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On Locally Invo-Regular Ring

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Abstract:

An associative ring with identity D claimed to be locally Invo- regular (L.I.Reg.Rings) if , for any element ƒ in D , either ƒ or $1-$ ƒ is Invo-regular in D , that is ƒ $=$ ƒ v ƒ or $1-$ ƒ $=$ $(1-$ ƒ$)$ v $(1-$ ƒ$)$ for some involution element v in D , these rings due to Danchev [5] . In this article several instances, and properties of L.I.Reg.Rings are introduced and invest gules the behavior of these properties , as well as certain relation between L.I.Reg.Rings and other rings are discussed.

Keywords: Invo-regular ring, local ring, locally Invo-Regular Rings.

1. Introduction

Every where the current article, all rings are assumed to be associative, having the identity element 1, in general differs from the zero element let D be a ring , then an element ƒ $\in D$ is regular if ƒ $\in$ ƒDƒ [1] . D is referred to unit-regular if , for each $\tau \in D$ there is $\mu \in U(D)$ such that τ = τ μ τ [2] , [3] , [4] .A ring D is called Invo-Regular if , for each $\tau \in D$, there exists $v \in Invo.(D)$ such that $\tau = \tau v \tau$. and $\tau = v$ę , where ę $= v\tau \in Idem.(D)$ [5]. A ring D is said to clean if each ж $\in D$ can be expressed as ж $= \mu +$ ę ,where $\mu \in U(D)$ and ę $\in Idem.(D)$ [6]. A ring D is called a local ring if it has exactly one maximal ideal [7]. A ring D is said to be tripotent if any its element satisfies the equation ж^3 = ж. [8], [9].

U(D), Idem(D), Reg.(D), Invo. (D), C(D) and J(D) stands for the set of all unit elements (the group of units), all idempotent elements, a set of regular elements, the set of all involutions (square roots of units), central of D. and the Jacobson radical of D respectively.

2. L.I. Reg.Rings

This paragraph about certain properties over L.I.Reg.Rings. Danchev in [5] introduced the following basic notion.

Definition 2.1.

A ring D is claimed locally Invo-Regular (L.I.Reg.Ring) if , for any ƒ $\in D$, either ƒ or $1-$ƒ is Invo-Regular element in D (either ƒ $\in$ Invo.Reg.(D) or $1-$ ƒ $\in$ Invo.Reg.(D)) , that is either ƒ $=$ ƒ v ƒ or $1-$ ƒ $= (1-$ƒ$)$ v $(1-$ƒ$)$ for some $v \in$ Invo.(D).

Examples:

- Boolean rings.

- The field Z_3 and $Z_3 \times Z_3 \times \dots . \times Z_3$ or $Z_3 \times Z_3 \times \dots . \times Z_3 \times$ ....
- Z_4 is L.I.Reg.Ring but $Z_4 \times Z_4 \times$ .... is not L.I.Reg.Rings.

Definition 2.2.

An ideal I is claimed Invo – Regular if for each element ℓ in I, there exists an element $v \in$ Invo. (D) , such that $\ell = \ell\ v\ \ell$ .The following characteristic of L.I.Reg.Rings.

Proposition 2.3.

If D is L.I.Reg.Ring , then for each element ƒ in D we have:

1. ƒ = ƒẹ , where ẹ $\in Idem.(D)$.
2. When D is abelian ring, ƒ is tripotent element, if D of characteristic 3.

Proof: 1- Let ƒ $\in D$, then either ƒ = ƒ v ƒ or $(1-$ ƒ$) = (1-$ ƒ$)\ v\ (1-$ ƒ$)$, if ƒ = ƒ v ƒ , so $(v$ ƒ$)^2 = v$ ƒ, that is vƒ $\in Idem.(D)$, when $(1-$ ƒ$) = (1-$ ƒ$)\ v\ (1-$ ƒ$)$, we have $(v(1-$ ƒ$))^2 = v\ (1-$ ƒ$)$, so $v\ (1-$ ƒ$) \in Idem.(D)$.

2- If ƒ = ƒ v ƒ , then ƒ = ƒ$^2\ v$, implies that ƒ^2 = ƒ$^3 v$ and so ƒ$^2 v$ = ƒ$^3\ v$, thus ƒ = ƒ^3 .Now, if $(1-$ ƒ$) = (1-$ƒ$)\ v\ (1-$ ƒ$)$, then $(1-$ ƒ$) = (1-$ƒ$)^3 = 1 - 3$ƒ $+ 3$ƒ$^2 -$ ƒ^3. since 3 = 0, then $1-$ ƒ $= 1 -$ ƒ^3 , consequently, ƒ = ƒ^3 . therefore D is tripotent.

Proposition 2.4.

Every surjective homeomorphic image of L.I.Reg.Ring is also L.I.Reg.Ring.

Proof: let $g: D \longrightarrow S$ homomorphic function from L.I.Reg.Ring D onto ring S.let ж, ¥ ∈ S, then, there exists ƒ ∈ D,$v_1 \in$ Invo.(D), such that $g($ƒ$)$ = ж, $g(v_1)$ = ¥, to prove ¥ ∈ Invo(S), let $v_1 \in$ Invo.(D) , we must prove $(g(v_1))^2 = 1$, $(g(v_1))^2 = g^2(v_1) = g(v_1^2) = g(1) = 1_s$, since D is L.I.Reg.Ring then , either ƒ or $(1-$ƒ$)$ is Invo-regular in D .If ƒ is Invo-regular element , then there exists $v_1 \in$ Invo.(D) , such that ƒ = ƒ v_1 ƒ . Now ж = $g($ƒ$) = g($ƒ v_1 ƒ$)$ = $g($ƒ$)\ g(v_1)\ g($ƒ$)$ = ж ¥ ж, ¥ ∈ Invo. (S). if $(1-$ƒ$)$ is Invo -regular element. then there exists $v_2 \in$ Invo.(D) , such that $(1-$ƒ$) = (1-$ƒ$)v_2(1-$ƒ$)$, $g(1-$ƒ$) = g\ [(1-$ƒ$)v_2(1-$ƒ$)]$ = $g(1-$ ƒ$)\ g(v_2)\ g(1-$ ƒ$) = (g(1) - g($ƒ$))\ g(v_2)\ (g(1) - g($ƒ$))$ = $(1_s -$ ж$)$ ꭒ $(1 -$ ж$)$, where ꭒ= $g(v_2) \in$ Invo.(S).

Proposition 2.5.

Let I be an ideal of a ring D, if D is L.I.Reg.Ring , then $D\ /\ I$ is L.I. Reg.Ring.

Proof: Let ƒ $\in D$. then ƒ $+ I \in D\ /\ I$, since D is L.I.Reg.Ring , then either ƒ or$(1-$ƒ$) \in$ Invo.Reg. (D) in D . If ƒ ∈ Invo.Reg. (D), then there exists $v_1 \in$ Invo. (D) , such $that$ ƒ = ƒ v_1 ƒ . Now, ƒ $+ I$ = ƒ v_1 ƒ $+ I = ($ƒ $+ I)\ (v_1 + I)($ƒ $+ I)$, that is ƒ $+ I$ is Invo. (D/I) . If $(1-$ƒ$) \in$ Invo.Reg.(D) , there exists $v_2 \in$ Invo.(D) , such that $(1-$ƒ$) = (1-$ƒ$)\ v_2\ (1-$ƒ$)$, thus $(1-$ƒ$) + I = \left((1-$ƒ$)v_2\ (1-$ ƒ$)\right) + I = \left((1-$ ƒ$) + I\right)(v_2 + I)\ ((1-$ ƒ$) + I)$. Hence $((1-$ ƒ$) + I) \in$ Invo.Reg $(D\ /\ I)$, therefore $D\ /\ I$ is L.I.Reg.Ring.

Example:

Let $D = Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be L.I.Reg.Ring and let I be an ideal in D, $I = (2) = \{0, 2, 4, 6\}$, Invo. $(Z_8) = \{1, 3, 5, 7\}$, $D/I = Z_8/(2) = \{0 + (2), 1 + (2)\}$ D / I is L.I.Reg.Ring.

Proposition 2.6.

Let I be an Invo-regular ideal in a ring D,then D is L.I.Reg.Ring if, D / I is L.I.Reg.Ring.

Proof: Let D / I is L.I.Reg.Ring and $ƒ \in D$ then , either $ƒ + I$ or $(1 - ƒ) + I$ is Invo-regular in D/I .$ƒ + I = (ƒ + I)(v_1 + I)(ƒ + I)$ for some $v_1 \in$ Invo.(D) , that is $ƒ - ƒ v_1 ƒ \in I$. or $((1 - ƒ) + I) = ((1 - ƒ) + I)(v_2 + I)((1 - ƒ) + I)$, that is $(1 - ƒ) - (1 - ƒ)v_2(1 - ƒ) \in I$ for some $v_2 \in$ Invo.(D) , since I is Invo-regular then , either $(ƒ - ƒ v_1 ƒ) \in$ Invo.Reg.(D) or $((1 - ƒ) - (1 - ƒ) v_2 (1 - ƒ)) \in$ Invo.Reg.(D) for some $v_1, v_2 \in$ Invo.(D). hence either $ƒ$ or$(1 - ƒ)$ is Invo - regular in D. therefore D is L.I.Reg.Ring.

Proposition 2.7.

If D is L.I.Reg.Ring , then $ȩDȩ$ is L.I.Reg.Ring for central idempotent element ȩ in D.

Proof: Let $ƒ \in ȩDȩ$ then, either $ƒ$ or $1 - ƒ \in$ Invo.Reg.(D) . If $ƒ$ is regular, then there exists $v_1 \in$ Invo(D),such that $ƒ = ƒ v_1 ƒ$ and $ȩƒȩ = ȩƒv_1ƒȩ = ƒ ȩ v_1 ȩ ƒ = ƒ (ȩ v_1 ȩ) ƒ$, Follows that $ƒ \in$ Invo.Reg.$(ȩDȩ)$when $(1 - ƒ) \in$ Invo.Reg.(D),we have $(1 - ƒ) = (1 - ƒ)v_2(1 - ƒ)$ Where $v_2 \in$ Invo.(D) and $ȩ(1 - ƒ)ȩ = ȩ (1 - ƒ)v_2(1 - ƒ)ȩ = ȩ. ȩ (1 - ƒ) v_2(1 - ƒ) ȩ. ȩ = ȩ (1 - ƒ) ȩ v_2 ȩ (1 - ƒ) ȩ$, hence $(1 - ƒ) \in$ Invo.Reg.$(ȩDȩ)$, therefore .$ȩDȩ$ is L. I.Reg.ring.

Corollary 2.8.

If D is L.I.Reg.Ring, then $(1 - ȩ)D(1 - ȩ)$ is L.I.Reg.Ring for every central idempotent ȩ in D.

Proof: Since D is L.I.Reg.Ring, then $ƒ$ or $1 - ƒ \in$ Invo.Reg.(D) , if $ƒ \in$ Invo.Reg.(D), then there exist $v_1 \in$ Invo.(D), such that $ƒ = ƒv_1ƒ$, $(1 - ȩ) ƒ(1 - ȩ) = (1 - ȩ) ƒv_1ƒ (1 - ȩ)$, since $(1 - ȩ) \in Idem.(D)$ then , $(1 - ȩ)$ is central then , $((1 - ȩ) ƒ (1 - ȩ) = ƒ(1 - ȩ)v_1(1 - ȩ)ƒ$, hence $ƒ \in$ Invo. Reg.$(1 - ȩ)D(1 - ȩ)$, if $(1 - ƒ) \in$ Invo.Reg.(D), then there exists $v_2 \in$ Invo.(D), Let $ƒ \in ȩDȩ$ then, either $ƒ$ or $1 - ƒ \in$ Invo.Reg.(D). If $ƒ$ is regular, then there exists $v_1 \in$ Invo(D),such that $ƒ = ƒ v_1 ƒ$ and $ȩƒȩ = ȩƒv_1ƒȩ = ƒ ȩ v_1 ȩ ƒ = ƒ (ȩ v_1 ȩ) ƒ$, Follows that $ƒ \in$ Invo.Reg.$(ȩDȩ)$when $(1 - ƒ) \in$ Invo.Reg.(D),we have $(1 - ƒ) = (1 - ƒ)v_2(1 - ƒ)$ Where $v_2 \in$ Invo.(D) and $ȩ(1 - ƒ)ȩ = ȩ (1 - ƒ)v_2(1 - ƒ)ȩ = ȩ. ȩ (1 - ƒ) v_2(1 - ƒ) ȩ. ȩ = ȩ (1 - ƒ) ȩ v_2 ȩ (1 - ƒ) ȩ$, hence $(1 - ƒ) \in$ Invo.Reg.$(ȩDȩ)$, therefore .$ȩDȩ$ is L. I.Reg.ring. such that $(1 - ƒ) = (1 - ƒ) v_2(1 - ƒ)$ so$(1 - ȩ) (1 - ƒ) (1 - ȩ) = (1 - ȩ) (1 - ƒ) v_2(1 - ƒ) (1 - ȩ) = (1 - ȩ)(1 - ȩ) (1 - ƒ)v_2(1 - ƒ)(1 - ȩ)(1 - ȩ) = \big((1 - ȩ)(1 - ƒ)\big)((1 - ȩ)v_2(1 - ȩ) \big((1 - ƒ)(1 - ȩ)\big)$ Now (1-ȩ) $- ƒ = [(1 - ȩ) - ƒ] (1 - ȩ) v_2(1 - ȩ) [(1 - ȩ) - ƒ]$, therefore, $(1 - ȩ)D(1 - ȩ)$ is L.I.Reg.Ring.

Proposition 2.9.[5]

The center of L.I.Reg.Ring, is also L.I.Reg.

Proof: Assume $ɀ \in C(D)$, we write $ɀ = ɀ v_1 ɀ$ or $1 - ɀ = (1 - ɀ)\, v_2\, (1 - ɀ)$,for some v_1 , $v_2 \in$ Invo.(D) , If $ɀ = ɀ v_1 ɀ$ it follows that $ɀ = ɀ^2\, v_1$, Hence $ɀ^2 = ɀ^3\, v_1$, and so $ɀ = ɀ^3\, v_1^2 = ɀ^3$. thus $ɀ$ is a tripotent element and hence it can be written as $ɀ = ɀ\,(1 + ɀ - ɀ^2)\, ɀ$, where $1 + ɀ - ɀ^2$ is an Invo.$C(D)$, that is $(1 + ɀ - ɀ^2)^2 = 1$, when $(1 - ɀ) \in C(D)$, we write $(1 - ɀ) = (1 - ɀ)\, v_2\, (1 - ɀ)$ for some $v_2 \in$ Invo.(D) , Now , $(1 - ɀ) = (1 - ɀ)^2\, v_2$, whence $(1 - ɀ)^2 = (1 - ɀ)^3\, v_2$, and so $(1 - ɀ) = (1 - ɀ)^3\, v_2^2 = (1 - ɀ)^3$. Thus $(1 - ɀ)$ is tripotent element and hence it can be written as $(1 - ɀ) = (1 - ɀ)(1 + ɀ - ɀ^2)(1 - ɀ)$, where $(1 + ɀ - ɀ^2) \in$ Invo. $(C(D))$ Where $(1 + ɀ - ɀ^2)^2 = 1$. Therefore $C(D)$ is L.I.Reg.

Definition 2.10.

Let $(D, +, .)$ be any ring , a function $g(x) = \sum_{i=0}^{\infty} a_i ж^i = a_0 ж^0 + a_1 ж^1 + a_2 ж^2$ +……+$a_n ж^n$ + ….. , where $a_0, a_1, a_2, \ldots . a_n, \ldots \in D$ and the powers $0, 1, 2, \ldots . n, \ldots$ are non negative integers , then $g(x)$ is called a polynomial .

Definition 2.11.[10]

The polynomial $p(ж) = a_n ж^n + a_{n-1} ж^{n-1} + \cdots a_1 ж + a_0$ is called monic if $a_n = 1$, where a_n is the leading coefficient of the term with the highest degree n .

Definition 2.12.

A ring D is said to be integral over a subring S is for each $\boldsymbol{r} \in D$, there is a monic polynomial $g(ж) \in S[ж]$ such that $g(r) = 0$.

Proposition 2.13.

Let S be a subring of D such that D is integral over S , If $s \in S$, with $s \in$ Invo.Reg.(D), then $s \in$ Invo.Reg.(S).

Proof: By assumption (D) is integral over S , there exists $g(x) = ж^n + a_{n-1} ж^{n-1} + a_{n-2} ж^{n-2} + \cdots + a_0 \in$ S[ж] such that $g(s) = 0$, If we multiply the equation $g(s) = 0$ by s^n and use the fact that $s = s^3 = ss^2 = s(s^3 v)$, for $v \in$ Invo. (D) , then we deduce that $(ss^2)^n \in S$ or $s\,(s^3 v) \in S$, but $(s^3 v)^2 = (s^3 v)(s^3 v) = s^6 v = s^6 = s^3 . s^3 = s . s = s^2 = s^3 v$, so $(s^3 v) \in S$. multiply the equation $g(s) = 0$ by s^{n-1} this yields that $s^{n-1}(s)^n \in S$ but $s^{n-1}\,(s)^n = (s(s^3 v))^{n-1}\, s^n = (\,s\,(\,s^3 v\,)\,)\, s = s$. Hence $s^3 v \in$ Idem. (D) , therefore $s \in$ Invo.Reg. (S).

Example: Let $D = Z_3 = \{0, 1, 2\}$ and $g(x) = ж^3 + 2ж$, $g(0) = 0$, $g(1) = 3 = 0, g(2) = 12 = 0$.

Proposition 2.14.

Let D be a commutative ring, then $\{0, 1\}$ + Invo. (D)= Invo. $(D) \cup (1 +$ Invo. $(D)) \subseteq$ L.I.Reg.(D).

Proof: Since $(1 - ғ) \in$ Invo.Reg. (D) , iff $(ғ - 1) \in$ Invo.Reg.(D) , follows that $ғ \in 1 +$ Invo.Reg.(D) , we have L.I.Reg. (D) = Invo.Reg. $(D) \cup$ (1 + Invo.Reg. (D)) , and this is true , since Invo. $(D) \subseteq$ Invo.Reg.(D) .

Example: Let$D = Z_4 = \{0,1,2,3\}$,Invo. $(Z_4) = \{1, 3\}$, Now $\{0,1\} + \{1,3\} = \{1,3\} \cup (1 + \{1,3\}) = \{1,3\} \cup \{0,2\}$, $\{0,1,2,3\} = \{0,1,2,3\} \subseteq$ Invo.Reg.(D) .

Proposition 2.15.

Let D be abelian ring with $2 \in U(D)$, Then every $ғ \in$ Invo.Reg.(D) can be written as a sum of two invertable elements in D.

Proof: Let $ғ \in$ Invo.Reg. (D) , then $ғ = ғ\, v_1\, ғ$ for some $v_1 \in$ Invo.(D) , that is $ғ\, v^2 = ғ$, also $ғ\, v$ = ȩ $\in$ $Idem.(D)$ and $v_1 = v_1^{-1} \in U(D)$,then v_1ȩ = $v_1^{-1}(v_1\, ғ) = ғ$, since $(2ȩ - 1)^2 = 4ȩ^2 - 4ȩ + 1 = 1$, then we can assume $v_1 = 2ȩ - 1$ with ${v_1}^2 = 1$, hence ȩ $= 2^{-1}\, v_1 + 2^{-1}$, thus $ғ = v_1\, ȩ = v_1\, (2^{-1}\, v_1 + 2^{-1}) = 2^{-1} v_1 . v_1 + 2^{-1} . v_1 = 2^{-1} + 2^{-1}\, v_1$ If $(1 - ғ) \in$ Invo.Reg.(D) , then $(1 - ғ) = (1 - ғ)\, v_2\, (1 - ғ)$ for some $v_2 \in$ Invo.(D), that is $. v_2 = (1 - ғ)$, since also , $(1 - ғ)\, v_2 =$ ȩ $\in$ Idem.(D) and $v_2 = v_2^{-1} \in U(D)$, thus v_2 ȩ $= {v_2}^{-1}\, (1 - ғ)\quad (1 - ғ)\, v_2 = {v_2}^{-1}\, v_2\, (1 - ғ) = (1 - ғ)$ And $(2ȩ - 1)^2 = 4ȩ^2 - 4ȩ + 1 = 1$. Suppose that $v_2 = 2ȩ - 1$ with ${v_2}^2 = 1$, hence ȩ $= 2^{-1} v_2 + 2^{-1}$ thus $(1 - ғ) = v_2\, ȩ = v_2\, (2^{-1} v_2 + 2^{-1}) = 2^{-1} v_2 . v_2 + 2^{-1} . v_2 = 2^{-1} + 2^{-1} v_2$.

Proposition 2.16.

Every element of abelian L.I.Reg.Ring can be written as a sum of three units , if $2 \in U(D)$.
Proof: Let $ғ \in D$, then by proposition 2.13 we have $ғ = ж + (1 + ¥)$ for some ж , ¥ $\in$ Invo.Reg.(D) , Now by proposition 2.14 , we have ж $= 2^{-1} + 2^{-1}\, v_1$, ¥ $= 2^{-1} + 2^{-1}\, v_2$, where $v_1 , v_2 \in$ Invo.(D), $ғ = 2^{-1} + 2^{-1} v_1 + (1 + 2^{-1} + 2^{-1} v_2)$, hence $ғ = 2^{-1}\, (1 + v_1) + 2^{-1}\, (1 + v_2) + 1$. therefore $ғ$ is a sum of three units elements of D.

Example: Let $D = Z_3 = \{0 , 1 , 2\}$ is L.I.Reg.Ring, $U(Z_3) = \{1,2\}$, $ғ = 0 = 1 + 1 + 1$, $ғ = 1 = 1 + 1 + 2$, $ғ = 2 = 2 + 2 + 1$.

3. The relation between L. I.Reg.Rings and other rings

In this part we introduce some relation between L.I.Reg.Rings and local rings , Invo.Reg.Rings and clean rings. Now, we will give the main results.

Proposition 3.1.

Every local ring is L. I.Reg.Ring when $U(D)$ = Invo. (D).

Proof: Since D is local , then either $ғ$ or $1 - ғ$ has unit , If $ғ$ has unit , then there exist $b \in D$, such that $ғ . b = 1$, follows to $ғ = ғ\, b\, ғ$, $b \in U(D)$ = Invo.(D) , $b \in$ Invo.(D) . If $(1 - ғ)$ has unit then there exist $c \in D$, such that $(1 - ғ)c = 1$, that is $(1 - ғ)\, c\, (1 - ғ) = (1 - ғ)$, since $c \in U(D) =$ Invo.(D) , then $c \in$ Invo.(D) . therefore D is L.I.Reg.(D).

Proposition 3.2.

Let D be abelian ring and P is primitive ideal, then D is local ring, if $D \,/\, P$ is L.I.Reg.Ring.

Proof: $Let\ ƒ + P \in D/P$ for all $ƒ \in D$, since D/P is L.I.Reg.Ring , then there exists $v \in$ Invo.(D) , such that either $ƒ + P = (ƒ + P)(v_1 + P)(ƒ + P)\ or\ (1-ƒ) + P = ((1-ƒ) + P)(v_2 + P)((1-ƒ)$ +P), for the first case , that is $ƒ + P = (ƒ + P)(v_1 + P)(ƒ + P)$ = $(ƒ\, v_1\, ƒ) + P$, implies that $(ƒ - ƒ\, v_1\, ƒ) \in P$, that is $(1 - v_1\, ƒ) \in P$. Assume that $ƒ \notin P$, then $(1-v_1ƒ)^n \in P$such that $(1-v_1\, ƒ)^n$= $\mathbf{1} - [\ \sum_{k=0}^{n} (-1)^k\, (K)^n\, (1)^{n-k}\, (v_1\, ƒ)^k\ ]$. $ƒ \in P$, where $C_k^n = \frac{n!}{k!\,(n-k)!}$. Now assume that $ꝁ = [\ \sum_{k=0}^{n}(-1)^k\, (K)^n\, (1)^{n-k}\, (v_1\, ƒ)^k\ ]$. then $1 - ꝁ\, ƒ \in$ P. implies that $1 + P = (ꝁ + P)(ƒ + P)$. Hence $(ƒ + P)$ is invertable. Now, if $(1-ƒ) + P \in D/P$ for all $ƒ \in D$, Since D/P is L.I.Reg.Ring, then there exists $v_2 \in$ Invo.(D) such that $(1-ƒ) + P = \big((1-ƒ) + P\big)(v_2 + P)\big((1-ƒ) + P\big), (1-ƒ)\ +P$ = $[\,(1-ƒ)\, v_2\, (1-ƒ)\,] + P = (1-ƒ) - ((1-ƒ)\, v_2\, (1-ƒ)) \in$ P,$(1-ƒ)\,(1 - v_2(1-ƒ)) \in P$. Assume that $(1-ƒ) \notin P$, then $(1 - v_2\,(1-ƒ))^n \in P$ such that $(1 - v_2\,(1-ƒ))^n$= $1 - [\ \sum_{k=0}^{n} (-1)^k\, (K)^n\, (1)^{n-k}\, (v_2(1-ƒ)^k)\,]$. $(1-ƒ) \in P$. Let $Ş = [\sum_{k=0}^{n} (-1)^k\, (K)^n\, (1)^{n-k}\, (v_2(1-ƒ))^k$. then $1 - S(1-ƒ) \in P, 1 + P = (S + P)((1-ƒ) + P)$. Hence $((1-ƒ) + P)$ is invertable. Thus $D \,/\, P$ is divition ring, that is P is maximal ideal and unique. Therefore,D is local ring.

Proposition 3.3.

Every Invo.Reg.Ring is locally Invo.Reg.Ring.

Proof: By definition of Invo.Reg.Ring (we say D is Invo.Reg.Ring if for each $ƒ \in D$ there exists $v \in$ Invo.(D) such that $ƒ = ƒ\, v\, ƒ$) . Then every Invo.Reg.Ring is locally Invo.Reg.Ring.

Example:

We say D is Invo.Reg.Ring if for each $ƒ \in D$ there exists $v \in$ Invo.(D) such that $ƒ = ƒ\, v\, ƒ$. Let $D = Z_6 = \{0,1,2,3,4,5\}$, $U(Z_6)$ =Invo$(Z_6) = \{1,5\}$, $1 = 1(1)1$, $2 = 2\,(5)2$, $3 = 3\,(1)\,3$, $4 = 4\,(1)\,4$, $5 = 5\,(5)\,5$, Z_6 is Invo.Reg.Ring , Z_6 is L.I.Reg.Ring , since for each $ƒ \in D$ there exists $v \in$ Invo.(D) such that , either $ƒ = ƒ\, v\, ƒ$ or $(1-ƒ) = (1-ƒ)\, v\, (1-ƒ)$

Proposition 3.4.

Let D be a ring without zero divisors. then D is Invo.Reg.Ring , if D is L.I.Reg.Ring

Proof: Since D is L.I.Reg.Ring , then either $ƒ$ or $1-ƒ$ is Invo.Reg. in D . if $ƒ$ is Invo.Reg.element , then there exists $v_1 \in$ Invo.(D) such that $ƒ = ƒ\, v_1\, ƒ = ƒ - ƒ\, v_1\, ƒ = 0 = ƒ\,(1 - ƒ\, v_1) = 0$, $ƒ \neq 0$, $ƒ\, v_1 = 1$. Now if $(1-ƒ)$ is Invo.Reg.element , then there exists $v_2 \in$ Invo.(D)such that $(1-ƒ) = (1-ƒ)v_2\,(1-ƒ) = (1-ƒ) - (1-ƒ)\, v_2\,(1-ƒ) = 0 = (1-ƒ)[\,1 - v_2(1-ƒ)\,]$, $(1-ƒ) \neq 0$, $(1-ƒ)\, v_2 = 1$. D is Invo.Reg.Ring .

Example:

Let $D = Z_5 = \{0,1,2,3,4\}$ is L.I.Reg.Ring,for each $ғ \in Z_5$ either $ғ = ғ\, v_1\, ғ$ or $(1 - ғ) = (1 - ғ)v_2(1 - ғ)$, for some $v_1, v_2 \in$ Invo.(Z_5), $U(Z_5)$ = Invo.$(Z_5) = \{1,2,3,4\}$, $1 = 1\,(1)1$, $2 = 2(3)2$, $3 = 3(2)3$, $4 = 4\,(4)\,4$, Z_5 is L.I.Reg.Ring and Invo.Reg.Ring .

Proposition 3.5.

Let D be a belian L.I.Reg.Ring , then D is clean ring.

Proof: By [11] if $ғ \in D$, then $either\, ғ = vę$ or $ғ = 1 + vę$. If $ғ = vę$, $ғ = (vę + ę - 1) + (1 - ę)$ where $(vę + ę - 1)$has Invo such that $(vę + ę - 1)(vę + ę - 1) = 1$, $v^2ę^2 + vę^2 - vę + vę^2 + ę^2 - ę - vę - ę + 1 = 1$, $where\, (vę + ę - 1) \in U(D)$ and $(1 - ę)^2 = (1 - ę)(1 - ę) = 1 - ę - ę + ę^2 = 1 - 2ę + ę = 1 - ę \in Idem.(D)$. If $ғ = 1 + vę = 1 + vę - ę + ę = (1 + vę - ę) + ę$ where $(1 + vę - ę)^2 = (1 + vę - ę)(1 + vę - ę) = 1 + vę - ę + vę + v^2ę^2 - vę^2 - ę - vę^2 + ę^2 = 1 \in$ Invo. $(D) \subseteq U(D)$ and $ę \in$ Idem. (D) , $ғ$ is clean element. D is a clean ring.

Example:

$D = Z_6 = \{0,1,2,3,4,5\}$, $U(Z_6) = \{1, 5\}$, $Idem.(D) = \{0, 1, 3, 4\}$, $2 = 2\,(5)\,2$, $3 = 3\,(1)\,3$, $4 = 4\,(1)\,4$, $5 = 5(5)5$. $Z_6$is abelian L.I.Reg.Ring .We say Z_6 is clean ring if each element in Z_6 can be written as $(\mu + ę)$Where $\mu \in U(Z_6)$ and $e \in Idem.(Z_6)$. $2 = 1 + 1$, $3 = 5 + 4 = 9 = 3$, $4 = 1 + 3$, $5 = 1 + 4$. Therefore Z_6 is clean ring

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