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A New Extension of Combination of Linear and Bilinear Generating Relations in Function Spaces Associated with Hypergeometric Polynomials; Some New Applications of Two Signals in Special Functions

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Abstract:

For a certain class of generalized hypergeometric polynomials, we first derive some special cases on linear and bilinear generating functions and then apply these generating functions in order to reduce the corresponding results for the classical Jacobi, Hermite, Laguerre and Gegenbauer Polynomials, hypergeometric functions of Gauss and functions of Bessel and Kelvin. We also consider several linear generating functions for these polynomials as well as for some multivariable Jacobi and multivariable Laguerre polynomials which were investigated in recent years. Some of the linear and bilinear generating functions, presented in this paper, are associated with the hypergeometric polynomials.

Keywords: Generating functions, Jacobi, Hermite, Laguerre and Gegenbauer polynomials, etc.

1. Introduction:

In 1920, Humbert [1] has defined seven confluent forms of the Four Appell functions and denoted them by $\psi_1,\psi_2,\psi_3,\zeta_1,\zeta_2,\chi_1,\chi_2$ some of them are given below:

$$\zeta_1\left[\alpha;\beta;\gamma,r;a,b\right]=\sum_{f,g=0}^{\infty}\frac{(\alpha)_{f+g}(\beta)_f}{(\gamma)_f(r)_g}\cdot\frac{a^f}{f!}\cdot\frac{b^g}{g!} \tag{1}$$

$$\zeta_2\left[\alpha;\beta;\gamma;a,b\right]=\sum_{f,g=0}^{\infty}\frac{(\alpha)_{f+g}}{(\beta)_f(\gamma)_g}\cdot\frac{a^f}{f!}\cdot\frac{b^g}{g!} \tag{2}$$

Horn's (see [2]-[4]) defined the following terms:

$$H_1(\alpha,\beta,\gamma,r;s;a,b)=\sum_{f,g=0}^{\infty}\frac{(\alpha)_{f+g}(\beta)_f(\gamma)_g(r)_g}{(s)_f}.\frac{a^f}{f!}.\frac{b^g}{g!},(j+1)k=1 \tag{3}$$

$$H_2(\alpha,\beta;\gamma;a,b)=\sum_{f,g=0}^{\infty}\frac{(\alpha)_{2f+g}(\beta)_f}{(\gamma)_f(r)_g}.\frac{a^f}{f!}.\frac{b^g}{g!}\ 4j=(k-1)^2 \tag{4}$$

$$H_7(\alpha;\beta,\gamma;a,b)=\sum_{f,g=0}^{\infty}\frac{(\alpha)_{2f+g}}{(\beta)_f(\gamma)_g}.\frac{a^f}{f!}.\frac{b^g}{g!}\ 4|a|<1 \tag{5}$$

where the positive quantities j and k are the associated radii of absolute convergence of the concerned double power series $\sum_{f,g=0}^{\infty}B_{f,g}a^f.b^g$ such that $|a|<j,|b|<k$.

The parabolic cylindrical function $D_f(c)$ (see [5]; P. 117 (4) or (also see [6]), defined as the linear combination of confluent hypergeometric function 1^F1 is

$$D_f(c)=2^{\frac{m}{2}}\exp\left(-\frac{c^2}{4}\right)\left\{\frac{\sqrt{\left(\frac{1}{2}\right)}}{\sqrt{\frac{(1-f)}{2}}}.1^F1\left(-\frac{f}{2};\frac{1}{2};\frac{c^2}{2}\right)+\frac{c.\sqrt{\left(\frac{-1}{2}\right)}}{2^{\frac{1}{2}}.\sqrt{\left(\frac{-m}{2}\right)}}.1^F1\left(\frac{1-f}{2};\frac{c}{2};\frac{c^2}{2}\right)\right\} \tag{6}$$

where f is not an integer. Sometimes it is also known as Weber functions.

In 1893, Lauricella [7] defined the following terms in the following way

$$F_A^{(g)}(\alpha,\beta_1,\ldots,\beta_g;\gamma_1,\gamma_2,\ldots,r_g;a_1,a_2,\ldots,a_g)=$$

$$\sum_{f_1,f_2,\ldots,f_g=0}^{\infty}\frac{(\alpha)_{f_1+f_2+\ldots+f_g}(\beta_1)_{f_1}(\beta_g)_{f_g}}{(\gamma_1)_{f_1}\ldots(\gamma_g)_{f_g}}.\frac{a_1^{f_1}}{f_1!}\ldots\frac{a_g^{f_g}}{f_g!} \tag{7}$$

$$F_B^{(g)}(\alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g;\gamma;a_1,\ldots,a_g)=$$

$$\sum_{f_1,f_2,\ldots,f_g=0}^{\infty}\frac{(\alpha_1)_{f_1}\ldots(\alpha_g)_{f_g}(\beta_1)_{f_1}\ldots(\beta_g)_{f_g}}{(\gamma)_{f_1+\ldots+f_g}}.\frac{a_1^{f_1}}{f_1!}\ldots\frac{a_g^{f_g}}{f_g!} \tag{8}$$

$$F_C^{(g)}(\alpha,\beta;\gamma_1,\ldots,\gamma_g;a_1,\ldots,a_g)=$$

$$\sum_{f_1,f_2,\ldots,f_g=0}^{\infty}\frac{(\alpha)_{f_1+\ldots+f_g}(\beta)_{f_1+\ldots+f_g}}{(\gamma_1)_{f_1}\ldots(\gamma_g)_{f_g}}.\frac{a_1^{f_1}}{f_1!}\ldots\frac{a_g^{f_g}}{f_g!} \tag{9}$$

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and

$$F_D^{(g)}\left(\alpha,\beta_1,...,\beta_g;\gamma;a_1,...,a_g\right)=$$

$$\sum_{f_1,...,f_g=0}^{\infty}\frac{(\alpha)_{f_1+...+f_g}(\beta_1)_{f_1}...(\beta_g)_{f_g}}{(\gamma)_{f_1+...+f_g}}\cdot\frac{a_1^{f_1}}{f_1!}...\frac{a_g^{f_g}}{f_g!} \tag{10}$$

Let $X(a,x)$ be two variable functions having a formal power series expansion in x such that

$$X(a,x)=\sum_{g=0}^{\infty}\psi_g(a)x^g \tag{11}$$

where each member of the coefficient set $\left\{\psi_g(a)\right\}_{g=0}^{\infty}$ is independent of a and x. Then the expansion (7) of $X(a,x)$ is said to have generated the set $\left\{\psi_g(a)\right\}$ and $X(a,x)$ is known as generating function of the set $\left\{\psi_g(a)\right\}$.

The extended form of linear generating function is

$$X_1(a,x)=\sum_{g=0}^{\infty}\gamma_g\psi_g(a)x^g \tag{12}$$

where γ_g is a function of g that may contain the parameters of the set $\left\{\psi_g(a)\right\}$.

Generalized Rice and related polynomial:

The generalized Rice (see [8]) polynomial $H_g^{(\in,\theta_1)}(\ell,i,a)$ are defined as follows:

$$H_g^{(\in,\theta_1)}(\ell,i,a)=\frac{(1+\in)_g}{g!}\,{}_3F_2\left[\begin{matrix}-g, g+\in+\theta_1+1,\ell; \\ 1+\in, \quad i \quad ;\end{matrix}\ a\right] \tag{13}$$

$$\operatorname{Re}(\in)>-1,\ \operatorname{Re}(\theta_1)>-1$$

if $\in=\theta_1=0$, the equation (8) whose study was initiated by Khandekar, would reduce to ordinary Rice polynomial (see [8]);

$$H_g(\ell,i,a)={}_3F_2\left[\begin{matrix}-g,\ g+1,\ell; \\ 1, \quad i \quad ;\end{matrix}\ a\right] \tag{14}$$

<u>Jacobi Polynomials:</u>

Put $\ell=i$ and $a=\dfrac{1-a}{2}$ in equation (13) then it becomes Jacobi Polynomial $L_g^{(\in,\theta_1)}(a)$ (see [9], p. 254]

$$L_g^{(\in,\theta_1)}(a) = \frac{(1+\in)_g}{g!}\, {}_2F_1\left[\begin{matrix} -g, 1+\in+\theta_1+g; \\ 1+\in\,; \quad \dfrac{1-a}{2} \end{matrix}\right] \tag{15}$$

$$= \frac{(1+\in)_g}{g!}\left(\frac{a+1}{2}\right)^g {}_2F_1\left[\begin{matrix} -g, -\theta_1-g; \\ 1+\in\,; \quad \dfrac{a-1}{a+1} \end{matrix}\right] \tag{16}$$

where in each of the above equations $\mathrm{Re}(\in) > -1,\ \mathrm{Re}(\theta_1) > -1$ and g being a non-negative integer. An equivalent form of equation (16), given in Rainville (see [9], p. 255 [8])

$$L_g^{(\in,\theta_1)}(a) = \frac{(1+\theta_1)_g}{g!}\left(\frac{a-1}{2}\right)^g {}_2F_1\left[\begin{matrix} -g, -\in-g; \\ 1+\theta_1\,; \quad \dfrac{a+1}{a-1} \end{matrix}\right] \tag{17}$$

which is obtained by expanding the equation (16) in finite series form and then reversing the order of summation.

Legendre Polynomials:

Put $\in = \theta_1 = 0$ in equation (15), we get Legendre polynomial $L_g(a)$ given in Reinville [see [9]; p. 166 (2)] by

$$L_g(a) = {}_2F_1\left(-g,\ g+1;\ 1; \frac{1-a}{2}\right) \tag{18}$$

where

$$L_g(a) = \begin{cases} = \dfrac{\left(\frac{1}{2}\right)_g (2a)^g}{g!}\, {}_2F_1\left(\dfrac{-g}{2}, \dfrac{-g+1}{2}; \dfrac{-g}{2}; \dfrac{1}{a^2}\right) \\ = \left(\dfrac{a-1}{2}\right)^g {}_2F_1\left(-g; -g; 1; \dfrac{a+1}{a-1}\right) \\ = a^g\, {}_2F_1\left(\dfrac{-g}{2}, \dfrac{-g+1}{2}; 1; \dfrac{a^2-1}{a^2}\right) \end{cases} \tag{19}$$

The Legendre polynomial $L_g(a)$ of order g is generated by means of relation;

$$\sum_{g=0}^{\infty} L_g(a)\, x^g = \left(1-2ax+a^2\right)^{-\frac{1}{2}} \tag{20}$$

and has its series representation as follows:

$$L_g(a)=\sum_{g=0}^{\left[\frac{g}{2}\right]}\frac{(-1)^t(2g-2t)!\,a^{g-2t}}{2^g t!(g-t)!(g-2t)!}$$

where

$$\left[\frac{g}{2}\right]=\begin{cases}\frac{g}{2}, & if\ \ g\ is\ even\\ \frac{g-1}{2}, & if\ \ g\ is\ odd\end{cases}$$

Some works on linear and bilinear generating functions for hypergeometric polynomials have been done [10, 23]. In this note we obtain some generating functions for hypergeometric polynomials using some special techniques. It is the generalization of the results of author [19; pp. 133-134] for Hermite polynomial.

We consider the generating function for hypergeometric polynomials see [9; p. 244 (Ex. 7), p. 252 (17)]

$$\sum_{g=0}^{\infty}\theta_g(a).\frac{x^g}{g!}=e^x 0^F 1\left[\begin{matrix}-; & \\ 1; & \frac{-x(a-1)}{2}\end{matrix}\right]0^F 1\left[\begin{matrix}-; \frac{-x(a+1)}{2}\\ 1; \end{matrix}\right] \tag{21}$$

where $$\theta_g(a)=\sum_{t=0}^{g}\frac{(-1)^t.g!.L_t(a)}{(t!)^2.(g-t)!} \tag{22}$$

and $L_t(a)$ is the Legendre polynomials defined by the given equation (18).

Without loss of generating, we may assume that

$$\sum_{g=0}^{\infty}B(g)=\sum_{g=0}^{\infty}B(2g)+\sum_{g=0}^{\infty}B(2g+1) \tag{23}$$

and using result of author [20; p. 591 (1)]

$$\ell^F i\left[\begin{matrix}\alpha_1,\alpha_2,...,\alpha_\ell; \\ \beta_1,\beta_2,...,\beta_i;\end{matrix}\,a\right]=$$

$$2\ell^F\, 2i+1\left[\begin{matrix}\frac{\alpha_1}{2},\frac{\alpha_1+1}{2},\frac{\alpha_2}{2},\frac{\alpha_2+1}{2},...,\frac{\alpha_\ell}{2},\frac{\alpha_\ell+1}{2}; \\ \frac{1}{2},\frac{\beta_1}{2},\frac{\beta_1+1}{2},\frac{\beta_2}{2},\frac{\beta_2+1}{2},...,\frac{\beta_i}{2},\frac{\beta_i+1}{2};\end{matrix}\;4^{(\ell-i-1)}.a^2\right]+$$

$$\frac{\alpha_1.\alpha_2...\alpha_\ell}{\beta_1.\beta_2...\beta_i}2\ell^F\, 2i+1\left[\begin{matrix}\frac{\alpha_1+1}{2},\frac{\alpha_1+2}{2},\frac{\alpha_2+1}{2},\frac{\alpha_2+2}{2},...\frac{\alpha_\ell+1}{2},\frac{\alpha_\ell+2}{2}; \\ \frac{3}{2},\frac{\beta_1+1}{2},\frac{\beta_2+2}{2},\frac{\beta_2+1}{2},\frac{\beta_2+2}{2},...\frac{\beta_i+1}{2},\frac{\beta_i+2}{2};\end{matrix}\;4^{(\ell-i-1)}.a^2\right] \tag{24}$$

for generalized function in (21), replacing x by $i\,x$, separating real and imaginary parts and changing again x by $i\,x^{\frac{1}{2}}$, we obtain the generating functions for even and odd hypergeometric polynomials $\theta_{2g}(a)$ and $\theta_{2g+1}(a)$ in the form

$$\sum_{g=0}^{\infty}\theta_{2g}(a)\frac{x^{g}}{(2g)!}=\cosh\left(x^{\frac{1}{2}}\right)\left\{0^{F}3\left[\begin{matrix}\overline{\quad\quad};\\ \frac{1}{2},\frac{1}{2},1;\end{matrix}\ \frac{x(a-1)^{2}}{64}\right]\right.$$

$$0^{F}3\left[\begin{matrix}\overline{\quad\quad};\\ \frac{1}{2},\frac{1}{2},1;\end{matrix}\ \frac{x(a+1)^{2}}{64}\right]+\frac{\alpha\left(a^{2}-1\right)}{4}\ 0^{F}3\left[\begin{matrix}\overline{\quad\quad};\\ 1,\frac{3}{2},\frac{3}{2};\end{matrix}\ \frac{x(a-1)^{2}}{64}\right]$$

$$\left.0^{F}3\left[\begin{matrix}\overline{\quad\quad};\\ 1,\frac{3}{2},\frac{3}{2};\end{matrix}\ \frac{x(a+1)^{2}}{64}\right]\right\}-\frac{x^{\frac{1}{2}}}{2}.$$

$$\sinh\left(x^{\frac{1}{2}}\right)\left\{(a+1)0^{F}3\left[\begin{matrix}\overline{\quad\quad};\\ \frac{1}{2},\frac{1}{2},1;\end{matrix}\ \frac{x(a-1)^{2}}{64}\right]\right.$$

$$0^{F}3\left[\begin{matrix}\overline{\quad\quad};\\ \frac{1}{2},\frac{3}{2},\frac{3}{2};\end{matrix}\ \frac{x(a+1)^{2}}{64}\right]+(a-1)\ 0^{F}3\left[\begin{matrix}\overline{\quad\quad};\\ 1,\frac{3}{2},\frac{3}{2};\end{matrix}\ \frac{x(a-1)}{64}\right]$$

$$\left.0^{F}3\left[\begin{matrix}\overline{\quad\quad};\\ \frac{1}{2},\frac{1}{2},1;\end{matrix}\ \frac{x(a+1)^{2}}{64}\right]\right\} \tag{25}$$

and

$$\sum_{g=0}^{\infty}\theta_{2g+1}(a).\frac{x^{g}}{(2g+1)!}=\frac{\cosh\left(x^{\frac{1}{2}}\right)}{2}\left\{(1-a)0^{F}3\left[\begin{matrix}\overline{\quad\quad};\\ 1,\frac{3}{2},\frac{3}{2};\end{matrix}\ \frac{x(a-1)^{2}}{64}\right]\right.$$

$$0^{F}3\left[\begin{matrix}\overline{\quad\quad};\\ \frac{1}{2},\frac{1}{2},1;\end{matrix}\ \frac{x(a+1)^{2}}{64}\right]-(1+a)\ 0^{F}3\left[\begin{matrix}\overline{\quad\quad};\\ \frac{1}{2},\frac{1}{2},1;\end{matrix}\ \frac{x(a-1)^{2}}{64}\right]$$

$$\left.0^{F}3\left[\begin{matrix}\overline{\quad\quad};\\ 1,\frac{3}{2},\frac{3}{2};\end{matrix}\ \frac{x(a+1)^{2}}{64}\right]\right\}+\frac{\sinh\left(x^{\frac{1}{2}}\right)}{x^{\frac{1}{2}}}\left\{0^{F}3\left[\begin{matrix}\overline{\quad\quad};\\ \frac{1}{2},\frac{1}{2},1;\end{matrix}\ \frac{x(a-1)^{2}}{64}\right]\right.$$

$$0^{F}3\left[\begin{matrix}\overline{\quad\quad};\\ \frac{1}{2},\frac{1}{2},1;\end{matrix}\ \frac{x\left(a+1\right)^{2}}{64}\right]+\frac{x\left(a^{2}-1\right)}{4}0^{F}3\left[\begin{matrix}\overline{\quad\quad};\\ \frac{1}{2},\frac{3}{2},\frac{3}{2};\end{matrix}\ \frac{x\left(a-1\right)^{2}}{64}\right]$$

$$0^{F}3\left[\begin{matrix}\overline{\quad\quad};\\ 1,\frac{3}{2},\frac{3}{2};\end{matrix}\ \frac{x\left(a+1\right)^{2}}{64}\right]\Bigg\} \tag{26}$$

replace x by xk, multiplying both sides by $e^{-k}.k^{(\alpha-1)}$ and integrating with respect to k from 0 to ∞ in equation (18), we obtain new generating function

$$\sum_{g=0}^{\infty}\left(\alpha\right)_{g}\theta_{g}\left(a\right).\frac{x^{g}}{g!}=\left(1-x\right)^{-\alpha}.\zeta_{2}\left[\alpha;1,1;\frac{-x\left(a-1\right)}{2\left(1-x\right)},\frac{-x\left(a+1\right)}{2\left(1-x\right)}\right] \tag{27}$$

where ζ_2 is Humbert's function, given by equation (2).

Let us write the polynomial $P_g\left(a\right)$, given by see [9; p. 237 (26)]

$$P_{g}\left(a,x,\beta\right)=\left(2a\right)^{g}\ 3^{F}1\left[\begin{matrix}\frac{-g}{2},\frac{-g+1}{2},1+\alpha; & \frac{-1}{a^{2}}\\ 1+b; & \end{matrix}\right] \tag{28}$$

A generating function for $P_g\left(a,x,\beta\right)$ [see [9]; p. 239 (40)] is in the form

$$\sum_{g=0}^{\infty}P_{g}\left(a,x,\beta\right).\frac{x^{g}}{g!}=\ e^{2\alpha x}1^{F}1\left[\begin{matrix}1+\alpha;\\ 1+\beta;\end{matrix}-x^{2}\right] \tag{29}$$

Using the result (24) for exponential term in equation (29) and proceeding on the some lines as in the proofs of (25) and (26), we get

$$\sum_{g=0}^{\infty}P_{2g}\left(a,x,\beta\right).\frac{x^{g}}{\left(2g\right)!}=0^{F}1\left[\begin{matrix}\overline{\quad};\\ \frac{1}{2};\end{matrix}\ xa^{2}\right]1^{F}1\left[\begin{matrix}1+\alpha;\\ 1+\beta;\end{matrix}-x\right] \tag{30}$$

and

$$\sum_{g=0}^{\infty}P_{2g+1}\left(a,x,\beta\right).\frac{x^{g}}{\left(2g+1\right)!}=2a\ 0^{F}1\left[\begin{matrix}\overline{\quad};\\ \frac{3}{2};\end{matrix}\ xa^{2}\right]1^{F}1\left[\begin{matrix}1+\alpha;\\ 1+\beta;\end{matrix}-x\right] \tag{31}$$

respectively.

With the help of the equations (30) and (31), we obtain four generating functions in the form

$$\sum_{g=0}^{\infty} P_{4g}\left(a,x,\beta\right).\frac{x^{g}}{\left(4g\right)!}=0^{F}3\left[\begin{matrix}\overline{\qquad};\\ \frac{1}{2},\frac{1}{4},\frac{3}{4};\end{matrix}\ \frac{a^{4}x}{16}\right]2^{F}3\left[\begin{matrix}\frac{1+\alpha}{2},\frac{2+\alpha}{2}\quad ;\\ \frac{1}{2},\frac{1+\beta}{2},\frac{2+\beta}{2};\end{matrix}\ \frac{x}{4}\right]-$$

$$\frac{2.xa^{2}\left(1+\alpha\right)}{1+\beta}0^{F}3\left[\begin{matrix}\overline{\qquad};\\ \frac{3}{2},\frac{3}{4},\frac{5}{4};\end{matrix}\ \frac{xa^{4}}{16}\right]2^{F}3\left[\begin{matrix}\frac{2+\alpha}{2},\frac{3+\alpha}{2};\\ \frac{3}{2},\ \frac{2+\beta}{2}\ ;\end{matrix}\ \frac{x}{4}\right] \tag{32}$$

$$\sum_{g=0}^{\infty} P_{4g+2}\left(a,x,\beta\right)\frac{x^{g}}{\left(4g+2\right)!}=2a^{2}0^{F}3\left[\begin{matrix}\overline{\qquad};\\ \frac{3}{2},\frac{3}{4},\frac{5}{4};\end{matrix}\ \frac{xa^{4}}{16}\right]$$

$$2^{F}3\left[\begin{matrix}\frac{1+\alpha}{2},\frac{2+\alpha}{2}\quad ;\\ \frac{1}{2},\frac{1+\beta}{2},\frac{2+\beta}{2};\end{matrix}\ \frac{x}{4}\right]-\left(\frac{1+\alpha}{1+\beta}\right).$$

$$0^{F}3\left[\begin{matrix}\overline{\qquad};\\ \frac{1}{2},\frac{1}{4},\frac{3}{4};\end{matrix}\ \frac{xa^{4}}{16}\right]2^{F}3\left[\begin{matrix}\overline{\qquad\qquad\qquad};\\ \frac{3}{2},\frac{2+\beta}{2},\frac{3+\beta}{2};\end{matrix}\ \frac{x}{4}\right] \tag{33}$$

$$\sum_{g=0}^{\infty} P_{2g+1}\left(a,x,\beta\right).\frac{x^{g}}{\left(4g+1\right)!}=2a\ 0^{F}3\left[\begin{matrix}\overline{\qquad};\\ \frac{1}{2},\frac{3}{4},\frac{5}{4};\end{matrix}\ \frac{xa^{4}}{16}\right]2^{F}3\left[\begin{matrix}\frac{1+\alpha}{2},\frac{2+\alpha}{2}\quad ;\\ \frac{1}{2},\frac{1+\beta}{2},\frac{2+\beta}{2};\end{matrix}\ \frac{x}{4}\right]-$$

$$\frac{4xa^{3}\left(1+\alpha\right)}{3\left(1+\beta\right)}0^{F}3\left[\begin{matrix}\overline{\qquad};\\ \frac{3}{2},\frac{5}{4},\frac{7}{4};\end{matrix}\ \frac{xa^{4}}{16}\right]2^{F}3\left[\begin{matrix}\frac{2+\alpha}{2},\frac{3+\beta}{2}\quad ;\\ \frac{3}{2},\frac{2+\beta}{2},\frac{3+\beta}{2};\end{matrix}\ \frac{x}{4}\right] \tag{34}$$

and

$$\sum_{g=0}^{\infty} P_{2g+3}\left(a,x,\beta\right).\frac{x^{g}}{\left(4g+3\right)!}=\frac{4a^{3}}{3}\ 0^{F}3\left[\begin{matrix}\overline{\qquad};\\ \frac{3}{2},\frac{5}{4},\frac{7}{4};\end{matrix}\ \frac{a^{4}x}{16}\right]2^{F}3\left[\begin{matrix}\frac{1+\alpha}{2},\frac{2+\alpha}{2}\quad ;\\ \frac{1}{2},\frac{1+\beta}{2},+\frac{2+\beta}{2};\end{matrix}\ \frac{x}{4}\right]-$$

$$\frac{2a\left(1+\alpha\right)}{\left(1+\beta\right)}0^{F}3\left[\begin{matrix}\overline{\qquad};\\ \frac{1}{2},\frac{3}{4},\frac{5}{4};\end{matrix}\ \frac{xa^{4}}{16}\right]2^{F}3\left[\begin{matrix}\frac{2+\alpha}{2},\frac{3+\alpha}{2}\quad ;\\ \frac{3}{2},\frac{2+\beta}{2},\frac{3+\beta}{2};\end{matrix}\ \frac{x}{4}\right] \tag{35}$$

Using the Leplace transform technique as in the proof of (27), we obtain two more generating functions from (30) and (31) in the form

$$\sum_{g=0}^{\infty}(r)_g\, P_{2g}(a,x,\beta).\frac{x^g}{(2g)!}=\zeta_1\left[r;1+\alpha,1+\beta;\frac{1}{2};-x,xa^2\right] \tag{36}$$

and

$$\sum_{g=0}^{\infty}(r)_g\, P_{2g+1}(a,x,\beta).\frac{x^g}{(2g+1)!}=2a\,\zeta_1\left[r;1+\alpha,1+\beta;\frac{3}{2},-x,xa^2\right] \tag{37}$$

where ζ_1 is Humbert's function, defined by the equation (1).

Again replace x by xk^2, multiplying both sides by $e^{-k}.k^{\alpha-1}$, integrating with respect to k from 0 to ∞ and using the result of [21] (p. 146 (24)) in equation (18)

$$\int_0^{\infty} e^{-\ell x}.x^{\eta_1-1}.\exp\frac{-x^2}{8\eta_2}\,dx=\left|(\eta_1).2^{\eta_1}.\eta_2^{\frac{\eta_1}{2}}.\exp\left(\eta_2\ell^2\right).D_{-\eta_1}\left(2\ell\eta_2^{\frac{1}{2}}\right)\right. \tag{38}$$

where $\operatorname{Re}(\eta_1)>0,\ \operatorname{Re}(\eta_2)>0$,

$$\sum_{g=0}^{\infty}\frac{(\alpha)_{2g}}{2^g.g!}\eta_{2_g}(a)\,D_{-\alpha-2g}\left((2x)^{\frac{-1}{2}}\right)=$$

$$\frac{(8x)^{\frac{\alpha}{2}}}{2^{\alpha}}.\exp\left(\frac{-1}{8x}\right)F_4\left[\frac{\alpha}{2},\frac{\alpha+1}{2};1,1;-2x(a-1),-2x(a+1)\right] \tag{39}$$

where $D_g(a)$ is a parabolic cylindrical function, given by equation (6).

At last, in Mehler's generating relation (see [22]; p. 83-84 (13)]

$$\sum_{g=0}^{\infty}H_g(a)H_g(b)\frac{x^g}{g!}=\left(1-4x^2\right)^{\frac{-1}{2}}\exp.\left(\frac{4abx-4\left(a^2+b^2\right)x^2}{1-4x^2}\right) \tag{40}$$

Using equation (20) and equation (21), we get two generating functions for even and odd Hermite polynomials $H_{2g}(a)$ and $H_{2g+1}(a)$ in the form

$$\sum_{g=0}^{\infty}H_{2g}(a)H_{2g}(b).\frac{x^g}{(2g)!}=\frac{1}{\sqrt{1-4x}}0^F1\left[\begin{matrix}—;\\ \frac{1}{2};\end{matrix}\left(\frac{2ab}{1-4x}\right)^2x\right]$$

$$0^F1\left[\begin{matrix}—;\\ \frac{1}{2};\end{matrix}\left(\frac{2\left(a^2+b^2\right)x}{1-4x}\right)^2\right]-$$

$$\frac{4\left(a^2+b^2\right)x}{(1-4x)^{3/2}}\ {}_0F_1\left[\begin{matrix}\text{—};\\ \frac{1}{2}\ ;\end{matrix}\left(\frac{2ab}{1-4x}\right)^2 x\right]$$

$$ {}_0F_1\left[\begin{matrix}\text{—};\\ \frac{3}{2}\ ;\end{matrix}\left(\frac{2\left(a^2+b^2\right)x}{1-4x}\right)^2\right] \tag{41}$$

and

$$\sum_{g=0}^{\infty} H_{2g+1}(a)\ H_{2g+1}(b)\ \frac{x^g}{(2g+1)!}=\frac{4ab}{(1-4x)^{3/2}}\ {}_0F_1\left[\begin{matrix}\text{—};\\ \frac{3}{2}\ ;\end{matrix}\left(\frac{2ab}{1-4.x}\right)^2 x\right]$$

$$ {}_0F_1\left[\begin{matrix}\text{—};\\ \frac{1}{2}\ ;\end{matrix}\left(\frac{2\left(a^2+b^2\right)x}{1-4x}\right)^2\right]-$$

$$\frac{4ab\left(a^2+b^2\right)x}{(1-4x)^2}\ {}_0F_1\left[\begin{matrix}\text{—};\\ \frac{3}{2}\ ;\end{matrix}\left(\frac{2ab}{1-4x}\right)^2 x\right] {}_0F_1\left[\begin{matrix}\text{—};\\ \frac{3}{2}\ ;\end{matrix}\left(\frac{2\left(a^2+b^2\right)x}{1-4x}\right)^2\right] \tag{42}$$

respectively.

Special Cases:

Put $\beta = 1+2\alpha$ in equations (30) and (31), and using Kummer's second transform (see [9]; p. 126 (9)]

$${}_1F_1\left[\begin{matrix}\alpha\ ;\\ 2\alpha;\end{matrix}\ 2c\right]=e^c\ {}_0F_1\left[\begin{matrix}\text{——};\\ \alpha+\frac{1}{2}\ ;\end{matrix}\ \frac{c^2}{4}\right] \tag{43}$$

and using Laplace transform technique, we obtain

$$\sum_{g=0}^{\infty}(r)_g\ P_{2g}\left(a,\alpha,1+2\alpha\right)\frac{x^g}{(2g)!}=\left(\frac{2}{2+x}\right)^r\ H_7\left[r;\ \alpha+\frac{3}{2},\frac{1}{2};\frac{x^2}{4(2+x)^2},\frac{2xa^2}{2+x}\right] \tag{44}$$

and

$$\sum_{g=0}^{\infty}(r)_g\ P_{2g+1}\left(a,\alpha,1+2\alpha\right)\frac{x^g}{(2g+1)!}=$$

$$2a\left(\frac{2}{2+x}\right)^r\ H_7\left[r;\alpha+\frac{3}{2},\frac{3}{2};\frac{x^2}{4(2+x)^2},\frac{2ax^2}{2+x}\right] \tag{45}$$

respectively, where H_7 is a limit of H_4, defined by equation (5).

Put $\beta=\alpha$, a special case of $P_g\left(a,x,\beta\right)$ obtained from the equation (28) is the Hermite polynomial expressed by the equation (12).

Again, using above condition, each of the equation from (30) to (35) gives the generating relations for even and odd Hermite polynomials in the form

$$\sum_{g=0}^{\infty} H_{2g}\left(a\right)\frac{x^g}{(2g)!}=e^{-x}.{}_0F_1\left[\begin{matrix}\text{—};\\ \frac{1}{2};\end{matrix}\ a^2x\right]=e^{-x}\cosh\left(2a\sqrt{x}\right) \tag{46}$$

$$\sum_{g=0}^{\infty} H_{2g+1}\left(a\right)\frac{x^g}{(2g+1)!}=2ae^{-x}.\ {}_0F_1\left[\begin{matrix}\text{—};\\ \frac{3}{2};\end{matrix}\ a^2x\right]=\frac{e^{-x}}{\sqrt{x}}\sinh\left(2a\sqrt{x}\right) \tag{47}$$

$$\sum_{g=0}^{\infty} H_{4g}\left(a\right)\frac{x^g}{(4g)!}={}_0F_1\left[\begin{matrix}\text{—};\\ \frac{1}{2};\end{matrix}\ \frac{x}{4}\right]{}_0F_3\left[\begin{matrix}\text{———};\\ \frac{1}{2},\frac{1}{2},\frac{3}{4};\end{matrix}\ \frac{xa^4}{16}\right]$$
$$-3xa^2\,{}_0F_1\left[\begin{matrix}\text{—};\\ \frac{3}{2};\end{matrix}\ \frac{x}{4}\right]{}_0F_3\left[\begin{matrix}\text{———};\\ \frac{3}{2},\frac{3}{4},\frac{5}{4};\end{matrix}\ \frac{xa^4}{16}\right] \tag{48}$$

$$\sum_{g=0}^{\infty} H_{4g+2}\left(a\right)\frac{x^g}{(4g+2)!}=2a^2\ {}_0F_1\left[\begin{matrix}\text{—};\\ \frac{1}{2};\end{matrix}\ \frac{x}{4}\right]{}_0F_3\left[\begin{matrix}\text{———};\\ \frac{3}{2},\frac{3}{4},\frac{5}{4};\end{matrix}\ \frac{xa^4}{16}\right]$$
$$-{}_0F_1\left[\begin{matrix}\text{—};\\ \frac{3}{2};\end{matrix}\ \frac{x}{4}\right]{}_0F_3\left[\begin{matrix}\text{———};\\ \frac{1}{2},\frac{1}{4},\frac{3}{4};\end{matrix}\ \frac{xa^4}{16}\right] \tag{49}$$

$$\sum_{g=0}^{\infty} H_{4g+1}\left(a\right)\frac{x^g}{(4g+1)!}=2a\ {}_0F_1\left[\begin{matrix}\text{—};\\ \frac{1}{2};\end{matrix}\ \frac{x}{4}\right]{}_0F_3\left[\begin{matrix}\text{———};\\ \frac{1}{2},\frac{3}{4},\frac{5}{4};\end{matrix}\ \frac{xa^4}{16}\right]$$
$$-\frac{4xa^3}{3}\ {}_0F_1\left[\begin{matrix}\text{—};\\ \frac{3}{2};\end{matrix}\ \frac{x}{4}\right]\ {}_0F_3\left[\begin{matrix}\text{———};\\ \frac{3}{2},\frac{5}{4},\frac{7}{4};\end{matrix}\ \frac{xa^4}{16}\right] \tag{50}$$

and

$$\sum_{g=0}^{\infty} H_{4g+3}\left(a\right)\frac{x^g}{(4g+3)!}=\frac{4a^3}{3}\ {}_0F_1\left[\begin{matrix}\text{—};\\ \frac{1}{2};\end{matrix}\ \frac{x}{4}\right]{}_0F_3\left[\begin{matrix}\text{———};\\ \frac{3}{2},\frac{5}{4},\frac{7}{4};\end{matrix}\ \frac{xa^4}{16}\right]-$$

$$2a\ {}_0F_1\left[\begin{matrix}-; \\ \frac{3}{2};\end{matrix}\ \frac{x}{4}\right]\ {}_0F_3\left[\begin{matrix}\overline{\quad\quad}; \\ \frac{1}{2},\frac{3}{4},\frac{5}{4};\end{matrix}\ \frac{xa^4}{16}\right] \tag{51}$$

which are the required generating functions of author [19; pp. 133-134].

Conclusion:

In this research note we obtain very special cases of the result of the author [19] with the help of the motivated works [1-22].

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