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Coincidence Point Theorem on Parametric Cone Metric Space for AJ's Pair

S.Dhivya

Assistant Professor, PG & Research Department of Mathematics, Kandaswami Kandar’s College, Velur, Namakkal (Dt), Tamilnadu, India

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Abstract:

The article is based on the coincidence point theory especially in parametric cone metric space. Some coincidence point theorems of contractive mappings are proved by omitting the normality conditions.

Keywords: Fixed point, Coincidence point, cone metric space, parametric cone-metric space.

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1. Introduction

Huang and Zhang [2] generalized the concept of a metric space by incorporating vectorvalued metrics in an ordered real Banach space. Fixed point theory is a fundamental tool in nonlinear analysis, with numerous applications. In 2012, Azam [8] introduced the concept of coincidence points for mappings and obtained relevant results under a contractive condition in metric spaces. Moreover, Huang and Zhang [2] introduced cone-metric spaces, in which interior points were investigated using partial orders and normality conditions. In the present work, we exclude the assumption of normality conditions in parametric spaces, as introduced by V. Subash and M. Angayarkanni [11], and apply the resulting framework to the study of coincidence points.

2. Prelinimaries

Definition 2.1. [4] A neighborhood V of e is symmetric if $\mathrm{V}-1=\mathrm{V}$. If W is any neighborhood V of e such that $\mathrm{V} * \mathrm{V} \subset \mathrm{H}$

Definition 2.2. [2] Let (X, d) be a cone metric space and P be a cone with non empty interior. Suppose $\mathrm{f}, \mathrm{g}: \mathrm{X} \rightarrow \mathrm{X}$ are such that the range of g contains the range of f and $\mathrm{f}(\mathrm{X})$ or $\mathrm{g}(\mathrm{X})$ is a complete subspace of X. The pair (f, g) is AJ’s pair.

Definition 2.3. [8] Let E be a Banach Space with norm $||.||$ and $\mathcal{A}$ be a subset of E. The $\mathcal{A}$is said to be a cone if it satisfies

- $\mathcal{A}$ is closed, non empty and $\mathcal{A} \neq \theta$, where θ is a zero vector in $\mathcal{A}$.
- If $\lambda, \mu \geq 0$ and $\mathrm{p}, \mathrm{q} \in \mathcal{A}$ then $\mathrm{p}+\mu \mathrm{q} \in \mathcal{A}$.
- If $\mathrm{p},-\mathrm{p} \in \mathcal{A}$ then $\mathrm{p}=\theta$.

3. Main Result

Proposition 3.1.

If $c \in \operatorname{int}\mathcal{P}$, $0 \le a_n$ and $a_n \to 0$ then there exists a number such that for all $n > n_0$ we have $a_n << c$.

Proof:

Given $0 << c$, choose a symmetric neighbourhood $\mathcal{V}$ such that $c + \mathcal{V} \subset \mathcal{P}$. As $a_n \to 0$, there is a n_0 such that $a_n \in \mathcal{V} = -\mathcal{V}$ for $n > n_0 \Rightarrow c + a_n \subset c + \mathcal{V} \subset \mathcal{P}, n > n_0 \Rightarrow a_n << c$.

Proposition 3.2.

Let $0 << c$. If $0 \le d(a_n, a, t) \le b_n$ and $b_n \to 0$ then eventually $d(a_n, a, t) << c$, where a_n, a are sequence and given point in $\mathcal{X}$, parametric cone metric space.

Proof:

Let $0 << c$. By Proposition 3.1, we have $b_n << c$. But $d(a_n, a, t) \le b_n << c \implies d(a_n, a, t) << c$.

Proposition 3.3.

If E is a real Banach space with cone $\mathcal{P}$ and if $a \le \mu a$ where $a \in \mathcal{P}$ and $0 < \mu < 1$ then $a = 0$.

Proof:

We have $a \le \mu a$ Then $\mu a - a \in \mathcal{P} \Rightarrow (\mu - 1)a \in \mathcal{P} \Rightarrow -(1 - \mu)a \in \mathcal{P} \Rightarrow (1 - \mu)a \in -\mathcal{P}$. Also $a \in \mathcal{P}$ and $1 - \mu > 0$ then $(1 - \mu)a \in \mathcal{P}$. Thus $(1 - \mu)a \in \mathcal{P} \cap -\mathcal{P} = \{0\}$. $\therefore a = 0$

Theorem 3.1.

Suppose that (f, g) is AJ's pair and that for some constant $\mu \in (0,1]$ and for every $a, b \in \mathcal{X}$, a PCMS, there exists

$$u = u(a, b, c) \in \left\{d(ga, gb, t), d(f\,a, ga, t), d(f\,b, gb, t), \frac{d(f\,a, gb, t) + d(f\,b, ga, t)}{2}\right\} \ldots (3.1)$$

such that

$$d(f\,a, f\,b, t) \le \mu u \qquad \ldots\ldots.. (3.2)$$

Then (f, g) have unique coincidence point in $\mathcal{X}$.

Proof: Choose $a_0, a_1 \in \mathcal{X}$, satisfying that $ga_1 = f\,a_0 = b_0$. By this $a_n, a_{n+1} \in \mathcal{X}$ such that $ga_{n+1} = f\,a_n = b_n$. We claim that,

$$d(b_n, b_{n+1}, t) \le \mu d(b_{n-1}, b_n, t), n \ge 1 \qquad \ldots\ldots\ldots (3.3)$$

By Equation 3.2, $d(b_n, b_{n+1}, t) = d(f\,a_n, f\,a_{n+1}, t)$

$\le \mu u$ $\ldots\ldots\ldots.$ (3.4)

where

$$u \in \left\{d(ga_n, ga_{n+1}, t), d(f\,a_n, ga_n, t), d(f\,a_{n+1}, ga_{n+1}, t), \frac{d(f\,a_n, ga_{n+1}, t) + d(f\,a_{n+1}, ga_n, t)}{2}\right\}$$

$u \in \left\{d(b_n, b_{n-1}, t), d(b_{n+1}, b_n, t), \frac{d(b_{n+1}, b_{n-1}, t)}{2}\right\}$ ………… (3.5)

Now consider,

Case (i): If $u = d(b_{n-1}, b_n, t)$ then $d(b_n, b_{n+1}, t) \leq \mu d(b_{n-1}, b_n, t)$ holds.

Case (ii): If $u = d(b_n, b_{n+1}, t)$ then

$$d(b_n, b_{n+1}, t) \leq d(b_n, b_{n+1}, t), \mu \in (0,1]$$

$\Rightarrow d(b_n, b_{n+1}, t) = 0$; By Proposition (3.3)

and so Equation (3.3) holds.

Case (iii): If $u = d(b_{n-1}, b_{n+1}, t)$ then

$d(b_n, b_{n+1}, t) \leq \frac{\mu}{2} d(b_{n-1}, b_{n+1}, t)$

$$\leq \frac{\mu}{2} \; [d(b_{n-1}, b_n, t) + d(b_n , b_{n+1}, t)]$$

$$\leq \frac{\mu}{2} (1 - \mu\, 2)^{-1} d(b_{n-1}, b_n, t)$$

$< \mu d(b_{n-1}, b_n, t)$

and so Equation (3.3) holds.

By iteration, $d(b_n, b_{n+1}, t) \leq \mu^2 d(b_{n-2}, b_{n-1}, t)$

$\leq \mu^3 d(b_{n-3}, b_{n-2}, t)$

$\vdots$

$\leq \mu^n d(b_0, b_1, t)$

Now we assert that $\{b_n\}$ is Cauchy sequence. For $n > m$ we have,

$d(b_n, b_m, t) \leq d\left(b_n, b_{(n-1)}, t\right) + d(b_{n-1}, b_m, t)$

$\leq d(b_n, b_{n-1}, t) + d(b_{n-1}, b_{n-2}, t) + \cdots + d(b_{m+1}, b_m, t)$

$\leq [\mu^{n-1} + \mu^{n-2} + \cdots + \mu^m] d(b_0, b_1, t)$

$= \mu^m \frac{1}{1 - \mu} d(b_0, b_1, t) \to 0$ as $n \to \infty$

$\Rightarrow d(b_n, b_m, t) << c$.

Thus $\{b_n\}$ is Cauchy in P C M - metric space.

Since $f(\mathcal{X}) \subseteq g(\mathcal{X})$ and $g(\mathcal{X})$ is complete, there exists a $q \in g(\mathcal{X})$ such that $g(a_n) \to q$ as $n \to \infty$.

To find: $p \in \mathcal{X}$ such that $gp = q = f p$ we have $gp = q$ where $p \in \mathcal{X}$.

Now, $d(f p, q, t) \leq d(fp, fa_n, t) + d(fa_n, q, t)$

$\leq \mu u + d(fa_n, a, t)$

where $u \in \left\{d(ga_n, gp, t), d(fa_n, ga_n, t), d(fp, gp, t, \frac{d(fa_n,gp,t)+d(f p,ga_n,t)}{2}\right\}$

Let $0 \ll c$, for infinitely many n,

Case (i): If $u = d(ga_n, gp, t)$ then,

$d(gp, q, t) \quad \leq \mu d(ga_n, gp, t) + d(fa_n, q, t)$

$\leq \mu d(ga_n, q, t) + d(fa_n, q, t)$

$\ll \mu \frac{c}{2\mu} + \frac{c}{2}$

$d(f p, q, t) \ll c$

Case (ii): If $u = d(fa_n, ga_n, t)$ then

$d(fp, q, t) \leq \mu d(fa_n, ga_n, t) + d(fa_n, q, t)$

$\leq (\mu + 1)d(fa_n, q, t) + \mu d(q, ga_n, t)$

$\ll c$

Case (iii): If $u = d(fp, gp, t)$ then

$d(fp, q, t) \leq \mu d(fp, gp, t) + d(fa_n, q, t)$

$\leq \frac{1}{1 - \mu} d(fa_n, q, t)$

$\ll c$

Case (iv): If $u = \frac{d(fa_n,gp,t)+d(fp,ga_n,t)}{2}$ then

$$d(f p, q, t) \leq \mu \left\{\frac{d(fa_n,gp,t)+d(fp,ga_n,t)}{2}\right\} + d(fa_n, q, t)$$

$\leq \left\{\frac{2 + \mu}{2 - \mu}\right\} d(fa_n, q, t)$

$\ll c$

$\therefore d(fp, q, t) = 0 \Rightarrow fp = q$

Hence f and g have a coincidence point $p \in \mathcal{X}$ and a point of coincidence $q \in \mathcal{X}$ with $fp = gp = q$.

Uniqueness: Let q_1 be another point of coincidence then there is $p_1 \in \mathcal{X}$ with $q_1 = fp_1 = gp_1$.

Now, $d(q, q_1, t) = d(f p, fp_1, t)$

$\leq \mu u$

Where $u \in \left\{d(gp, gp_1, t), d(f p, gp, t), d(fp_1, gp_1, t), \frac{d(f p,gp_1,t)+d(fp_1,gp,t)}{2}\right\}$

$u \in \{0, d(q, q_1, t)\}$

$\therefore d(q, q_1, t) \leq \mu d(q, q_1, t)$

$\Rightarrow\ q = q_1$

Thus q is the unique point of coincidence for f and g.

Theorem 3.2.

Suppose that (f, g) is AJ's pair and for some constant $\mu \in (0,1]$; for every $a, b \in \mathcal{X}$ there exists $u = u(a, b, t) \in \left\{ d(ga, gb, t), \frac{d(fa,ga,t)+d(fb,gb,t)}{2}, \frac{d(fa,gb,t)+d(fb,ga,t)}{2} \right\}$ such that $d(f a, f b, t) \le \mu u$. Then f, g have a unique coincidence point in $\mathcal{X}$.

Proof:

Let $a_0, a_1 \in \mathcal{X}$. Then $ga_1 = fa_0 = b_0$.

In general, if $a_{n+1} \in \mathcal{X}$ then we can have $ga_{n+1} = fa_n = b_n$.

To Prove: $d(b_n, b_{n+1}, t) \le d(b_{n-1}, b_n, t), n \ge 1$

Now $d(b_n, b_{n+1}, t) = d(fa_n, fa_{n+1}, t) \le \mu u$

Where $u \in \left\{ d\left(ga_n, ga_{n+1}, t\right), \frac{d(fa_n,ga_n,t)+d(fa_{n+1},ga_{n+1},t)}{2}, \frac{d(fa_n,ga_{n+1},t)+d(fa_{n+1},ga_n,t)}{2} \right\}$

$\Rightarrow\ u \in \left\{ d(b_{n-1}, b_n, t), \frac{d(b_n,b_{n-1},t)+d(b_{n+1},b_n,t)}{2}, \frac{d(b_{n+1},b_{n-1},t)}{2} \right\}$

Case(i): If $u = d(b_{n-1}, b_n, t)$ then, $d(b_n, b_{n+1}, t) \le \mu d(b_{n-1}, b_n, t)$ holds.

Case(ii): If $u = \frac{d(b_n,b_{n-1},t)+d(b_{n+1},b_n,t)}{2}$ then

$$d(b_n, b_{n+1}, t) \le \mu \frac{d(b_n,b_{n-1},t)+d(b_{n+1},b_n,t)}{2}$$

$$\left(1 - \frac{\mu}{2}\right) d(b_n, b_{n+1}, t) \le d(b_n, b_{n-1}, t)$$

$$d(b_n, b_{n+1}, t) \le \left(\frac{\mu}{2 - \mu}\right) d(b_n, b_{n-1}, t)$$

$\le \mu d(b_n, b_{n-1}, t)$ holds

Case(iii): If $u = \frac{d(b_{n+1},b_{n-1},t)}{2}$ then $d(b_n, b_{n+1}, t) \le \mu \frac{d(b_{n+1},b_{n-1},t)}{2}$

$$\left(1 - \frac{\mu}{2}\right) d(b_n, b_{n+1}, t) \le \frac{\mu}{2} d(b_n, b_{n-1}, t)$$

$\le \mu d(b_n, b_{n-1}, t)$ holds.

By theorem (3.1), $\{b_n\}$ is a Cauchy sequence.

As f, g is a AJ's pair, there exists a $q \in g(\mathcal{X})$ $p \in \mathcal{X}$ such that $g(x_n) \to q$ as $n \to \infty$ and $gp = q$.

To prove: $fp = gp = q$

Now consider $d(fp, q, t) \le d(fp, f a_n, t) + d(fa_n, q, t)$

$\le \mu u + d(fa_n, q, t)$

where u $\in \left\{ d(ga_n, gp, t), \frac{d(fa_n,ga_n,t)+d(fp,gp,t)}{2}, \frac{d(fa_n,gp,t)+d(fp,ga_n,t)}{2} \right\}$

Let c $>>$ 0 for infinitely many n, we have

Case (i): If $u = d(ga_n, gp, t)$ then $d(fp, q, t) \leq \mu d(ga_n, gp, t) + d(fa_n, q, t) << \mu$

Case (ii): If $u = \frac{d(fa_n,ga_n,t)+d(fp,gp,t)}{2}$ then

$$d(fp, q, t) \leq \mu \left(\frac{d(fa_n, ga_n, t) + d(fp, gp, t)}{2} \right) + d(fa_n, q, t)$$

$$= \mu \left(\frac{d(fa_n,q,t)+d(fp,q,t)}{2} \right) + d(fa_n, q, t)$$

$$= \left(\frac{\mu}{2} + 1 \right) d(fa_n, q, t) + \frac{\mu}{2}\, d(fp, q, t)$$

$$\Rightarrow d(fp, q, t)[1 - \mu\ 2\,] \leq \left(\frac{\mu + 2}{2} \right) d(fa_n, q, t)$$

$$\Rightarrow d(f\,p, q, t) \leq \left(\frac{\mu+2}{2-\mu} \right) d(fa_n, q, t)$$

$$\ll \left(\frac{\mu + 2)}{2 - \mu} \right) \left(\frac{2-\mu}{\mu + 2} \right) c$$

$$\Rightarrow d(f\,p, q, t) << c.$$

Case (iii): If $u = \frac{d(fa_n,gp,t)+d(fp,ga_n,t)}{2}$ then

$$d(f\,p, q, t) \leq \mu \left(\frac{d(fa_n,gp,t)+d(fp,ga_n,t)}{2} \right) + d(fa_n, q, t)$$

$$= \frac{\mu}{2} d(fa_n, gp, t) + \frac{\mu}{2} d(fp, ga_n, t) + d(fa_n, q, t)$$

$$\leq \frac{(2 + \mu)}{2} d(fa_n, q, t) + \frac{\mu}{2} d(fp, q, t) + \frac{\mu}{2} d(q, ga_n, t)$$

$$d(fp, q, t) << c$$

$\therefore d(f\,p, q, t) << c$ for every $c \in \operatorname{int} \mathcal{P}$

$\therefore fp = q.$

Thus q is the point of coincidence.

Uniqueness: Let q_1 be another point of coincidence with $fp_1 = gp_1 = q_1$.

To prove: $= q_1$.

Consider $d(q, q_1, t) \leq \mu u \Rightarrow d(fp, fp_1, t) \leq \mu u$

where $u \in \left\{ d(gp, gp_1, t), \frac{d(fp,gp,t)+d(fp_1,gp_1,t)}{2}, \frac{d(fp,gp_1,t)+d(fp_1,gp,t)}{2} \right\}$

$$u \in \left\{ d(q, q_1, t), \frac{d(q, q, t) + d(q, q_1, t)}{2}, \frac{d(q, q_1, t) + d(q_1, q, t)}{2} \right\}$$

$u \in \{0, d(q, q_1, t)\}$

If $u = 0$, then $d(q, q_1, t) \leq 0$ (absurd)

If $u = d(q, q_1, t)$ then $d(q, q_1, t) \leq \mu d(q, q_1, t)$

$(1 - \mu) d(q, q1, t) \leq 0$

$1 - \mu \leq 0$ (Impossible)

$d(q, q_1, t) = 0.$

$\therefore q = q_1$

Corollary 3.1. Suppose that (f, g) is AJ's pair and for some constant $\mu \in (0,1]$ and for every $x, y \in \mathcal{X}$, $d(fa, fb, t) \leq \mu d(ga, gb, t)$ then f, g have a unique coincidence point.

Proof: Choose $a_0 \in \mathcal{X}$. Then $ga_{n+1} = fa_n = b_n$.

Now $d(b_n, b_{n+1}, t) = d(fa_n, fa_{n+1}, t) \leq \mu d(ga_n, ga_{n+1}, t) = \mu d(b_{n-1}, b_n, t)$.

From theorem (3.1), $\{b_n\}$ is a Cauchy and has a unique coincidence point.

By the theorem (3.2) the following corollaries get true.

Corollary 3.2. Suppose that (f, g) is AJ's pair and for some constant $\mu \in (0,1]$ and for every $a, b \in \mathcal{X}$; $d(fa, fb, t) \leq \mu \left[\frac{d(fa,ga,t)+d(fb,gb,t)}{2}\right]$ then f, g have unique coincidence point.

Corollary 3.3. Suppose that (f, g) is AJ's pair and for some constant $\mu \in (0,1]$ and for every $a, b \in \mathcal{X}$; $d(fa, fb, t) \leq \mu \left[\frac{d(fa,gb,t)+d(fb,ga,t)}{2}\right]$ then f, g have unique coincidence point.

Corollary 3.4. Suppose that (f, g) is AJ's pair and for some constant $\mu \in (0,1]$ and for every $, b \in \mathcal{X}$; there exists $u = u(a, b, t) \in \{d(ga, gb, t), d(fa, ga, t), d(fb, gb, t)\}$ such that $d(fa, fb, t) \leq \mu u$ then f, g have unique coincidence point.

Corollary 3.5. Suppose that (f, g) is AJ's pair and for some constant $\mu \in (0,1]$ and for every $a, b \in \mathcal{X}$; there exists $u = u(a, b, t) \in \{d(fa, ga, t), d(fb, gb, t)\}$ such that $d(fa, fb, t) \leq u$ then f, g have unique coincidence point

Theorem 3.3. Suppose that (f, g) is AJ's pair and there exists non-negative constants ai satisfying $\sum_{i=1}^{5} a_i < 1$ such that for each $a, b \in \mathcal{X}$;

$d(fa, fb, t) \leq a_1 d(ga, gb, t) + a_2 d(ga, fa, t) + a_3 d(gb, fb, t) + a_4 d(ga, fb, t) + a_5 d(gb, fa, t)$. Then f, g have a unique coincidence point in $\mathcal{X}$.

Proof: Choose $a_0, a_1 \in \mathcal{X}$ such that $ga_1 = fa_0 = b_0$.Then a_(n + 1) $\in \mathcal{X}$ for $a_n \in \mathcal{X}$ such that $ga_{n+1} = fa_n = b_n$.

Now we assert that, $d(b_n, b_{n+1}, t) \leq \mu d(b_n, b_{n+1}, t)$; $\mu \in (0,1], n \leq 1$.

Consider,

$d(b_n, b_{n+1}, t) = d(fa_n, fa_{n+1}, t) \leq a_1 d(ga_n, ga_{n+1}, t) + a_2 d(ga_n, fa_n, t)$

$$+a_3 d(ga_{n+1}, fa_{n+1}, t) + a_4 d(ga_n, fa_{n+1}, t) + a_5 d(ga_{n+1}, fa_n, t)$$

$$= a_1 d(b_{n-1}, b_n, t) + a_2 d(b_{n-1}, b_n, t) + a_3 d(b_n, b_{n+1}, t)$$

$$+a_4 d(b_{n-1}, b_{n+1}, t) + a_5 d(b_n, b_n, t)$$

$$= (a_1 + a_2) d(b_{n-1}, b_n, t) + a_3 d(b_n, b_{n+1}, t) + a_4 d(b_{n-1}, b_{n+1}, t) \quad .. (3.6)$$

$$d(b_{n+1}, b_n, t) = d(fa_{n+1}, fa_n, t)$$

$$\leq a_1 d(ga_{n+1}, ga_n, t) + a_2 d(ga_{n+1}, fa_{n+1}, t) + a_3 d(ga_n, fa_n, t)$$

$$+a_4 d(ga_{n+1}, fa_n, t) + a_5 d(ga_n, fa_{n+1}, t)$$

$$= a_1 d(b_n, b_{n-1}, t) + a_2 d(b_n, b_{n+1}, t) + a_3 d(b_{n-1}, b_n, t)$$

$$+a_4 d(b_n, b_n, t) + a_5 d(b_{n-1}, b_{n+1}, t)$$

$$= (a_1 + a_3) d(b_n, b_{n-1}, t) + a_2 d(b_n, b_{n+1}, t) + a_5 d(b_{n-1}, b_{n+1}, t) \quad ..(3.7)$$

From Equations (3.6) and (3.7),

$$2d(b_n, b_{n+1}, t) \leq (2a_1 + a_2 + a_3) d(b_n, b_{n-1}, t) + (a_2 + a_3) d(b_n, b_{n+1}, t)$$

$$+ (a_4 + a_5) d(b_n, b_{n+1}, t)$$

$$\leq (2a_1 + a_2 + a_3) d(b_n, b_{n-1}, t) + (a_2 + a_3) d(b_n, b_{n+1}, t)$$

$$+ (a_4 + a_5) d(b_{n-1}, b_n, t) + d(b_n, b_{n+1}, t)$$

$$= (2a_1 + a_2 + a_3 + a_4 + a_5) d(b_n, b_{n-1}, t) + (a_2 + a_3 + a_4 + a_5) d(b_n, b_{n+1}, t) d(b_n, b_{n+1}, t)$$

$$\leq \left[\frac{(2a1 + a2 + a3 + a4 + a5)}{(2 - a2 - a3 - a4 - a5)}\right] d(b_n, b_{n-1}, t)$$

$$< d(b_n, b_{n-1}, t)$$

$$= \mu d(b_n, b_{n-1}, t)$$

where $\mu = 1 \in (0,1]$

From theorem (3.1) $\{b_n\}$ is Cauchy.

Thus there exists $q \in g(\mathcal{X})$ with $p \in \mathcal{X}$ such that $ga_n \to q$ as $n \to \infty$ and $gp = q$.

Now we assert that $fp = q$.

$$d(f p, q, t) \leq d(fp, fa_n, t) + d(fa_n, q, t)$$

$$\leq a_1 d(gp, ga_n, t) + a_2 d(gp, fp, t) + a_3 d(ga_n, fa_n, t) + a_4 d(gp, fa_n, t) + a_5 d(ga_n, fp, t) + d(fa_n, q, t)$$

$$= a_1 d(q, ga_n, t) + a_2 d(q, fp, t) + a_3 d(ga_n, fa_n, t) + a_4 d(q, fa_n, t) + a_5 d(ga_n, fp, t)$$
$$+ d(fa_n, q, t)(1 - a_2) d(q, fp, t)$$

$$\leq a_1 d(q, ga_n, t) + a_3 d(ga_n, fa_n, t) + a_5 d(ga_n, fp, t) + (1 + a_4) d(fa_n, q, t)(1 - a_2) d(q, fp, t)$$

$$\leq a_1 d(q, ga_n, t) + a_3 d(ga_n, q, t) + a_3 d(fa_n, q, t) + a_5 d(ga_n, q, t) + a_5 d(fp, q, t)$$
$$+ (1 + a_4) d(fa_n, q, t)(1 - a_2 - a_5) d(fp, q, t)$$

$$\leq (a_1 + a_3 + a_5) d(ga_n, q, t) + (1 + a_3 + a_4) d(fa_n, q, t) d(fp, q, t)$$

$\leq (a_1 + a_3 + a_5)(1 - a_2 - a_5)\, d(ga_n, q, t) + (1 + a_3 + a_4)(1 - a_2 - a_5)\, d(fa_n, q, t)$

As $n \to \infty$,

$$d(f\,p, q, t) \ll \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5}\frac{c(1 - a_2 - a_5)}{2(a_1 + a_3 + a_5)} + \frac{1 + a_3 + a_4}{1 - a_2 - a_5}\frac{c(1 - a_2 - a_5)}{2(1 + a_3 + a_4)} = c$$

$$d(f\,p, q, t) \ll c$$

$\therefore f\,p = q$

Thus f, g have a point of coincidence q.

Uniqueness:

If q_1 is another point of coincidence then there exists $p_1 \in \mathcal{X}$ such that $fp_1 = q_1 = gp_1$. Now $d(q, q_1, t) = d(fp, fp_1, t)$

$\leq a_1 d(gp, gp_1, t) + a_2 d(gp, fp, t) + a_3 d(gp_1, fp_1, t) + a_4 d(gp, fp_1, t) + a_5 d(gp_1, fp, t)$

$= a_1 d(q, q_1, t) + a_2 d(q, q, t) + a_3 d(q_1, q_1, t) + a_4 d(q, q_1, t) + a_5 d(q_1, fp, t) d(q, q_1, t)$

$\leq (a_1 + a_4 + a_5) d(q, q_1, t)$

$$1 \leq a_1 + a_4 + a_5$$

This is absurd.

$\therefore q = q_1$.

Hence f, g have a unique point of coincidence.

Example 3.1. If $\mathcal{X} = R, E = C_R^1\,[0,1]$ and $\mathcal{P} = \{\varphi \in E | \varphi \geq 0\}$ the we define $d : \mathcal{X}\, x \mathcal{X}\, x[0, \infty] \to E$ by $d(a, b, t) = t|a - b|\phi$ where $\phi : [0,1] \to R$ such that $\phi(p) = e^p$. It can easily be verified that $(\mathcal{X}, d)$ is a parametric cone metric space. Consider the mappings $f, g : \mathcal{X} \to \mathcal{X}$ defined as $fa = \begin{cases} \frac{a}{(1+\alpha)} + \beta, a \neq 0, \\ 0, a = 0 \end{cases}$; $ga = \begin{cases} a + (1 + \alpha)\beta, a \neq 0 \\ 0, a = 0 \end{cases}$

Where$\alpha > 1, \beta \in R$

$d(fa, fb, t) = d\left(\frac{a}{1 + \alpha} + \beta, \frac{b}{1 + \alpha}, t\right)$

$= t\left|\frac{a}{1+\alpha} + \beta - \frac{b}{1+\alpha}\right| e^p$

$= t\left|\frac{a - b}{1 + \alpha}\right| e^p$

$d(ga, gb, t) = d(a + (1 + \alpha)\beta, b + (1 + \alpha)\beta, t)$

$= t|\, a + (1 + \alpha)\beta - b - (1 + \alpha)\beta | \varphi(t)$

$= t\ |\ a - b|\ e^p$

$\Rightarrow \frac{1}{1+\alpha} \leq \lambda$

$\Rightarrow (1+\alpha)\lambda \geq 1$

Thus $d(f a, f b, t) \leq \lambda d(ga, gb, t)$, where $\lambda = \frac{1}{1+\alpha} \in (0,1]$

Thus f and g coincide at the point 0.

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