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Time-Dependent Analysis of a Single-Server Queueing System with Disasters and Unreliable Repairs: A Probabilistic Approach to System Resilience

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Abstract:

A single server Markovian queueing paradigm is studied. The server is prone to disaster when delivering service or during idle periods. When the server encountered problems, all waiting customers will be flushed away and the repair procedure began immediately. Arrivals are allowed to join the queue during the repair time. The likelihood of restoring the server is p(p<1) after an exponential repair time. All waiting users leave the system if the server cannot be fixed. The time-dependent likelihood for a single server queue with disaster and an unreliable repairer is explicitly estimated for the first time. To demonstrate the significance of parameter values, performance measures and numerical figures were also supplied.

Keywords: Disaster; Unreliable Repairer; Time dependent Probabilities; Idle Period; Abandon

1.1 Introduction:

Queueing systems with server failures have extensively examined catastrophic events, a failure type that results in the immediate clearance of the entire system. Catastrophes, such as a complete machine breakdown, cause all queued tasks to be lost at once. For instance, a malfunction in an ATM or public telephone forces all waiting users to abandon their positions, while in computer systems, malware can initiate a wipe, erasing all stored data.

In early studies, Towsley and Tripathi (1991) explored a single-server model where all customers are removed from the system during a catastrophic event. Later, Artalejo (2000) conducted a detailed survey on queueing models involving such catastrophic scenarios.

Following this, Yang, Kim, and Chae (2002) examined M/G/1 queueing systems affected by disasters, describing system size and wait time distributions. Di Crescenzo et al. (2003, 2012) modeled queueing systems experiencing catastrophes using a Wiener process with random jumps. Baumann and Sandmann (2012) provided steady-state analysis for M/M/c queues in a random environment with catastrophes and rates dependent on system state, applying matrix-analytic techniques. Kalidass et al. (2012) analyzed an M/M/1/N queue with catastrophic events, deriving time-dependent and steady-state probabilities, along with performance measures. Ammar (2014) studied transient behavior in a two-processor heterogeneous system with catastrophes, calculating time-dependent probabilities and statistical measures such as mean and variance. Giorno, Nobile, and Spina (2014) examined queueing systems with time-varying catastrophic events modeled by a non-homogeneous Poisson process, determining transition probabilities and the time density function for reaching the zero state. Jiang, Liu, and Li (2015) investigated M/G/1 queues in a multi-phase random environment with catastrophes, detailing queue length and wait time distributions. Mytalas and Zazanis (2015) explored a batch arrival queue with catastrophic events under a multi-vacation policy, providing queue length and wait time distributions and calculating the fraction of customers who complete service.

Recent research has greatly enriched our understanding of queueing systems with disaster events. For example, Jain, Singh, and Meena (2022) derived time-dependent probabilities for an M/M/1 queue under disaster failure, with provisions for multiple working vacations. Janani (2022) performed a transient-state analysis on a model with differentiated breakdowns, obtaining both transient-state probabilities and steady-state probabilities through the Laplace transform’s final value theorem. Seenivasan, Chakravarthy, and Abinaya (2022) analyzed a Markovian queueing model that includes server breakdowns, single working vacations, and catastrophes, deriving steady-state probabilities with the matrix geometric method.

In another study, Sudhesh, Shapique, and Dharmaraja (2022) investigated an M/M/1 dual-stage vacation queueing system, where the server alternates between two types of vacation stages (stage-I and stage-II). Disasters can occur at any point, with immediate repairs considered afterward. They provided explicit formulas for both transient and steady-state probabilities. Karthick, Suvitha, and Sudhesh (2024) examined a two-server heterogeneous queue model incorporating differentiated vacation periods and potential disaster events during busy or vacation periods, immediately followed by repair. Their work includes explicit analytical expressions for time-dependent probabilities. Jeyakumar and Logapriya (2024) focused on production control in manufacturing environments with compulsory server vacations, modeling disasters and repair processes to derive probability-generating functions for queue size and related performance metrics using the supplementary variable technique.

Despite these advancements, there remains a lack of research on explicit time-dependent probabilities for single-server queues facing disasters and unreliable repairs. Understanding these transient probabilities is essential for capturing the dynamic behavior of such systems, supporting real-time decision-making, evaluating risks, assessing system recovery processes, enhancing performance, validating model accuracy, managing customer satisfaction, meeting SLAs, and studying emergent system behavior. While most studies have focused on steady-state conditions, gaining insights into transient dynamics is crucial for a full appreciation of system behavior under varying conditions.

1.2 Importance of the considered model:

- **Novel Contribution**: The paper presents the first explicit estimation of time-dependent probabilities for a single server queue with disasters and an unreliable repairer. This addresses a significant gap in the literature, as previous studies have primarily focused on steady-state analyses.

- **Relevance**: The study is highly relevant to various real-world systems where server reliability is a critical concern. For instance, public telephones, ATMs, and computer systems are all susceptible to catastrophic failures that can disrupt service.
- **Enhanced Understanding**: By examining time-dependent probabilities, the research provides a deeper understanding of the dynamic behavior of queueing systems. This is essential for making real-time decisions, optimizing system performance, and ensuring effective recovery strategies.
- **Model Validation**: The explicit estimation of transient probabilities allows for better validation of queueing models, ensuring that they accurately reflect the behavior of real-world systems under different conditions.
- **Risk Assessment**: The insights gained from this study can be used to assess the risks associated with server failures and to develop strategies to mitigate these risks, thereby enhancing system resilience and reliability.
- **Performance Measures**: The paper provides numerical figures and performance measures that highlight the significance of different parameter values. This aids in the practical application of the model to various scenarios, allowing for tailored solutions to specific problems.
- **1.3 Application:**

This research paper has broad applicability, especially in service-oriented areas like banking, IT, and telecommunications, where the ability of systems to withstand disruptions effectively is critical to preserving client happiness. The results offer a disaster recovery planning framework that makes it possible to analyze and improve recovery plans in the event of server failures. The study's time-dependent probability are essential for making decisions in real time and enabling prompt action to lessen the effects of failures. These insights can be used by engineers and system designers to optimize queueing system designs for robustness and efficiency. Furthermore, by providing a better knowledge of the transitory behavior of queueing systems, the research helps manage customer expectations and enhance service level offered risk assessment capabilities, organizations can improve their risk management methods. Furthermore, by examining transient dynamics, the work makes it easier to analyze emergent behaviors in queueing systems and advances our understanding of how the system functions in a variety of scenarios.

Table 1. Importance of the queueing model under study

Factor	Without This Study	With This Study	Importance
System Failure Prediction	Failure likelihood is based on rough estimations or past events.	Time-dependent probabilities give precise predictions for failures based on server load and repair mechanisms.	Accurate failure prediction enables proactive risk mitigation, preventing critical service outages and minimizing financial losses for cloud providers.

Recovery Speed	Recovery time is unpredictable and often delayed.	Optimal recovery strategies can be evaluated, reducing recovery times.	Quicker recovery minimizes downtime, improving user experience and reducing penalties from unmet SLAs.
Infrastructure Investment	Investment decisions are based on general trends or steady-state assumptions.	Providers can invest based on time-sensitive performance data and disaster probabilities.	Data-driven decisions lead to optimized capacity planning, reducing unnecessary costs and ensuring reliable service during peak times.
Service Continuity	Unpredictable service disruptions can lead to customer dissatisfaction.	Continuity plans can be tailored to mitigate the effects of predicted downtimes.	Cloud providers can ensure near-continuous service, maintaining customer trust and meeting SLAs consistently.
Operational Efficiency	No clear understanding of how various parameters affect downtime.	The impact of arrival rates, service rates, and repair success on downtime is clearly quantified.	This clarity allows providers to fine-tune their operations for maximum efficiency, balancing cost and performance.
Customer Satisfaction	Prolonged downtimes negatively impact customer trust and engagement.	Reduced downtime ensures a smoother customer experience.	Enhancing recovery speed and predicting failures help maintain high customer satisfaction, crucial for long-term retention in competitive markets.

2.Overview of the Model:

The queueing model of a single server is considered. Arrivals join the queue at a Poisson distributed rate λ, and service occurs at an exponentially distributed rate μ. The server is subjected to disaster. The disaster can occur at any time either while providing service or even the server is idle. The disaster occurs at the rate η which is exponentially distributed. Once the server met with disaster it will send to repair immediately. The repairer will start the repair process immediately. During repair process arrivals are allowed to join the queue with parameter λ which is Poissonally distributed. Here the repairer is unreliable. The repair will set back the server with parameter $p\gamma$ or sometimes the repairer unable to set back the server then all waiting customers in the queue will be move away and system become empty.

Let's use the notation $X(t)$ for the number of customers in the service line and $J(t)$ for the system's current state. Any value between 0 and 1 may be used for the status. If it is taken to be zero, the system is thought to be in need of repair. If assumption 1 is true, the system is said to be in a functional state. Consequently, the process $\{X(t), J(t)\}$ is a continuous time Markov chain with the transition diagram shown below.

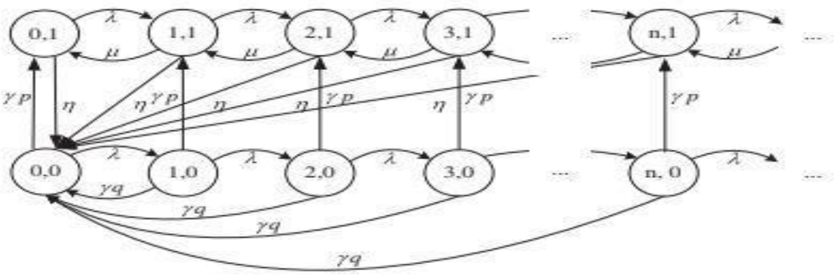

Figure 1. Transition of the Queueing Model Under Study

Following is the model's system of governing equations.

$$P_{0,1}^{'}(\mathrm{t}) = \gamma p\, P_{0,0}(t) - (\lambda+\eta)P_{0,1}(t) + \mu\, P_{1,1}(t)\,, \qquad (2.1)$$

$$P_{n,1}^{'}(\mathrm{t}) = \gamma p\, P_{n,0}(t) - (\lambda + \mu + \eta)P_{n,1}(t) + \lambda P_{(n-1),1}(t) + \mu P_{(n+1),1}(t), \quad n = 1, 2, 3, \ldots \qquad (2.2)$$

$$P_{0,0}^{'}(\mathrm{t}) = -(\lambda+\gamma p)P_{0,0}(t) + \eta \textstyle\sum_{n=0}^{\infty} P_{n,1}(t) + \gamma q \sum_{n=1}^{\infty} P_{n,0}(t), \qquad (2.3)$$

and

$$P_{n,0}^{'}(\mathrm{t}) = -(\lambda+\gamma p+\gamma q)P_{n,0}(t) + \lambda P_{(n-1),0}(t)\,, n = 1,2,3,\ldots \quad . \qquad (2.4)$$

with the initial condition $P_{0,1}(0) = 1$.

3 Transient Analysis:

Apply Laplace transform on equation (2.1) we get,

$$\hat{P}_{0,1}(s) = \frac{1}{s+\lambda+\eta} + \frac{\mu}{s+\lambda+\eta}\hat{P}_{1,1}(s) + \frac{\gamma p}{s+\lambda+\eta}\hat{P}_{0,0}(s)\,. \qquad (3.1)$$

Taking inverse Laplace transform for equation (3.1) we get,

$$P_{0,1}(t) = e^{-(\lambda+\eta)t} + \mu e^{-(\lambda+\eta)t} * P_{1,1}(t) + \gamma p e^{-(\lambda+\eta)t} * P_{0,0}(t)\,. \qquad (3.2)$$

Apply Laplace transform on both sides of equation (2.3) we get,

$$(\mathrm{s} + \lambda + \gamma p)\hat{P}_{0,0}(s) = \eta \sum_{n=0}^{\infty} \hat{P}_{n,1}(s) + \gamma q \sum_{n=1}^{\infty} \hat{P}_{n,0}(s). \qquad (3.3)$$

Apply Laplace transform on both sides of equation (2.4) we get,

$$\hat{P}_{n,0}(s) = \left(\frac{\lambda}{s+\lambda+\gamma}\right)^{n} \hat{P}_{0,0}(s), \qquad n = 1,2,3,\ldots \qquad (3.4)$$

Taking inverse Laplace transform for the equation (3.4) we get,

$$P_{n,0}(\mathrm{t}) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-(\lambda+\gamma)t} * P_{0,0}(\mathrm{t}), \qquad \mathrm{n} = 1,2,3,\ldots \quad . \qquad (3.5)$$

Apply Laplace transform on both sides of equation (2.2) we get,

$$(s+\lambda+\mu+\eta)\hat{P}_{n,1}(s)=\gamma p\hat{P}_{n,0}(s)+\lambda\hat{P}_{(n-1),1}(s)+\mu\hat{P}_{(n+1),1}(s), \qquad (3.6)$$

Taking summation n varies from 1 to ∞ on both sides of the equation (3.6) we get,

$$(s+\lambda+\mu+\eta)\sum_{n=1}^{\infty}\hat{P}_{n,1}(s)=\gamma p\sum_{n=1}^{\infty}\hat{P}_{n,0}(s)+\lambda\sum_{n=1}^{\infty}\hat{P}_{(n-1),1}(s)+\mu\sum_{n=1}^{\infty}\hat{P}_{(n+1),1}(s),$$

which can be simplified as,

$$(s+\eta)\sum_{n=1}^{\infty}\hat{P}_{n,1}(s)=\gamma p\sum_{n=1}^{\infty}\hat{P}_{n,0}(s)+\lambda\hat{P}_{0,1}(s)-\mu\hat{P}_{1,1}(s). \qquad (3.7)$$

Substituting the equation (3.1) in the equation (3.7) we get,

$$\sum_{n=1}^{\infty}\hat{P}_{n,1}(s)=\frac{\gamma p}{s+\eta}\sum_{n=1}^{\infty}\hat{P}_{n,0}(s)+\frac{\lambda}{(s+\eta)(s+\lambda+\eta)}-\frac{\mu}{s+\lambda+\eta}\hat{P}_{1,1}(s)$$
$$+\frac{\lambda\gamma p}{(s+\eta)(s+\lambda+\eta)}\hat{P}_{0,0}(s). \qquad (3.8)$$

Let us define,

$$\mathrm{Q}(\mathrm{z},\mathrm{t})=\sum_{n=1}^{\infty}P_{n,1(t)}z^n.$$

Then,

$$\frac{\partial Q}{\partial t}=\sum_{n=1}^{\infty}P'_{n,1}(\mathrm{t})\,z^n. \qquad (3.9)$$

Substituting the equation (2.2) in the equation (3.9) we get,

$$\frac{\partial Q(z,t)}{\partial t}-\left(-(\lambda+\mu+\eta)+\lambda z+\frac{\mu}{z}\right)Q(z,t)$$
$$=\gamma p\sum_{n=1}^{\infty}P_{n,0}(t)z^n-\mu P_{1,1}(\mathrm{t})+\lambda zP_{0,1}(t). \qquad (3.10)$$

Integrating the equation (3.10) we get,

$$Q(z,t)=\gamma p\int_0^t\left(\sum_{n=1}^{\infty}P_{n,0}(y)z^n\right)e^{-(\lambda+\mu+\eta)(t-y)}\,e^{\left(\lambda z+\frac{\mu}{z}\right)(t-y)}dy$$
$$-\mu\int_0^t P_{1,1}(y)\,e^{-(\lambda+\mu+\eta)(t-y)}e^{\left(\lambda z+\frac{\mu}{z}\right)(t-y)}\mathrm{dy}$$
$$+\lambda\int_0^t P_{0,1}(y)\,z\,e^{-(\lambda+\mu+\eta)(t-y)}e^{\left(\lambda z+\frac{\mu}{z}\right)(t-y)}\mathrm{dy}. \qquad (3.11)$$

We know that if $\alpha = 2\sqrt{\lambda\mu}$ and $\beta = \sqrt{\frac{\lambda}{\mu}}$ then,

$$e^{\left(\lambda z+\frac{\mu}{z}\right)t} = \sum_{n=-\infty}^{\infty} \beta^n z^n I_n(\alpha t),$$

where $I_n(\alpha t)$ is a modified Bessel's function of the first kind.

Therefore equation (3.11) gives,

$$\sum_{n=1}^{\infty} P_{n,1}(t)z^n = \gamma p \int_0^t \left(\sum_{m=1}^{\infty} P_{m,0}(y)\, z^m\right) e^{-(\lambda+\mu+\eta)(t-y)} \left(\sum_{n=-\infty}^{\infty} \beta^n I_n[\alpha(t-y)]z^n\right) dy$$
$$- \mu \int_0^t P_{1,1}(y) e^{-(\lambda+\mu+\eta)(t-y)} \left(\sum_{n=-\infty}^{\infty} \beta^n I_n[\alpha(t-y)]z^n\right) dy$$
$$+ \lambda \int_0^t P_{0,1}(y)\, e^{-(\lambda+\mu+\eta)(t-y)} \left(\sum_{n=-\infty}^{\infty} \beta^n I_n[\alpha(t-y)]z^{n+1}\right) dy. \quad (3.12)$$

Comparing the coefficients of z^n of equation (3.12) on both sides we get,

$$P_{n,1}(t) = \gamma p \int_0^t \sum_{m=1}^{\infty} P_{m,0}(y)\, \beta^{n-m} I_{n-m}[\alpha(t-y)] e^{-(\lambda+\mu+\eta)(t-y)} dy$$
$$- \mu \int_0^t P_{1,1}(y)\, \beta^n I_n[\alpha(t-y)] e^{-(\lambda+\mu+\eta)(t-y)} dy$$
$$+ \lambda \int_0^t P_{0,1}(y)\, \beta^{n-1} I_{n-1}[\alpha(t-y)] e^{-(\lambda+\mu+\eta)(t-y)} dy \quad (3.13)$$

Comparing the coefficient of z^{-n} of equation (3.12) on both sides we get,

$$0 = \gamma p \int_0^t \sum_{m=1}^{\infty} P_{m,0}(y)\, \beta^{-n-m} I_{n+m}[\alpha(t-y)] e^{-(\lambda+\mu+\eta)(t-y)} dy$$
$$- \mu \int_0^t P_{1,1}(y) \beta^{-n}\ I_n[\alpha(t-y)] e^{-(\lambda+\mu+\eta)(t-y)} dy$$
$$+ \lambda \int_0^t P_{0,1}(y) \beta^{-n-1}\, I_{n+1}[\alpha(t-y)] e^{-(\lambda+\mu+\eta)(t-y)} dy. \quad (3.14)$$

Multiply by β^{2n} on both sides of equation (3.14) we get,

$$0 = \gamma p \int_0^t \sum_{m=1}^{\infty} P_{m,0}(y)\, \beta^{n-m} I_{n+m}[\alpha(t-y)] e^{-(\lambda+\mu+\eta)(t-y)} dy$$
$$- \mu \int_0^t P_{1,1}(y) \beta^n\, I_n[\alpha(t-y)] e^{-(\lambda+\mu+\eta)(t-y)} dy$$
$$+ \lambda \int_0^t P_{0,1}(y)\, \beta^{n-1} I_{n+1}[\alpha(t-y)] e^{-(\lambda+\mu+\eta)(t-y)} dy. \quad (3.15)$$

Subtracting equation (3.15) from equation (3.13) we get,

$$P_{n,1}(t) = \gamma p \int_0^t \sum_{m=1}^{\infty} P_{m,0}(y)\, \beta^{n-m}\{I_{n-m}[\alpha(t-y)] - I_{n+m}[\alpha(t-y)]\} e^{-(\lambda+\mu+\eta)(t-y)} dy$$
$$+ \lambda \int_0^t P_{0,1}(y)\, \beta^{n-1}\{I_{n-1}[\alpha(t-y)] - I_{n+1}[\alpha(t-y)]\} e^{-(\lambda+\mu+\eta)(t-y)} dy. \quad (3.16)$$

Using the equation (3.5) in the equation (3.16) we get,

$$P_{n,1}(t) = \gamma p \int_0^t \left(\sum_{m=1}^{\infty}\left(\frac{\lambda^m t^{m-1}}{(m-1)!} e^{-(\lambda+\gamma)t} * (P_{0,0}(t)\right)\right) \beta^{n-m} \{I_{n-m}[\alpha(t-y)] - I_{n+m}[\alpha(t-y)]\}\, e^{-(\lambda+\mu+\eta)(t-y)} dy$$
$$+ \lambda \int_0^t P_{0,1}(y) \beta^{n-1} \{(I_{n-1}[\alpha(t-y)]) - I_{n+1}[\alpha(t-y)]\}\; e^{-(\lambda+\mu+\eta)(t-y)} dy. \quad (3.17)$$

Put $n = 1$ in the equation (3.16) we get,

$$P_{1,1}(t) = \gamma p \int_0^t \sum_{m=1}^{\infty} P_{m,0}(y)\, \beta^{1-m}\{I_{1-m}[\alpha(t-y)] - I_{1+m}[\alpha(t-y)]\} e^{-(\lambda+\mu+\eta)(t-y)} dy$$
$$+ \lambda \int_0^t P_{0,1}(y)\, \{(I_0[\alpha(t-y)]) - I_2[\alpha(t-y)]\}\; e^{-(\lambda+\mu+\eta)(t-y)} dy. \quad (3.18)$$

Using $I_{-n}(\mathrm{t}) = I_n(\mathrm{t})$ in the equation (3.18) we get,

$$P_{1,1}(t) = \gamma p \int_0^t \sum_{m=1}^{\infty} P_{m,0}(y)\, \beta^{1-m}\{I_{m-1}[\alpha(t-y)] - I_{m+1}[\alpha(t-y)]\} e^{-(\lambda+\mu+\eta)(t-y)} dy$$
$$+ \lambda \int_0^t P_{0,1}(y)\, \{(I_0[\alpha(t-y)]) - I_2[\alpha(t-y)]\}\; e^{-(\lambda+\mu+\eta)(t-y)} dy, \quad (3.19)$$

which on further simplification gives,

$$P_{1,1}(t) = \gamma p \int_0^t \sum_{m=1}^{\infty} P_{m,0}(y)\, \beta^{1-m} \left\{\frac{2m I_m[\alpha(t-y)]}{\alpha(t-y)}\right\} e^{-(\lambda+\mu+\eta)(t-y)} dy$$
$$+ \lambda \int_0^t P_{0,1}(y) \left\{\frac{2I_1[\alpha(t-y)]}{\alpha(t-y)}\right\} e^{-(\lambda+\mu+\eta)(t-y)} dy. \quad (3.20)$$

Applying Laplace transform on both sides of the equation (3.20) we get,

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$$\hat{P}_{1,1}(s) = 2\gamma p \sum_{m=1}^{\infty} \hat{P}_{m,0}(s)\, \beta^{1-m} \left(\alpha^{m-1} \frac{1}{\left(p+\sqrt{p^2-\alpha^2}\right)^m} \right) + 2\,\lambda \hat{P}_{0,1}(s) \frac{1}{p+\sqrt{p^2-\alpha^2}} \;, \qquad (3.21)$$

where $p = s + \lambda + \mu + \eta$.

Using the equation (3.4) in the equation (3.21) we get,

$$\hat{P}_{1,1}(s) = 2\gamma p \sum_{m=1}^{\infty} \frac{\lambda^m}{(s+\lambda+\gamma)^m} \frac{1}{\left(p+\sqrt{p^2-\alpha^2}\right)^m} \left(\frac{\alpha}{\beta}\right)^m \left(\frac{\beta}{\alpha}\right) \hat{P}_{0,0}(s) + 2\,\lambda \hat{P}_{0,1}(s) \frac{1}{p+\sqrt{p^2-\alpha^2}}, \qquad (3.22)$$

which on further simplification gives,

$$\hat{P}_{1,1}(s) = \frac{\gamma p}{\mu} \sum_{m=1}^{\infty} \left(\frac{1}{(s+\lambda+\gamma)^m}\right) \left(\frac{\lambda}{\beta}\right)^m \left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)^m \hat{P}_{0,0}(s) + \frac{2\,\lambda}{\alpha} \left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right) \hat{P}_{0,1}(s)\,. \qquad (3.23)$$

Taking inverse Laplace transform of the equation (3.23) we get,

$$P_{1,1}(\mathrm{t}) = \frac{\gamma p}{\mu} \sum_{m=1}^{\infty} \left(\frac{\lambda}{\beta}\right)^m \frac{e^{-(\lambda+\gamma)t} t^{m-1}}{(m-1)!} * \frac{m I_m(\alpha t) e^{-(\lambda+\mu+\eta)t}}{t} * P_{0,0}(\mathrm{t}) \qquad + \frac{\beta I_1(\alpha t) e^{-(\lambda+\mu+\eta)t}}{t} * P_{0,1}(\mathrm{t}). \qquad (3.24)$$

Using the equation (3.23) in the equation (3.1) we get,

$$\hat{P}_{0,1}(s) = \frac{1}{s+\lambda+\eta} + \frac{\gamma p}{s+\lambda+\eta} \sum_{m=1}^{\infty} \left(\frac{1}{(s+\lambda+\gamma)^m}\right) \left(\frac{\lambda}{\beta}\right)^m \left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)^m \hat{P}_{0,0}(s) + \frac{\mu}{s+\lambda+\eta} \frac{2\,\lambda}{\alpha} \left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right) \hat{P}_{0,1}(s) + \frac{\gamma p}{s+\lambda+\eta} \hat{P}_{0,0}(s)\,. \quad (3.25)$$

The above equation can be rewritten as,

$$\hat{P}_{0,1}(s) = \frac{1}{s+\lambda+\eta} + \frac{\gamma p}{s+\lambda+\eta} \hat{P}_{0,0}(s) + \frac{\gamma p}{s+\lambda+\eta} \sum_{m=1}^{\infty} \left(\frac{\lambda}{\beta}\right)^m \left(\frac{1}{(s+\lambda+\gamma)^m}\right) \left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)^m \hat{P}_{0,0}(s) + \frac{\sqrt{\lambda\mu}}{s+\lambda+\eta} \left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right) \hat{P}_{0,1}(s) \quad . \qquad (3.26)$$

The equation (3.26) can be simplified as,

$$\hat{P}_{0,1}(s) = \sum_{j=0}^{\infty}(F(s))^j \left\{ \frac{1}{s+\lambda+\eta} + \frac{\gamma p}{s+\lambda+\eta}\ \hat{P}_{0,0}(s) + \frac{\gamma p}{s+\lambda+\eta} \sum_{m=1}^{\infty} \left(\frac{\lambda}{\beta}\right)^m \left(\frac{1}{(s+\lambda+\gamma)^m}\right) \left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)^m \hat{P}_{0,0}(s) \right\}, \quad (3.27)$$

where

$$\mathrm{F(s)} = \frac{\sqrt{\lambda\mu}}{s+\lambda+\eta}\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right).$$

Taking inverse Laplace transform of equation (3.27) we get,

$$P_{0,1}(\mathrm{t}) = \sum_{j=0}^{\infty}\big(F(t)\big)^{*j} * \left(e^{-(\lambda+\eta)t} + \gamma p e^{-(\lambda+\eta)t} * \sum_{m=1}^{\infty}\left(\frac{\lambda}{\beta}\right)^m \frac{e^{-(\lambda+\gamma)t}t^{m-1}}{(m-1)!} * \frac{mI_m(\alpha t)e^{-(\lambda+\mu+\eta)t}}{t} * P_{0,0}(\mathrm{t}) + \gamma p e^{-(\lambda+\eta)t} * P_{0,0}(\mathrm{t}) \right), \quad (3.28)$$

where

$$F(t) = \frac{\alpha}{2}e^{-(\lambda+\eta)t} * \frac{I_1(\alpha t)e^{-(\lambda+\mu+\eta)t}}{t}.$$

To find $P_{0,0}$(t):

By using the Normalization condition we get,

$$P_{0,0}(\mathrm{t}) + \sum_{n=1}^{\infty} P_{n,0}(t) + P_{0,1}(\mathrm{t}) + \sum_{n=1}^{\infty} P_{n,1}(t) = 1. \quad (3.29)$$

Taking Laplace transform for the equation (3.29) we get,

$$\hat{P}_{0,0}(s) + \sum_{n=1}^{\infty}\hat{P}_{n,0}(s) + \hat{P}_{0,1}(s) + \sum_{n=1}^{\infty}\hat{P}_{n,1}(s) = \frac{1}{s}. \quad (3.30)$$

Using the equations (3.4),(3.8) & (3.27) in (3.30) we get,

$$\hat{P}_{0,0}(s) + \frac{1}{s+\lambda+\eta}\sum_{j=0}^{\infty}\left(F(s)\right)^j + \frac{\gamma p}{s+\lambda+\eta}\ \hat{P}_{0,0}(s)\sum_{j=0}^{\infty}\left(F(s)\right)^j$$
$$+\frac{\gamma p}{s+\lambda+\eta}\sum_{m=1}^{\infty}\left(\frac{\lambda}{\beta}\right)^m\frac{1}{(s+\lambda+\gamma)^m}\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)^m\ \hat{P}_{0,0}(s)\sum_{j=0}^{\infty}\left(F(s)\right)^j$$
$$+\sum_{n=1}^{\infty}\left(\frac{\lambda}{s+\lambda+\gamma}\right)^n\hat{P}_{0,0}(s) + \frac{\gamma p}{s+\eta}\sum_{n=1}^{\infty}\left(\frac{\lambda}{s+\lambda+\gamma}\right)^n\hat{P}_{0,0}(s) + \frac{\lambda}{(s+\eta)(s+\lambda+\eta)}$$
$$-\frac{\mu}{s+\lambda+\eta}\hat{P}_{1,1}(s)\ +\frac{\lambda\gamma p}{(s+\eta)(s+\lambda+\eta)}\ \hat{P}_{0,0}(s)$$
$$= \frac{1}{s}. \qquad (3.31)$$

Using the equation (3.23) in the equation (3.31) we get,

$$\hat{P}_{0,0}(s) + \frac{1}{s+\lambda+\eta}\sum_{j=0}^{\infty}\left(F(s)\right)^j + \frac{\gamma p}{s+\lambda+\eta}\ \hat{P}_{0,0}(s)\sum_{j=0}^{\infty}\left(F(s)\right)^j$$
$$+\frac{\gamma p}{s+\lambda+\eta}\sum_{m=1}^{\infty}\left(\frac{\lambda}{\beta}\right)^m\frac{1}{(s+\lambda+\gamma)^m}\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)^m\ \hat{P}_{0,0}(s)\sum_{j=0}^{\infty}\left(F(s)\right)^j$$
$$+\sum_{n=1}^{\infty}\left(\frac{\lambda}{s+\lambda+\gamma}\right)^n\hat{P}_{0,0}(s) + \frac{\gamma p}{s+\eta}\sum_{n=1}^{\infty}\left(\frac{\lambda}{s+\lambda+\gamma}\right)^n\hat{P}_{0,0}(s) + \frac{\lambda}{(s+\eta)(s+\lambda+\eta)}$$
$$+\frac{\lambda\gamma p}{(s+\eta)(s+\lambda+\eta)}\ \hat{P}_{0,0}(s)$$
$$-\frac{\gamma p}{s+\lambda+\eta}\ \sum_{m=1}^{\infty}\left(\frac{1}{(s+\lambda+\gamma)^m}\right)\left(\frac{\lambda}{\beta}\right)^m\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)^m\ \hat{P}_{0,0}(s)$$
$$-\frac{\mu}{s+\lambda+\eta}\frac{2\,\lambda}{\alpha}\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)\hat{P}_{0,1}(s) = \frac{1}{s}. \qquad (3.32)$$

Using the equation (3.27) & substituting $\frac{2\lambda\mu}{\alpha} = \frac{\alpha}{2}$ in the equation (3.32) we get,

$$\hat{P}_{0,0}(s) + \frac{1}{s+\lambda+\eta}\sum_{j=0}^{\infty}\big(F(s)\big)^j + \frac{\gamma p}{s+\lambda+\eta}\ \hat{P}_{0,0}(s)\sum_{j=0}^{\infty}\big(F(s)\big)^j$$

$$+\frac{\gamma p}{s+\lambda+\eta}\sum_{m=1}^{\infty}\left(\frac{\lambda}{\beta}\right)^m\frac{1}{(s+\lambda+\gamma)^m}\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)^m\ \hat{P}_{0,0}(s)\sum_{j=0}^{\infty}\big(F(s)\big)^j$$

$$+\sum_{n=1}^{\infty}\left(\frac{\lambda}{s+\lambda+\gamma}\right)^n\hat{P}_{0,0}(s) + \frac{\gamma p}{s+\eta}\sum_{n=1}^{\infty}\left(\frac{\lambda}{s+\lambda+\gamma}\right)^n\hat{P}_{0,0}(s) + \frac{\lambda}{(s+\eta)(s+\lambda+\eta)}$$

$$+\frac{\lambda\gamma p}{(s+\eta)(s+\lambda+\eta)}\ \hat{P}_{0,0}(s) - \frac{\gamma p}{s+\lambda+\eta}\ \sum_{m=1}^{\infty}\left(\frac{1}{(s+\lambda+\gamma)^m}\right)\left(\frac{\lambda}{\beta}\right)^m\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)^m\ \hat{P}_{0,0}(s)$$

$$-\frac{\alpha}{2}\frac{1}{(s+\lambda+\eta)^2}\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)\sum_{j=0}^{\infty}\big(F(s)\big)^j$$

$$-\frac{\alpha}{2}\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)\frac{\gamma p}{(s+\lambda+\eta)^2}\sum_{j=0}^{\infty}\big(F(s)\big)^j\hat{P}_{0,0}(s)$$

$$-\frac{\alpha}{2}\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)\frac{\gamma p}{(s+\lambda+\eta)^2}\ \sum_{m=1}^{\infty}\left(\frac{1}{(s+\lambda+\gamma)^m}\right)\left(\frac{\lambda}{\beta}\right)^m\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)^m\ \sum_{j=0}^{\infty}\big(F(s)\big)^j\hat{P}_{0,0}(s)$$

$$= \frac{1}{s}\,. \tag{3.33}$$

The equation (3.33) can be written as,

$$\hat{P}_{0,0}(s)R(s) = G(s)\,, \qquad (3.34$$

where,

$$R(s)$$
$$= 1 + \frac{\gamma p}{s+\lambda+\eta}\ \sum_{j=0}^{\infty}\big(F(s)\big)^j + \frac{\gamma p}{s+\lambda+\eta}\sum_{m=1}^{\infty}\left(\frac{\lambda}{\beta}\right)^m\frac{1}{(s+\lambda+\gamma)^m}\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)^m\ \sum_{j=0}^{\infty}\big(F(s)\big)^j$$

$$+\sum_{n=1}^{\infty}\left(\frac{\lambda}{s+\lambda+\gamma}\right)^n + \frac{\gamma p}{s+\eta}\sum_{n=1}^{\infty}\left(\frac{\lambda}{s+\lambda+\gamma}\right)^n + \frac{\lambda\gamma p}{(s+\eta)(s+\lambda+\eta)}$$

$$-\frac{\gamma p}{s+\lambda+\eta}\ \sum_{m=1}^{\infty}\frac{1}{(s+\lambda+\gamma)^m}\left(\frac{\lambda}{\beta}\right)^m\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)^m$$

$$-\frac{\alpha\gamma p}{2}\frac{1}{(s+\lambda+\eta)^2}\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)\sum_{j=0}^{\infty}\big(F(s)\big)^j$$

$$-\frac{\alpha\gamma p}{2}\frac{1}{(s+\lambda+\eta)^2}\sum_{m=1}^{\infty}\left(\frac{1}{(s+\lambda+\gamma)^m}\right)\left(\frac{\lambda}{\beta}\right)^m\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha}\right)^{m+1}\ \sum_{j=0}^{\infty}\big(F(s)\big)^j, \tag{3.35}$$

and

$$G(s) = \frac{1}{s} - \frac{1}{s+\lambda+\eta}\sum_{j=0}^{\infty}\big(F(s)\big)^{j} - \frac{\lambda}{(s+\eta)(s+\lambda+\eta)} + \frac{\alpha}{2}\frac{1}{(s+\lambda+\eta)^{2}}\left(\frac{p-\sqrt{p^{2}-\alpha^{2}}}{\alpha}\right)\sum_{j=0}^{\infty}\big(F(s)\big)^{j} \quad . \qquad (3.36)$$

Taking inverse Laplace transform for the equation (3.34) we get,

$$P_{0,0}(t) = \big(R(t)\big)^{-1} * G(t), \qquad (3.37)$$

where,

$$\begin{aligned}R(t) = \delta(t) &+ \gamma p e^{-(\lambda+\eta)t} * \sum_{j=0}^{\infty}\big(F(t)\big)^{*j} + \gamma p e^{-(\lambda+\eta)t} * \sum_{m=1}^{\infty}\left(\frac{\lambda}{\beta}\right)^{m}\frac{e^{-(\lambda+\gamma)t}t^{m-1}}{(m-1)!}\\ &* \frac{mI_m(\alpha t)e^{-(\lambda+\mu+\eta)t}}{t} * \sum_{j=0}^{\infty}\big(F(t)\big)^{*j} + \sum_{n=1}^{\infty}\frac{\lambda^{n}e^{-(\lambda+\gamma)t}t^{n-1}}{(n-1)!} + \gamma p e^{-\eta t}\\ &* \sum_{n=1}^{\infty}\frac{\lambda^{n}e^{-(\lambda+\gamma)t}t^{n-1}}{(n-1)!} + \lambda\gamma p e^{-\eta t} * e^{-(\lambda+\eta)t} - \gamma p e^{-(\lambda+\eta)t} * \sum_{m=1}^{\infty}\left(\frac{\lambda}{\beta}\right)^{m}\frac{e^{-(\lambda+\gamma)t}t^{m-1}}{(m-1)!}\\ &* \frac{mI_m(\alpha t)e^{-(\lambda+\mu+\eta)t}}{t} - \frac{\alpha\gamma p}{2}e^{-(\lambda+\eta)t} * \frac{I_1(\alpha t)e^{-(\lambda+\mu+\eta)t}}{t}\\ &* \sum_{j=0}^{\infty}\big(F(t)\big)^{*j} - \frac{\alpha\gamma p}{2}e^{-(\lambda+\eta)t}t * \sum_{m=1}^{\infty}\left(\frac{\lambda}{\beta}\right)^{m}\frac{e^{-(\lambda+\gamma)t}t^{m-1}}{(m-1)!}\\ &* \frac{(m+1)I_{m+1}(\alpha t)e^{-(\lambda+\mu+\eta)t}}{t} * \sum_{j=0}^{\infty}\big(F(t)\big)^{*j}, \qquad (3.38)\end{aligned}$$

and

$$\begin{aligned}G(t) = 1 - e^{-(\lambda+\eta)t} &* \sum_{j=0}^{\infty}\big(F(t)\big)^{*j} - \lambda e^{-\eta t} * e^{-(\lambda+\eta)t} + \frac{\alpha}{2}t.\,e^{-(\lambda+\eta)t} * \frac{I_1(\alpha t)e^{-(\lambda+\mu+\eta)t}}{t}\\ &* \sum_{j=0}^{\infty}\big(F(t)\big)^{*j} \quad . \qquad (3.39)\end{aligned}$$

Hence $P_{0,0}$(t) is found explicitly.

4 Performance Measures

To find Mean:

Now

Mean = E(N(t)) = m(t)

$$m(t) = \sum_{n=1}^{\infty} n\,(P_{n,0}(t) + P_{n,1}(t))$$

$$m'(\mathrm{t}) = \sum_{n=1}^{\infty} n\,\left(P'_{n,0}(\mathrm{t}) + P'_{n,1}(\mathrm{t})\right) \quad (4.1)$$

Using the equations (2.2) and (2.4) in (4.1) we get,

$$m'(\mathrm{t}) = -(\lambda + \gamma p + \gamma q)\sum_{n=1}^{\infty} nP_{n,0}(\mathrm{t}) + \lambda\sum_{n=1}^{\infty} nP_{(n-1),0}(\mathrm{t}) + \gamma p\sum_{n=1}^{\infty} nP_{n,0}(t)$$
$$-(\lambda + \mu + \eta)\sum_{n=1}^{\infty} nP_{n,1}(t) + \lambda\sum_{n=1}^{\infty} nP_{(n-1),1}(\mathrm{t}) + \mu\sum_{n=1}^{\infty} nP_{(n+1),1}(\mathrm{t}),$$

$$m'(t) = (\eta - \gamma q)g(t) + (\lambda - \mu) - \eta\sum_{n=1}^{\infty} n + \mu f(\mathrm{t}) + \mu P_{0,1}(\mathrm{t}), \quad (4.2)$$

where

$$\mathrm{f}(\mathrm{t}) = \sum_{n=0}^{\infty} P_{n,0}\;(t),$$

and

$$\mathrm{g}(\mathrm{t}) = \sum_{n=0}^{\infty} nP_{n,0}(\mathrm{t}).$$

Let us find f(t):

Consider

$$\mathrm{f}(\mathrm{t}) = \sum_{n=0}^{\infty} P_{n,0}(\mathrm{t}),$$

$$f(t) = P_{0,0}(\mathrm{t}) + \sum_{n=1}^{\infty} P_{n,0}(\mathrm{t}), \quad (4.3)$$

Taking Laplace transform for the equation (4.3) we get,

$$\hat{f}\;(s) = \hat{P}_{0,0}(s) + \sum_{n=1}^{\infty}\left(\frac{\lambda}{s + \lambda + \gamma}\right)^{n}\;\hat{P}_{0,0}(s),$$

which on further simplification yields,

$$\hat{f}\;(s) = \hat{P}_{0,0}(s) + \hat{P}_{0,0}(s)\left(\frac{\lambda}{s + \gamma}\right). \quad (4.4)$$

Taking inverse Laplace transform for the equation (4.4) we get,

$$f(t) = P_{0,0}(\mathrm{t}) + \lambda\left(P_{0,0}(t) * e^{-\gamma t}\right),$$

which can be rewritten as,

$$f(t) = P_{0,0}(\mathrm{t}) + \lambda \int_0^t P_{0,0}(u) e^{-\gamma(t-u)}\, \mathrm{d}u. \qquad (4.5)$$

Let us find g(t):

Consider,

$$\mathrm{g(t)} = \sum_{n=1}^{\infty} nP_{n,0}(\mathrm{t}) \quad . \qquad (4.6)$$

Taking Laplace transform for equation (4.6) we get,

$$\hat{g}\ (s) = \sum_{n=1}^{\infty} n\, \hat{P}_{n,0}(s)\, . \qquad (4.7)$$

Substituting the equation (3.4) in equation (4.7) gives,

$$\hat{g}(s) = \sum_{n=1}^{\infty} \mathrm{n} \left(\frac{\lambda}{s+\lambda+\gamma}\right)^n \ \hat{P}_{0,0}(s) \quad , \qquad (4.8)$$

which on simplification gives,

$$\hat{g}(s) = \lambda \hat{P}_{0,0}(s) \frac{1}{s+\gamma} + \lambda^2 \hat{P}_{0,0}(s) \frac{1}{(s+\gamma)^2} \quad . \qquad (4.9)$$

Taking inverse Laplace transform for the equation (4.9) we get,

$$\mathrm{g(t)} = \lambda P_{0,0}(\mathrm{t}) * e^{-\gamma t} + \lambda^2 \big(P_{0,0}(t) * e^{-\gamma t}\, t\big),$$

which can be rewritten as,

$$g(t) = \lambda \int_0^t P_{0,0}(u)\ e^{-\gamma(t-u)} \mathrm{du} + \lambda^2 \int_0^t P_{0,0}(u)\ e^{-\gamma(t-u)} (\mathrm{t-u})\mathrm{du}. \qquad (4.10)$$

Integrating equation (4.2) we get,

$$m(t) = (\eta - \gamma q) \int_0^t g(x)\, dx + (\lambda - \mu)t - \eta t \sum_{n=1}^{\infty} \mathrm{n} + \mu \int_0^t f(x)\ dx$$
$$+ \mu \int_0^t P_{0,1}(x)\ dx. \qquad (4.11)$$

Now

$$\mathrm{w(t)} = \mathrm{E}(n^2) = \sum_{n=1}^{\infty} n^2 \left(P_{n,0}(t) + P_{n,1}(t)\right),$$
$$\mathrm{w(t)} = \sum_{n=1}^{\infty} n^2\, P_{n,0}(\mathrm{t}) + \sum_{n=1}^{\infty} n^2\, P_{n,1}(t),$$

$$w'(\mathrm{t}) = \sum_{n=1}^{\infty} n^2 P'_{n,0}(\mathrm{t}) + \sum_{n=1}^{\infty} n^2\, P'_{n,1}(t)\,. \qquad (4.12)$$

Substituting the equation (2.2) & the equation (2.4) in the equation (4.12) we get,

$$w'(\mathrm{t}) = (\lambda+\mu) + (\eta-\gamma q)h(t) + 2\lambda\, m(t) - \eta\sum_{n=1}^{\infty} n^2 - 2\mu\sum_{n=1}^{\infty} \mathrm{n} + \mu\, f(t)$$
$$- \mu P_{0,1}(\mathrm{t}), \qquad (4.13)$$

where

$$h(t) = \sum_{n=1}^{\infty} n^2 P_{n,0}(\mathrm{t}).$$

To find h(t)

Consider,

$$\mathrm{h}(t) = \sum_{n=1}^{\infty} n^2 P_{n,0}(\mathrm{t})\,. \qquad (4.14)$$

Taking Laplace transform of the equation (4.14) we get,

$$\hat{\mathrm{h}}(s) = \sum_{n=1}^{\infty} n^2 \hat{P}_{n,0}(\mathrm{s}). \qquad (4.15)$$

Substituting the equation (3.4) in the equation (4.15) we get,

$$\hat{\mathrm{h}}(s) = \textstyle\sum_{n=1}^{\infty} n^2 \left(\frac{\lambda}{s+\lambda+\gamma}\right)^n \hat{P}_{0,0}(s)\,, \qquad (4.16)$$

which on simplification yields,

$$\hat{\mathrm{h}}(s) = \hat{P}_{0,0}(s)\left(\frac{2\lambda}{s+\gamma} + \frac{4\lambda^2}{(s+\gamma)^2} + \frac{2\lambda^3}{(s+\gamma)^3}\right) - \hat{g}\ (s). \qquad (4.17)$$

Taking inverse Laplace transform for the equation (4.17) we get,

$$\mathrm{h(t)} = \mathrm{P}_{0,0}(\mathrm{t}) * (2\lambda e^{-\gamma t} + 4\lambda^2\, e^{-\gamma t} t + \lambda^3\, e^{-\gamma t} t^2) - \mathrm{g(t)}, \qquad (4.18)$$

which can be rewritten as,

$$\mathrm{h(t)} = \int_0^{\mathrm{t}} \mathrm{P}_{0,0}(\mathrm{u})\left[2\lambda e^{-\gamma(t-u)} + 4\lambda^2\, e^{-\gamma(t-u)}(t-u) + \lambda^3\, e^{-\gamma(t-u)}(t-u)^2\right]\mathrm{du}$$
$$- \mathrm{g(t)}, \qquad (4.19)$$

where

$$g(t) = \lambda\int_0^t P_{0,0}(u)\ e^{-\gamma(t-u)}\mathrm{du} + \lambda^2\int_0^t P_{0,0}(u)\ e^{-\gamma(t-u)}(\mathrm{t}-\mathrm{u})\mathrm{du}. \qquad (4.20)$$

Hence, we get from equation (4.19),

$$h(t) = \lambda \int_0^t P_{0,0}(u)\ e^{-\gamma(t-u)}\mathrm{du} + 3\lambda^2 \int_0^t P_{0,0}(u)\ e^{-\gamma(t-u)}(\mathrm{t}-\mathrm{u})\mathrm{du}$$
$$+\ \lambda^3 \int_0^t P_{0,0}(u)\ e^{-\gamma(t-u)}(\mathrm{t}-\mathrm{u})^2\mathrm{du}. \qquad (4.21)$$

Integrating equation (4.13) we get,

$$w(t) = (\lambda+\mu)t + (\eta-\gamma q)\int_0^t h(x)dx + 2\lambda\int_0^t m(x)dx - \eta t\sum_{n=1}^{\infty} n^2$$
$$-\ 2\mu t\sum_{n=1}^{\infty}\mathrm{n} + \mu\int_0^t f(x)dx - \mu\int_0^t P_{0,1}(x)\mathrm{dx}. \qquad (4.22)$$

To find Var(X(t)):

$$\mathrm{Var}\big(\mathrm{X(t)}\big) = w(t) - \big(m(t)\big)^2 \qquad (4.23)$$

By substituting equations (4.11) & (4.22) in equation (4.23) we get $\mathrm{Var}\big(\mathrm{X(t)}\big)$.

5 Numerical Illustrations

Let's discuss the numerical explanations for the figures in the context of their applications in service industries such as telecommunications, banking, and IT services. Each figure highlights specific probabilities associated with the queueing system's behavior under various conditions. The explanations will focus on how these probabilities can be interpreted for practical applications in real-time decision-making and system optimization.

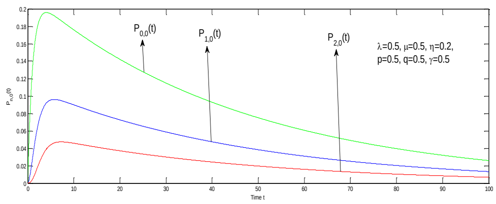

Figure 2. Variation of Probabilities $P_{n,0}(t)$ with respect to λ=0.5

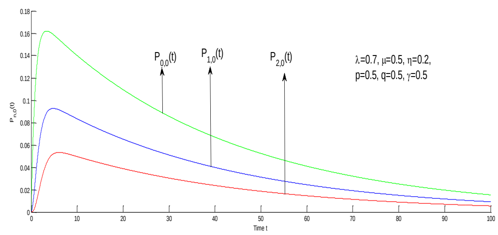

Figure 3. Variation of Probabilities $P_{n,0}(t)$ with respect to λ=0.7

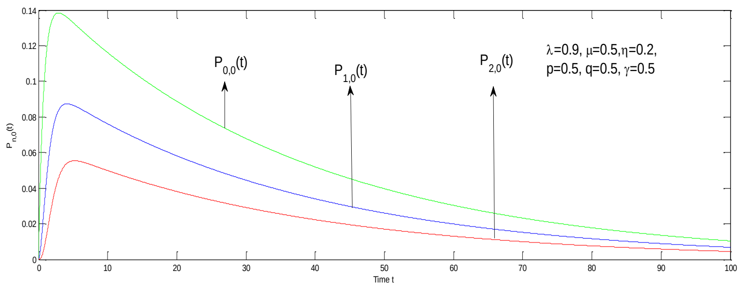

Figure 4. Variation of Probabilities $P_{n,0}(t)$ with respect to λ=0.9

The three figures ie., figure 2,3 and 4 illustrate the variation of probabilities $P_{n,0}(t)$ over time for different arrival rates λ (0.5, 0.7, and 0.9), where n denotes the number of customers and "0" represents the disaster state. As the arrival rate increases, the system experiences a greater likelihood of entering a disaster state, which reflects how increasing demand stresses the system and affects its ability to recover. In Figure 2 $\lambda = 0.5$, the system exhibits moderate demand, and the probability of being in the disaster state with zero customers $P_{0,0}(t)$ initially rises but eventually stabilizes. This indicates that the system, under lower arrival rates, can recover relatively quickly, leading to fewer customers being affected by disasters. However, as more customers join the system, probabilities like $P_{1,0}(t)$ and $P_{2,0}(t)$ begin to rise, showing that the likelihood of more customers experiencing the disaster state grows over time. The successful repair of the system, with a probability $p = 0.5$, plays a crucial role in mitigating the impact of these disasters, helping to prevent the system from remaining in a failed state for extended periods.

In Figure 3($\lambda = 0.7$) ,the increased arrival rate accelerates the transition into disaster states, and probabilities such as $P_{1,0}(t)$ and $P_{2,0}(t)$ rise more rapidly compared to Figure 1. This indicates that under higher demand, the system becomes more prone to breakdowns that affect multiple customers. Although the repair probability $p = 0.5$ remains constant, the system's ability to recover is more strained as the arrival rate increases, leading to a higher probability of multiple customers being caught in a disaster state. In real-world systems, such as public ATMs or telecommunication networks, this corresponds to peak usage periods when system failures become more likely due to higher user demand.

Figure 4 ($\lambda = 0.7$) further highlights the system's vulnerability under high demand, where the probabilities of being in a disaster state with more customers (e.g., $P_{2,0}(t)$) rise even more rapidly. As the arrival rate approaches saturation, the system's ability to handle customer load is overwhelmed, leading to frequent and prolonged periods in the disaster state. Even with a 50% chance of successful repair, the system struggles to return to an operational state quickly enough to mitigate the growing demand. The probability of $P_{n,0}(t)$ for multiple customers being affected continues to increase, showing that without a higher repair probability or more robust repair mechanisms, the system remains vulnerable to extended downtimes. This highlights the critical role of repair efficiency in managing disaster states. In high-demand scenarios, improving the repair probability p would significantly enhance system resilience, reduce downtime, and minimize the number of customers affected by failures.

Overall, the comparison of these three figures underscores the increasing likelihood of disaster states as the arrival rate λincreases, as well as the critical impact of successful repairs in mitigating these failures. At lower arrival rates, the system can manage occasional breakdowns, but as demand intensifies, the effectiveness of the repair process becomes vital for preventing extended system downtimes and reducing the number of customers affected by disasters.

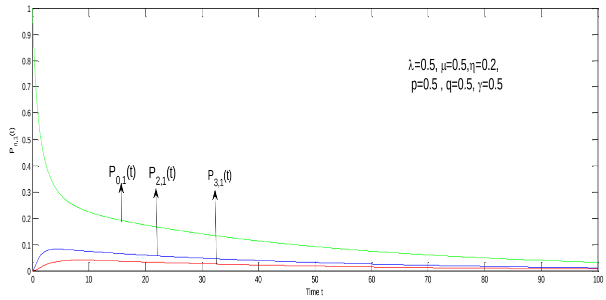

Figure 5 Variation of Probabilities $P_{n,1}(t)$ with respect to λ=0.5

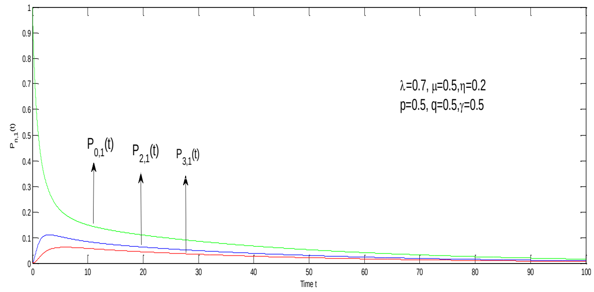

Figure 6 Variation of Probabilities $P_{n,1}(t)$ with respect to λ=0.7

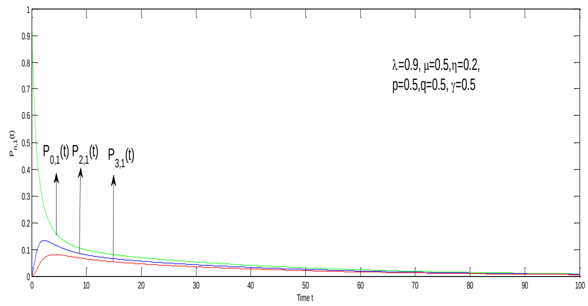

Figure 7 Variation of Probabilities $P_{n,1}(t)$ with respect to λ=0.9

Comparing the three figures 5, 6 & 7, it is clear that the arrival rate λ has a significant impact on the system's ability to remain operational. At lower arrival rates (λ=0.5), the system can manage customer demand effectively, with a 50% repair probability allowing it to recover quickly from disasters. As the arrival rate increases (λ=0.7), the system begins to experience more stress, but it remains capable of maintaining operations for a majority of the time, with the repair probability ensuring recovery. However, at the highest arrival rate (λ=0.9), the system struggles to stay operational, and despite the repair probability $p = 0.5$, the likelihood of remaining in an operational state decreases significantly.

The impact of successful repair is critical across all scenarios. With a 50% repair probability, the system is able to recover from disasters, but as the arrival rate increases, the system spends more time under stress. This comparison highlights the importance of optimizing the repair process to maintain system performance, particularly in high-demand scenarios where customer load is overwhelming. Increasing the repair probability could significantly improve system resilience, allowing it to better manage higher arrival rates and reduce the likelihood of prolonged downtimes.

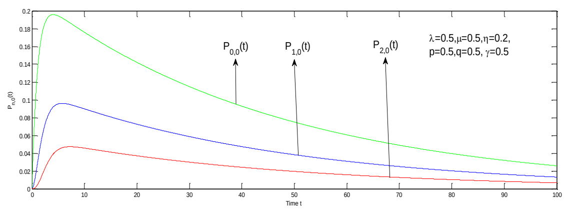

Figure 8 Variation of Probabilities $P_{n,0}(t)$ with respect to μ=0.5

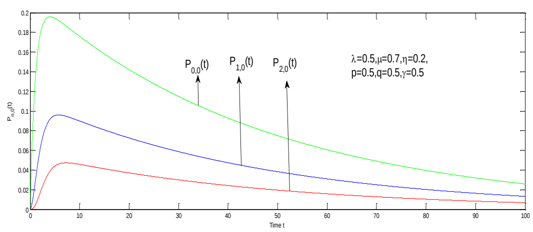

Figure 9 Variation of Probabilities $P_{n,0}(t)$ with respect to μ=0.7

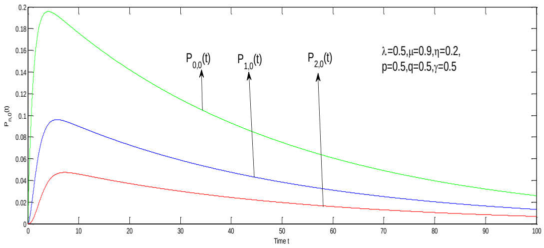

Figure 10 Variation of Probabilities $P_{n,0}(t)$ with respect to μ=0.9

The comparison of these three figures 8,9 & 10 demonstrates the critical role of the service rate μ in determining the system's resilience to disasters and its overall performance. As the service rate increases from 0.5 to 0.9, the system becomes increasingly capable of handling customer demand while reducing the probability of entering and remaining in disaster states. The repair probability $p = 0.5$ remains a key factor in the recovery process, but the higher service rates allow the system to rely less on repair interventions and more on its ability to quickly serve customers and transition out of disaster states.

At μ=0.5, the system experiences slower recoveries and higher disaster probabilities, particularly as the number of customers increases. However, with a service rate of μ=0.7, the system shows marked improvements in both recovery speed and overall resilience. Finally, at μ=0.9, the system demonstrates strong stability, with rapid recoveries and minimal time spent in disaster states, even when handling a larger number of customers.

The comparison clearly shows that increasing the service rate significantly enhances the system's ability to cope with disasters and maintain operational performance. While the repair success probability remains vital, a higher service rate reduces the system's dependence on repairs and ensures faster recoveries, improving overall system reliability and reducing the impact of disasters on customer service.

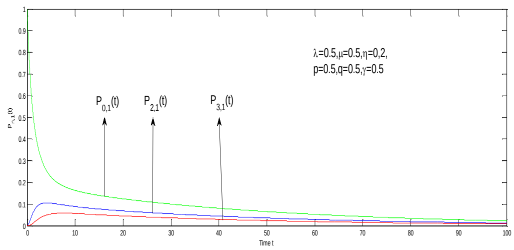

Figure 11 Variation of Probabilities $P_{n,1}(t)$ with respect to μ=0.5

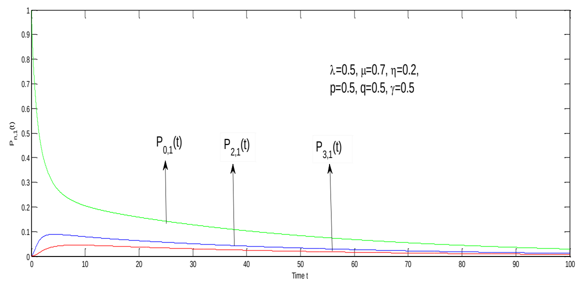

Figure 12 Variation of Probabilities $P_{n,1}(t)$ with respect to μ=0.7

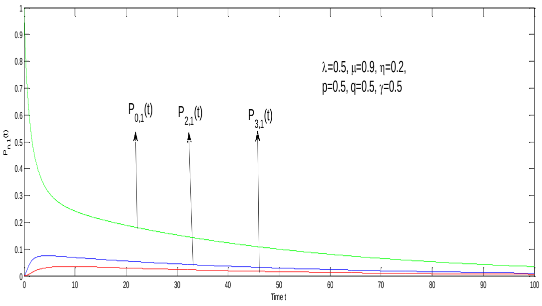

Figure 13 Variation of Probabilities $P_{n,1}(t)$ with respect to μ=0.9

Across the three figures 11, 12 &13, it is evident that increasing the service rate μ significantly improves the system's performance and resilience. At the moderate service rate of $\mu = 0.5$, the system is more prone to disasters and requires the repair process to restore operations frequently. However, as the service rate increases to $\mu = 0.7$and $\mu = 0.9$, the system becomes much more efficient at handling customer demand, recovering from disasters, and minimizing the time spent in failure states. The repair success probability $p = 0.5$ remains a key factor in ensuring recovery across all scenarios, but as the service rate improves, the system becomes less dependent on repairs and better equipped to stay operational without frequent breakdowns.

The overall comparison strongly emphasizes the importance of optimizing the service rate to improve system performance. While the repair mechanism is vital for recovery, a higher service rate enables the system to handle more customers with fewer interruptions, thereby reducing the need for repairs and enhancing system reliability. In real-world applications such as ATMs or telecommunication systems, increasing the service rate can significantly reduce downtime, maintain customer satisfaction, and ensure continuous service, even in high-demand scenarios.

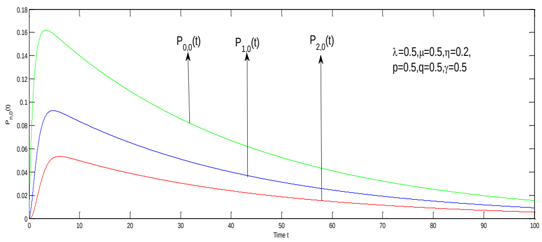

Figure 14 Variation of Probabilities $P_{n,0}(t)$ with respect to p=0.5

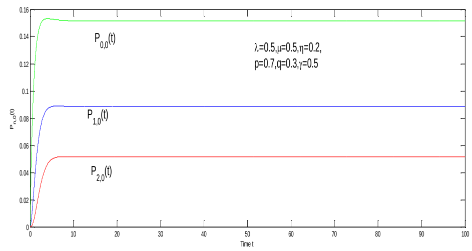

Figure 15 Variation of Probabilities $P_{n,0}(t)$ with respect to p=0.7

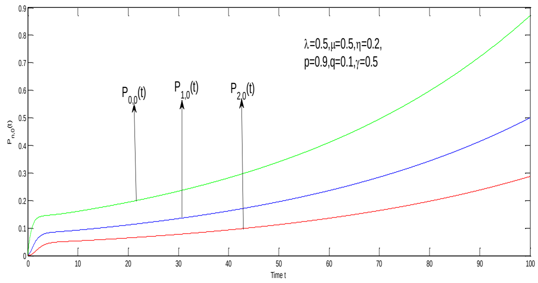

Figure 16 Variation of Probabilities $P_{n,0}(t)$ with respect to p=0.9

Comparing the three figures 13,15 & 16, the impact of the repair success probability p is clearly evident. At $p = 0.5$, the system experiences longer periods in disaster states and struggles to recover efficiently. However, as the repair probability increases to $p = 0.7$ and $p = 0.9$, the system's resilience improves drastically, allowing it to recover from disasters much faster and reducing the likelihood of extended downtimes. The higher the repair probability, the more robust the system becomes in handling failures, ensuring that it remains operational even when more customers are affected.

This comparison highlights the critical role that successful repair mechanisms play in maintaining system performance. While factors like arrival and service rates are important, the ability of the system to recover from disaster states is primarily driven by the repair success probability. In practical terms, improving the efficiency of repairs—whether through better maintenance processes, more reliable infrastructure, or faster recovery protocols—can significantly enhance the reliability and uptime of systems such as ATMs, telecommunications networks, or other critical services. As shown in these figures, optimizing repair probability is key to ensuring that systems can provide continuous service, even in the face of unexpected failures.

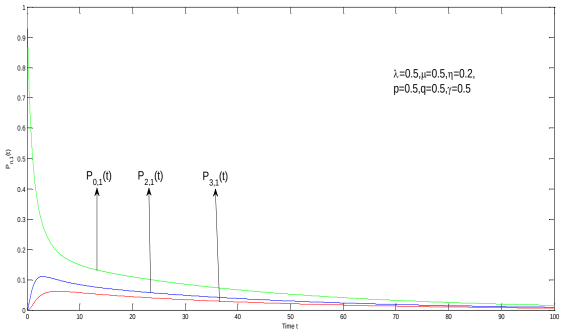

Figure 17 Variation of Probabilities $P_{n,1}(t)$ with respect to p=0.5

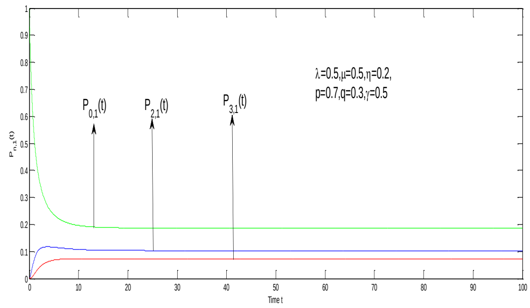

Figure 18 Variation of Probabilities $P_{n,1}(t)$ with respect to p=0.7

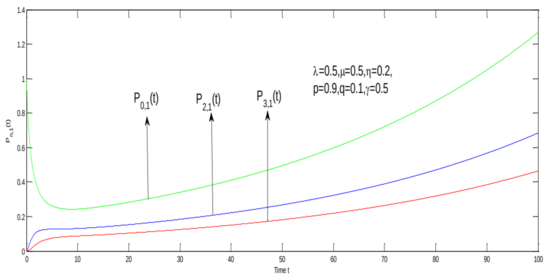

Figure 19 Variation of Probabilities $P_{n,1}(t)$ with respect to p=0.9

The comparison of the three figures 17,18 & 19 clearly demonstrates the critical influence of the repair success probability p on system performance. At p=0.5, the system struggles to recover efficiently, leading to frequent and prolonged downtimes, especially when multiple customers are present. As the repair probability increases to p=0.7 and p=0.9, the system's ability to recover improves significantly, allowing it to maintain operational states for longer periods and reducing the likelihood of extended failure periods.

The higher the repair probability, the more resilient the system becomes. At p=0.9, the system is able to recover from disasters almost immediately, reflecting the importance of optimizing repair mechanisms to maintain continuous operation. This comparison underscores the fact that while the arrival rate, service rate, and disaster rate affect system dynamics, the repair success probability is the most decisive factor in determining the system's long-term stability and performance. Ensuring a high repair success rate can dramatically reduce downtime, improve customer satisfaction, and keep the system operational even under stressful conditions.

In real-world applications, such as banking systems, telecommunications, or other critical infrastructures, improving repair mechanisms ensures minimal disruptions and enhances overall system reliability. The figures vividly demonstrate how optimizing this parameter can lead to a more resilient and dependable service, ensuring that systems recover quickly and provide seamless operations in the face of failures.

6 Conclusion

This study provides a comprehensive analysis of a single-server Markovian queueing model subject to potential disasters during both active service and idle periods. The model addresses scenarios where all waiting customers are removed upon a server breakdown, with the repair process beginning immediately. The unique feature of allowing arrivals to join the queue during repair and the probabilistic nature of the repair's success with probability $p < 1$ bring practical relevance to systems requiring resilience under uncertain repair conditions. The time-dependent probabilities derived in this work present a novel approach to evaluating the transient behavior of such a system, offering insights beyond traditional steady-state analyses. Through detailed performance measures and numerical illustrations, this study highlights the impact of key parameters on system resilience and efficiency, making the findings valuable for optimizing real-world queueing systems prone to failures and needing effective recovery strategies.

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