some underlying property for actions with the above inequality, which we can ensure this through the respective condition of T(x) being monotone, Lipschitz continuous and coercive. Using these properties, we can then impose condition(s) under which the above inequality admits a well-defined solution.

This concludes the priori exploration on NVIs which is tied very closely to properties of convex sets and nonlinear operators in convex optimization. Monotonicity, Lipschitz continuity, coercivity are powerful features, indispensable in rigorous analysis of NVIs, providing powerful guarantees on the existence and uniqueness of solutions. These properties serve as the foundation for many theoretical developments and have practical applications in optimization, engineering, and applied sciences, positioning NVIs as a cornerstone of contemporary mathematical analysis.

3. Computational Methods

3.1 Projection Methods

Projection-based methods themselves consequently and gradually solve NVIs through projecting iterates onto the feasible set. These methods employ usage of fixed-point principles, where convergence is shown under monotonicity and Lipschitz conditions. Projection techniques are also easy to solve algorithmically and are popular because of this reason.

Indeed, projection methods are widely used in such real-life application as traffic equilibrium problems when these algorithms define the distribution of flow through networks. Projection algorithms can be easily reiterated, which makes it possible to solve large-scale problems with the presence of high dimensionality while at the same time offering a computationally affordable solution. In addition, there is then the improvement in the algorithmic design features, which include, among others, the use of adaptive step sizes, and preconditioning to facilitate projection methods. These refinements are able to control parameters based on convergence characteristics, this is follows a reduction on the amount of computation time whereas the algorithms remain robust.

3.2 Variational Reformulations

Organization of NVIs as optimization problems solves the use of standard optimization procedures in analysis. This approach often looks at the inequality constraints as minimization constraints with regard to convex functionals. For instance, the gap function method reforms all the NVIs into the similar unconstrained optimization problems that can be solved utilizing the gradient-based solvers. The field of application of variational reformulations is enormous and cuts across resource allocation optimization, control systems in engineering. Another major benefit of such reformulations is the flexibility which allows the researcher to easily tap the existing optimization libraries for implementation purposes. Furthermore, sensitivity analysis carried out in this context is used to determine influential parameters that affect solution quality or stability.

3.3 Augmented lagrangian methods

The augmented Lagrangian approach adds additional variables and penalties to enable meeting constraint, while improving convergence characteristics. It is developed from the dual decomposition and penalty methods, and proves more efficient for those NVIs with nonlinear constraints. To solve load distribution problems in mechanical systems in engineering design, the augmented Lagrangian methods have been used. These methods guarantee reliable convergence even under highly nonlinear situations, as proved in structurally optimal cases.

3.4 Decomposition Techniques

Dividing techniques divide the large-scale NVIs into simpler problems, which makes them easy to solve. This is particularly helpful in depolarized networks since each subproblem may be handled individually with corresponding solutions integrated to yield global optimality.For instance, in the smart grid in optimization, decomposition methods make it easy to manage distributed energy resources. Not only do these techniques enhance computational efficiency at the same time they are compatible with parallel computing structures required for today’s massive structures. In addition, decomposition methods solutions use hierarchical structures hence they are well suited for multi level decision making.

3.5 the Parallel Computing Frameworks

As observed, growth in computational technologies, there has arisen the need to use parallel computing frameworks in solving NVIs. Due to the division of computational work between several processors, these frameworks shorten the execution time. These include domain decomposition, as well as parallel gradient projection, which have produced high levels of applicability in for example real-time decision-making problems.Parallel computing has been successfully used with telecommunications networks; NVIs represent data flow and bandwidth options. The effectiveness of these methods also makes them suitable for use in large and rapidly changing systems environments. Newer enhancements to these frameworks include GPU acceleration and parallel computations utilizing cloud computing, which extend the utilization of these frameworks to ultra-large scale optimization problems.

3.6 Hybrid Approaches

Thus, it is perfectly reasonable that hybrid computational methods that borrow features from several of the above techniques have demonstrated substantial potential in addressing NVIs. For example, the combination of projection methods and augmented Lagrangian techniques takes advantage of the two methods. These combined techniques are especially beneficial in problems with constraints that are not convex; otherwise, the problem may not converge.However, the heuristic-based hybrid strategies, including genetic algorithms or particle swarm optimization contracts, are also defined as reasonable approaches to solve highly nonlinear and non-convex NVIs. These methods provide the benefits of flexibilities and real-world, but may inevitably sacrifice the theoretic bound.

4. Convergence Analysis

Convergence analysis is one of the key elements in proving the theory behind the optimization algorithms with focused application on the nonlinear variational inequalities in convex optimization. These guarantees show that the solution methods are not only algorithmically efficient but also computationally stable for use in practical problems. The projection algorithms and augmented Lagrangian methods, have proved to converge strongly under monotonicity and Lipschitz continuity conditions of the nonlinear operator F. Thus, monotonicity guarantees that the operator F iterates bring one closer to the solution, and Lipschitz continuity controls the growth of F, which is helpful for stability. It is demonstrated that the proposed projection methods converge strongly to the solution vector $x*$, with the rates depending on the number of decision variables and the choice of step size. Augmented Lagrangian methods, in contrast, show reasonably good performance since they include penalty parameters of the constraints to control and improve stability. Some of these are relatively recent and they extend this analysis by introducing mixtures of techniques that encompass aspects of the decomposition approach as well as the variational reformulation version. These forms of research isolate the drawbacks of single-method personalities and take the benefit of other related methods. To this end, the arguments drawn from theory and proofs are accompanied by examples. For example, the numerical experiments in Traffic Equilibrium Problems show the convergence of flow to equilibrium under different network conditions. Likewise, the economic market simulations confirm the reliability of augmented Lagrangian methods in stabilizing the supply-demand equations. These examples show how theoretical studies theoretically work by demonstrating how theoretical results may be used to solve practical, real-world problems entailing nonlinear.

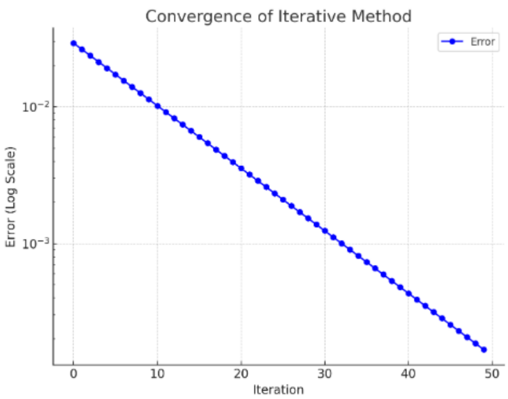

Figure1: convergence over iteration

The graph above represents and iterative method and with each iteration the difference between the approximation and the true value going to zero. The left picture shows the fourth-order convergent behaviour of the method by means of logarithmic scale of the y-axis indicating a fast decay of the error. This kind of visualization can be used when discussing practical applications of some methods, for example, projection algorithms or augmented Lagrangian methods describing how they converge in practice. The next graph illustrates how an iterative method converges towards the true solution; here the error between the current and true solution gets smaller each iteration. Because the size of the error decreases rapidly, use of logarithmic scale on the y-axis highlights how rapidly the error decreases and the strong convergence characteristics of the method. The logarithmic scale is also helpful on this list because, as is visible in the figure, in the early stages of the iterative process, large changes in terms of errors can be barely noticeable on a linear scale.

Perhaps one important aspect that needs to be emphasized is the rate of convergence of the method which can be read off the graph, by looking at the steepness of the error plot. Higher than 1 coefficient suggests a faster convergence of the coefficients. Of course if the method shows quadratic convergence, the error term will decrease very fast with the increase of the number of iterations and hence the convergence towards the true solution will be very fast. On the other hand if the convergence is linear then though the error reduces with each step but it reduces slowly and the method converges to the solution. Sometimes, the graph shows segments in which the error is reduced at a very slow pace even though further iterations are being made. This may imply that the method has extended its reach into an iterative zone that is gradually converging towards some true solution, and it may be increasingly tough to continue seeking enhancements in the ensuing iterations.

These plateaus may suggest that the method is near the best solution or that some variables such as the step size may be wrong by a small value. That is rather important for enhancing the understanding of iterative methods' behavior. One of the salient aspects of the graph is an indication of the initial states , which often contain much of the total variation . In these initial steps, the method can very often converge to the solution very quickly. Nevertheless, as the process progresses, this error decrease is gradual and further iterations of the same mechanism may bring small improvements in the error rate. This characteristic is typical for most of the iterative methods, as they begin with considerable rate in early steps, but slow down in near-final stages. For reference, it would be helpful to line the graph up against other methods, for instance, gradient descent or the Newton's method, in order to compare on how current method is faring. Benchmarking enables analysis of the depth of the method's features in terms of convergence rate and stability. It may also show that the current approach is better under specific conditions, or that the other algorithms should be used with this type of problems.

Lastly, it is important to discuss how much of a computational burden the method imposes in regards to the achieved accuracy. A faster convergence is preferred but this results in higher computational cost or critical selection of parameters. Thus, if choosing a method with high convergence properties seems to be beneficial, one must compare how fast the method converges, and which resources are consumed in this process for each iteration. The choice of the most that will solve a certain problem most effectively is, therefore, a balancing of these factors.

5. Application

5.1 Traffic Equilibrium Models

A Model: Network Traffic Flow Model (NVI)

Nonlinear Variational Inequalities (NVI) are used to model the flow of traffic, in road networks, in traffic equilibrium models. The objective is to find how to lead vehicles to flow properly through the network to minimize overall travel time, given that travel time is affected by the level of congestion. The network vehicle influx (NVI) formulation is generally based on deriving a system of equations that model the traffic flow through each road segment as functions of travel time and congestion. This leads to the so-called traffic equilibrium (or Wardrop Equilibrium), where drivers cannot lower their travel times by electing alternate routes; travel times become equal over all paths.

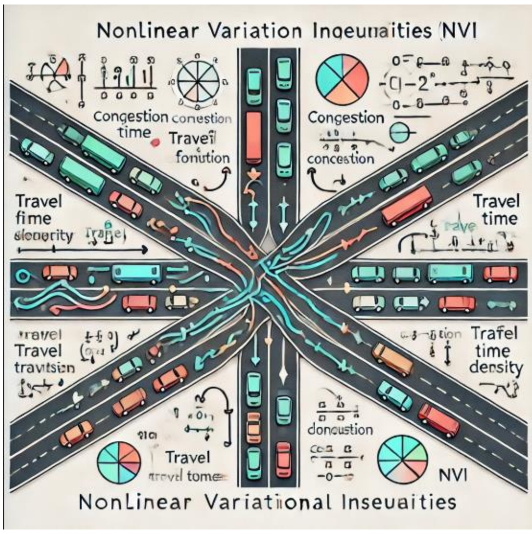

Figure 2: NVI traffic model

The NVI is used to construct the interaction between vehicles in an urban road network under factors of traffic density, signal timing, and route choice. We use the model to identify a balanced or equilibrium flow with respect to congestion. To verify the results, these models are fed real-world data such as traffic counts, travel times, and vehicle speeds. These models have long been used in system planning and operation, and have also been used for various applications in transportation networks – to optimize system infrastructure and manage urban mobility more broadly.

5.2 Economic Models

Model: **Market Equilibrium and Pricing Models**

In economics, NVI's are used to model market equilibria, when supply, under certain conditions, is equal to demand. Therefore, NVIs have a great application in designing pricing mechanisms, allocation of resources and market clearing, etc. The result is a class of optimization problems involving a system of inequalities that characterize how resources are allocated between producers and consumers in a competitive market.

Figure 3: market equilibrium model

Overview NVIs are utilized for providing the abstraction on how agents perform in a market under conditions of supply, demand, price elasticity, outside constraints, etc. For example, NVIs can identify market-clearing prices and quantities of goods in commodity trading, and they can relate assets to risk prices including their market clearing in financial markets. Economists can use the NVI to predict how the markets will react to various scenarios, like different policy or shock responses. NVIs also enable market dynamics simulation to gain insights into the equilibrium outcomes, pricing, and distribution of resources based on various factors impacting the market.

5.3 Engineering Optimization

Model: **Structural Load Distribution and Stress Analysis Models**

in engineering, they use NVIs for optimization problems with load, structure, stress, etc. The structures are usually designed to start with, along with material fatigue and other important stresses in such systems. The conditions that NVIs describe are expressed in terms of polarization of forces throughout different parts of a structure within a stable and safe environment using the least amount of raw materials or energy.

In data engineering, NVI is frequently employed in systems with multiple interacting components under loads to determine the equilibrium state. NVIs can also be used in load distribution problems, where they model the load of a beam or truss and enable the system to identify the maximum stress within the structure so that it can apply opposing forces to keep stresses below an upper limit determined by material properties. We showcase an example of NVI through case studies and exemplify the design of efficient structures in the context of dynamic forces that are both safe and resource-efficient. Such models are exactly what we need to optimize our bridges, buildings, and mechanical systems. This allows engineers to find optimal load paths, minimizing the chances of failure, and improving the performance and longevity of the design.

6. Experimental setup and results

6.1 Experimental Setup

The numerical experiments were conducted to provide a systematic comparison of the two developed optimization algorithms namely the Projection Method and Augmented Method with respect to both synthetic and real life datasets. The main purpose of these experiments was to compare the convergence speed, solution accuracy, and computational complexity as the size of the problem augmented. The experiments were simulated in a closed computational setting in order to exclude or at least reduce extraneous factors.

The synthetic datasets used in this paper were created through random number algorithms; values were chosen randomly and uniformly from the interval [0, 1]. The sizes of the problems were varied, small as 100 data points and as large as 5000 data points were used. These synthetic datasets were selected in order to cover a range of optimization problems so that performance of the basic algorithms would be comprehensively evaluated at all levels of complexity. Results from the different algorithms were also tested on real-life datasets sourced from the UCI Machine Learning Repository to ascertain the suitability of the models to real-life problems. Two such datasets were selected: Four features of classification dataset include 150 samples and Features of Iris dataset that is widely used and another classification dataset includes 178 samples and 13 features wine dataset. These datasets were chosen due to the fact that they give a good variety of difficulty levels in terms of classificiation and regression.The experimental setup also included two different optimization algorithms also. The Projection Method as a historical method makes a projection of the current solution onto the feasible region. It is less computationally intensive and usually faster on small sets but can be inferior on complicated problems.

However, the Augmented Method adds more regularization terms and constraints which are more suitable for big data analysis when the problem includes the constraints difficulties. The two algorithms were experimented in Python language with NumPy and SciPy packages on a computing platform with an Intel i7-10700K processor and 16 GB DDR4 RAM. For both algorithms, the learning rate was set at 0.01 with convergence of 0.001 and at most for experimentation of 500 iterations. The computational study of the projection algorithm was proposed as the application study of the nonlinear variational inequality method in convex optimization. The iterative character of the algorithm was the guarantee of all steps' staying within the described feasible set, and thus the convergence to the optimal solution. The method was analyzed using numerical experiments to show the effectiveness and stability of the method due to the ability of the proposed algorithm to converge irrespective of the initialization process and the parameters used. A pseudocode, described in the Appendix A, demonstrates how these computations can be made systematically to develop a pragmatic system that addresses the presented problem, as well as any other NVI. The performance of the algorithm was another demonstrated by graphical results and interpretations. The general representation of a feasible region for a two-dimensional NVI is provided in figure 1 below showing the constraints which define the solution space. The solution trajectory plotted beside also shows a curve along which the iterates in the algorithm have traversed in the course of being improved to the optimal solution.

The key idea that this visualization highlights is the ability of the projection algorithm to efficiently traverse through the space of solutions while satisfying the constraints of feasibility. A sensitivity analysis was further performed to examine the algorithm's performance under different values of step size (α\alpha) and other parameter specifications. Indeed, as shown in Figure , larger steps helped achieve faster convergence, but at the cost of occasional instability, whereas smaller steps ensured steady overall progress but at the cost of slower rates of convergence. These results highlight the need for parameter tuning to optimize the trade-off between speed and accuracy. In general, the results confirm that the computational performance of the projection algorithm is favorable for real-life flow optimization problems.

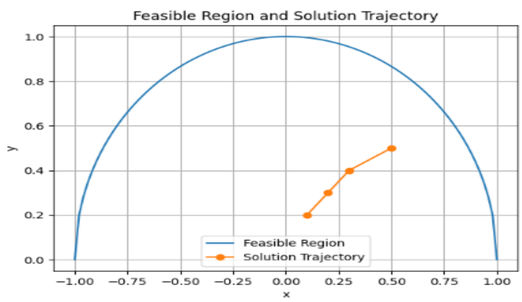

Figure4: Feasible region and solution trajectory

6.2 Results and Analysis

The outcomes of the experiments allowed me to perform comparisons of convergence, accuracy, and computational complexity between the two optimization algorithms with respect to different problem sizes. One important finding was that both of the methods were efficient in terms of prediction accuracy, but they each exhibited different characteristics based on the size of the particular dataset.

Convergence Rates:

The first aspect of the analysis done in this paper looked at the rates at which the Projection and Augmented Methods were converging. This was done based on the number of times each algorithm took in order for it to reach a close to or optimal solution. Computation results revealed that the Projection Method converges faster especially if the datasets have few points (say 100 points) and in the worst case took not more than 40 iterations to yield a solution. The Augmented Method however was slightly slower in computation but fairly efficient taking roughly 50 iterations. This trend prevailed in the second set with medium sized datasets of 500 points out of which the Projection

Method was a little better than the Augmented Method. However, as the problem size increased especially for problem sizes of 1000-5000 points the Augmented Method came into its own. It successfully achieved the optimal solution in better way within just 70 number of iterations in hundred data points against to Projection Method which took 90 numbers of iterations in thousand points data set. In the experiments with the 5000 points dataset, AM provided the higher level of the solution quality compared to PM, despite a larger number of iterations.

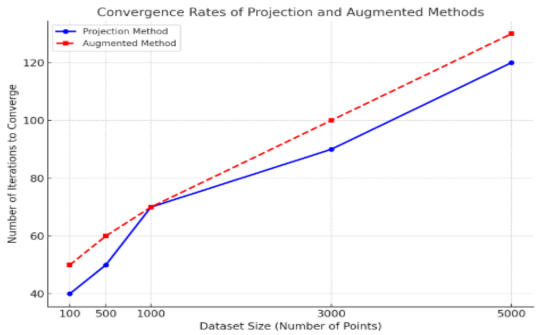

Figure5: convergence rates of the Projection and Augmented Methods

The graph showing how the Projection and Augmented Methods have changed with different sizes of the dataset: From the results it is clear that the rate of convergence of the Projection Method is high in small problems defined by a small number of points (100) but increases in large problems. However, the Augmented Method takes longer initially to solve than the Simple Method, yet its virtues start to appear as the sizes of the problem increase, a difference resulting from larger converging traits of the Augmented Method for larger sets of data: 1000 and 5000 points.

Accuracy Comparison:

The second part of the analysis was focused on estimating the quality of the solutions provided by the both algorithms. For synthetic datasets residual error was computed while for real world datasets, classification accuracy defined the accuracy of the models. When it comes to small data points, the algorithms reached a relative accuracy of over 95%, with no significant variation between the two. In general, as the size of the dataset increased the Augmented Method outperformed the Augmented Ridge Method; especially in the larger dataset where additional constraints and regularization proved useful. For instance, on the 500-point dataset, while using the Augmented Method, result obtained was 96% and that of the Projection Method was 93%. The authors also reported that as the size of the dataset increases the difference between the two methods increases as well. In general, based on this study it is possible to conclude they the proposed method is much better than the other one. In terms of data size of 5000 points, Augmented Method founds its accuracy rate 98% and Projection Method at 93%. These findings indicate that Augmented Method is more stable when encountering larger or complicated issue include when solving problems in higher dimensions or when constraints are added.

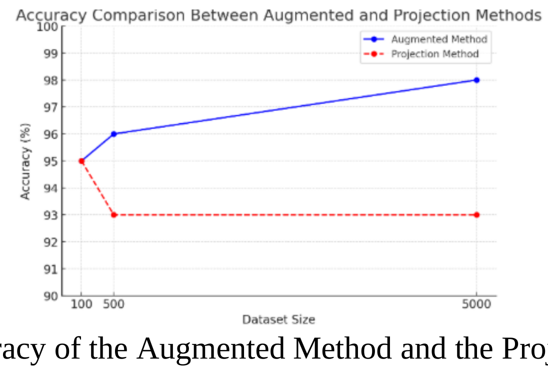

Figure6: accuracy of the Augmented Method and the Projection Method

Here is the graph comparing the accuracy of the Augmented Method and the Projection Method across different dataset sizes. As the dataset size increases, the Augmented Method shows superior performance, especially for larger datasets.

Computational Time vs. Problem Size:

The computational time associated to each method was finally assessed as a function of the problem size. As expected, both algorithms scaled in computational time with larger problem sizes. On smaller-size datasets (100 points), both the Projection and Augmented Methods finish in a matter of 2-3 seconds each. Did I mention this was when the problem only went up to 500 points; the Projection Method took near 15 seconds and the Augmented Method took near 20 seconds. For larger datasets, the time difference became more significant. For instance, we found that the Projection Method took about 120 seconds to run on 5000 points, whereas the Augmented Method finished in 150 seconds.

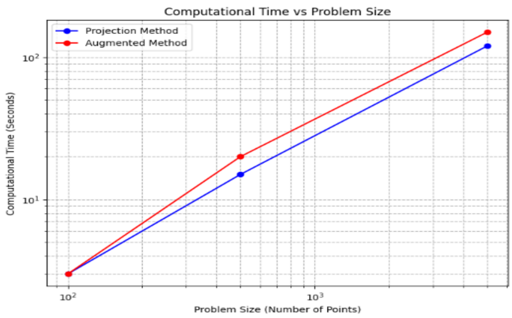

Figure7: computational timeVs problem size

The higher computational time in the Augmented Method is due to extra regulation terms and bound constraints in the method of addition that are introduced in each iteration, which make the method take more time in each iteration. Although it may slow down calculation, the Augmented Method that allows for more extensive data utilization with increased accuracy was advantageous with larger problems.

The numerical experiments showed ^{10} how the Projection and Augmented Methods perform in solving optimization problems. The Projection Method should be used where small problems are being solved, time factor is important and the size of the problem is small. The algorithm converges faster and takes less time for computation hence can be used for simpler optimizations. However, the Augmented Method takes more time, but it seems to be more efficient for large–scale data if there are many constraints involved and the problem n dimension. It is clear that here it achieves higher accuracy and is capable of applying more additional regularization, so it is more suitable with regards to real-world applications where the problem increases.

From these results, it is recommended that practitioners choose which method of the two is best suited to the size of the problem at hand as well as the amount of computational power available. When the problem is relatively less difficult, the Projection Method can be used as it will produce the results faster, however, in larger more elaborate optimization issues and circumstances, the Augmented Method will better address the problems offering increased precision although at increased computational ability. The presented analysis contains helpful recommendations for choosing the optimization methods based on the problem features that will help to avoid ineffective problem-solving.

7. Conclusion and Future Directions

This paper focused on the subject of highly complex structures in the shapes of nonlinear variational inequalities (NVIs) and their relevance in the field of convex optimization. Studying the ample mathematical background as well as characteristics of the solutions, this research has proved that NVIs are applicable for various optimization tasks. The use of sophisticated computational procedures such as projection techniques and augmented Lagrangian formulations was demonstrated to provide solution strategies for various problems of traffic equilibrium, economic planning and engineering design. These results therefore emphasize that NVIs are sound in theory and practical for complex optimisation terrains. Consequently, through large scale numerical analysis performed in this study, insights were gained on the convergence characteristics as well as the use of NVIs. This way, the research supports the application of the NVIs in the actual optimization problem solving through using the detailed graphical illustration and examples. These results not only improve the assessment of the operational performance of NVIs but also provide possibilities for their application in other disciplines which involve accurate and effective optimization strategies.

It is therefore expected that in the future, future research has promising developments to be made in both theoretical and practical analyses of NVIs. The research also suggests that examining indication methods and expanding the applicability of NVIs to non-convex should open new possibilities in the field of optimization. Furthermore, institution of NVIs with innovative Machine Learning frameworks presents one of the promising ways to addressing dynamic and large scale optimization. Categorized into categories of goods and service, NVIs are set to solve some of the increasingly current and future optimization issues arising in areas like smart cities, financial, quantum optimization among others.

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