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Exploring Fuzzy Nabla Dynamics on Time Scales with the Characterization Theorem

Hari Kishore. R[1], Leelavathi. R[2*], Mohana Rupa. S. V. D[3], Muneera. A[4], Suresh Kumar.G[5]

[1] Department of Mathematics, Vasavi College of Engineering, Ibrahimbagh, Hyderabad, Telangana-500031, India .

[2,5] Department of Mathematics, Koneru Lakshmaiah Education Foundation, Vaddeswaram, AP-522302, India.

[3]Department of Mathematics, Sri Durga Malleswara Siddhartha Mahila Kalasala, Vijayawada, AP, India

[4]Department of Mathematics, Vijaya Institute of Technology for Women Enikepadu,Vijayawada, AP, India.

*Correspondence:drrlv2024@gmail.com, rleelavathi@kluniversity.in

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Abstract:

This paper is committed to proving the local existence, and uniqueness theorem for fuzzy nabla-dynamic equations on time scales (FNDES-$\mathcal{T}$) under the Characterization theorem for fuzzy functions with the help of generalized ∇-H (nabla Hukuhara) differentiability, which aims to convert FNDES-$\mathcal{T}$ into a system of nabla-dynamic equations on time scales (NDES-$\mathcal{T}$). If we take $\mathcal{T}$ as integers then the FNDES-$\mathcal{T}$ transform to the system of difference equations, or if we decide $\mathcal{T}$ as Reals then the FNDES-$\mathcal{T}$ transform to the system of differential equations. Moreover, it is accompanied by appropriate examples that improve the suggested work.

Keywords: Time scales; Fuzzy functions; Hukuhara difference; Generalized Hukuhara derivative; Fuzzy Nabla dynamic equations

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1. Introduction

The Theory of time scales was developed by German mathematician S. Hilger [14] in 1988, which combines the two equations simultaneously namely difference, and differential equations. The dynamic equations on $\mathcal{T}$ play a fascinating role in many applications and applied sciences fields. In some real-world physical problems, cryptic situations may arise where impreciseness arises. To Specify these kinds of arguments, a vague set is an eminent technique for dealing with such dynamic models. The very first attempt to study the calculus of fuzzy in $\mathcal{T}$ was made by Fard and Bidgoli [11]. Moreover, integration on $\mathcal{T}$ was introduced in [13]. Further, Vasavi et al.[23], Leela et. al.[16, 17, 18, 19, 20] studied the theory for fuzzy delta and nabla DETs. Deng et. al.[10], studied conformable fractional ∇-H derivatives on $\mathcal{T}$.

In 1983 Puri, Ralescu [22], investigated f-functions (fuzzy functions), and also generalized the concept of H-differentiability(Hukuhara differentiability) for set-valued mapping. The concept of Fuzzy dynamic equations (FDEs), using H-differentiability was established by Kaleva [15]. The admired approach to study the existence, uniqueness results is using the Seikkala differentiability or

H-differentiability for fuzzy number-valued functions. The different strategies to investigate the solutions of FDEs are studied in [3, 4, 7]. In the paper [1], the author proposed solutions for FDEs under the interpretation based on H-derivative and demonstrated this method with two examples. It shows that it is possible to transform FDEs into a system of Differential equations (DEs). In a paper [6], Bede proved a characterization theorem that states that under certain conditions a FIVP is the same as a system of DEs. Bede et.al[4], Pederson et.al.[21] stated that this characterization theorem can help to solve FIVPs numerically by transforming them to systems of DEs which can then be solved by any numerical method. So to obtain numerical solutions of FDEs under H-differentiability, it is not necessary to rephrase the entire literature on numerical solutions of DEs in the vague setting, but instead any numerical technique can be used directly. This paper discusses the solutions for FNDES-$\mathcal{T}$. Firstly, we give a characterization theorem for f-functions on $\mathcal{T}$. Based on this theorem we prove the local existence, uniqueness theorem for FNDES-$\mathcal{T}$. This theorem is useful to replace the FNDES-$\mathcal{T}$ by NDES-$\mathcal{T}$ and study the solutions of FNDES-$\mathcal{T}$.

In Section 2, we present a basic definition and theorem related to strongly generalized derivability for fuzzy number-valued functions ($\mathscr{I}_{\Re} = \mathbb{E}_1$). A characterization theorem for generalized nabla H-derivable functions on $\mathcal{T}$ was obtained and the sufficient conditions for local existence, and uniqueness of solutions for FNDES-$\mathcal{T}$ are studied in Section 3. Further, in Section 4, we study solutions for FNDEs on $\mathcal{T}$ using the characterization theorem and also transform a FNDES-$\mathcal{T}$ to the system of differential equations or system of difference equations. In the end, numerical examples are given.

2. Preliminaries

This section deals with the strongly-generalized differentiable functions, definitions, notations and some other basic properties.

Definition 2.1 *[3] Let* $\mathcal{A}: (y, y_1) \to \mathscr{I}_{\Re}$ *and* $y_0 \in (y, y_1)$. *If there exists* $\mathcal{A}'(y_0) \in \mathscr{I}_{\Re} \ni$

• for $h > 0$, the H-difference $\mathcal{A}(y_0 + h) \ominus \mathcal{A}(y_0)$, $\mathcal{A}(y_0) \ominus \mathcal{A}(y_0 - h)$ and the limits exists

$$\lim_{h\to 0} \frac{\mathcal{A}(y_0+h)\ominus\mathcal{A}(y_0)}{h} = \lim_{h\to 0} \frac{\mathcal{A}(y_0)\ominus\mathcal{A}(y_0-h)}{h} = \mathcal{A}'(y_0),$$

Or

• for $h > 0$, the H-difference $\mathcal{A}(y_0) \ominus \mathcal{A}(y_0 + h)$, $\mathcal{A}(y_0 - h) \ominus \mathcal{A}(y_0)$ and the limits exists

$$\lim_{h\to 0} \frac{\mathcal{A}(y_0)\ominus\mathcal{A}(y_0+h)}{-h} = \lim_{h\to 0} \frac{\mathcal{A}(y_0-h)\ominus\mathcal{A}(y_0)}{-h} = \mathcal{A}'(y_0),$$

Or

• for $h > 0$, the H-difference $\mathcal{A}(y_0 + h) \ominus \mathcal{A}(y_0)$, $\mathcal{A}(y_0 - h) \ominus \mathcal{A}(y_0)$ and the limits exists

$$\lim_{h\to 0} \frac{\mathcal{A}(y_0+h)\ominus\mathcal{A}(y_0)}{h} = \lim_{h\to 0} \frac{\mathcal{A}(y_0-h)\ominus\mathcal{A}(y_0)}{-h} = \mathcal{A}'(y_0),$$

Or

• for $h > 0$, the H-difference $\mathcal{A}(y_0) \ominus \mathcal{A}(y_0 + h)$, $\mathcal{A}(y_0) \ominus \mathcal{A}(y_0 - h)$ and the limits exists

$$\lim_{h\to 0} \frac{\mathcal{A}(y_0)\ominus\mathcal{A}(y_0+h)}{-h} = \lim_{h\to 0} \frac{\mathcal{A}(y_0)\ominus\mathcal{A}(y_0-h)}{h} = \mathcal{A}'(y_0).$$

Then, $\mathcal{A}$ is strongly-generalized derivable at y_0.

Using the concept of fuzzy interval definition, we have $[a]^\alpha$ is the compact interval for any α in [0,1]. Denote α −level set of a as $[a]^\alpha = [a_-^\alpha, a_+^\alpha]$, where $a_+ : [0,1] \to \Re$ and $a_- : [0,1] \to \Re$ are crisp functions and $a_-(\alpha) = a_-^\alpha, a_+(\alpha) = b_+^\alpha$ are the end-point functions of a.

Theorem 2.1 *[12] A fuzzy set A is determined completely as $A = [a_-, a_+]$ of functions $a_-, a_+: [0,1] \to \Re$, defining the end points of the α-level sets, satisfying the following*

- $a_-: \alpha \to a_-^\alpha \in \Re$ is R-continuous at $\alpha = 0$ and L-continuous, bounded nondecreasing function $\forall \quad \alpha \in (0,1]$;
- $a_+: \alpha \to a_+^\alpha \in \Re$ is R-continuous at $\alpha = 0$ and L-continuous, bounded nonincreasing function $\forall \quad \alpha \in (0,1]$;
- $a_-^1 \le a_+^1$ at $\alpha = 1$.

From all of the above, $a_-^\alpha \le a_+^\alpha$, for $\alpha \in [0,1]$.

Define $D_H: \mathscr{J}_\Re \times \mathscr{J}_\Re \to [0,\infty)$ as $D_H(m,n) = \sup_{0\le\alpha\le1} \max\{|m_-^\alpha - n_-^\alpha|, |m_+^\alpha - n_+^\alpha|\}$, is the Hausdorff distance between fuzzy intervals and denote $[m]^\alpha = [m_-^\alpha, m_+^\alpha]$, $[n]^\alpha = [n_-^\alpha, n_+^\alpha]$.

3 Characterization Theorem of Generalized ∇-H-Differentiability

The below theorem characterizes the generalized fuzzy ∇-derivative of a f-function in $\mathscr{J}_\Re$ as the ∇-derivatives of its endpoint functions.

Theorem 3.1 *Let $\mathcal{A}: \mathcal{T}^{[a,b]} \to \mathscr{J}_\Re$. Then the following holds:*

- If $\mathcal{A}$ is ∇^g derivable function and y is L-dense point in $\mathcal{T}$, then one of the following holds
- a^α, b^α are ∇-derivable at y, uniformly in $\alpha \in [0,1]$ and either

$$[\mathcal{A}^{\nabla^g}(y)]^\alpha = [a_\alpha^\nabla(y), b_\alpha^\nabla(y)] \tag{1}$$

or

$$[\mathcal{A}^{\nabla^g}(y)]^\alpha = [b_\alpha^\nabla(y), a_\alpha^\nabla(y)]. \tag{2}$$

- If $(a_-^\alpha)^\nabla(y)$, $(a_+^\alpha)^\nabla(y)$ and $(b_-^\alpha)^\nabla(y)$, $(b_+^\alpha)^\nabla(y)$ exists, uniformly in $0 \le \alpha \le 1$, satisfies $(a_-^\alpha)^\nabla = (b_+^\alpha)^\nabla$ and $(a_+^\alpha)^\nabla = (b_-^\alpha)^\nabla$ with either

$$[\mathcal{A}^{\nabla^g}(y)]^\alpha = [(a_+^\alpha)^\nabla(y), (b_+^\alpha)^\nabla(y)] = [(b_-^\alpha)^\nabla(y), (a_-^\alpha)^\nabla(y)], \forall \quad 0 \le \alpha \le 1$$

or

$$[\mathcal{A}^{\nabla^g}(y)]^\alpha = [(b_+^\alpha)^\nabla(y), (a_+^\alpha)^\nabla(y)] = [(a_-^\alpha)^\nabla(y), (b_-^\alpha)^\nabla(y)], \forall \quad 0 \le \alpha \le 1.$$

- If $\mathcal{A}$ is ∇^g derivable at L-scattered point $y \in \mathcal{T}_k^{[a,b]}$, then a^α and b^α are ∇-derivable and any one of the following holds:

$$[\mathcal{A}^{\nabla^g}(y)]^\alpha = [a_\alpha^\nabla(y), b_\alpha^\nabla(y)], \forall \quad 0 \le \alpha \le 1 \tag{3}$$

or

$$[\mathcal{A}^{\nabla^g}(y)]^\alpha = [b_\alpha^\nabla(y), a_\alpha^\nabla(y)], \forall \quad 0 \le \alpha \le 1. \tag{4}$$

Proof:

- If y is leftdense-point, $\mathcal{A}$ is ∇^g derivable, then the proof is same to the proof of Theorem 3 in [9]
- If y is L-scattered then $\mathcal{A}$ is either GH_3 or GH_4 ∇-derivable and for any $0 \le \alpha \le 1$, then

$$[\mathcal{A}(y) \ominus_h \mathcal{A}(\rho(y))]^\alpha = [a_\alpha(y) - a_\alpha(\rho(y)), b^\alpha(y) - b^\alpha(\rho(y))] \tag{5}$$

or

$$[\mathcal{A}(\rho(y)) \ominus_h \mathcal{A}(y)]^\alpha = [a_\alpha(\rho(y)) - a_\alpha(y), b^\alpha(\rho(y)) - b^\alpha(y)] \tag{6}$$

on multiplying the equations (5) by $\frac{1}{\upsilon(y)}$ and (6) by $\frac{-1}{\upsilon(y)}$, we get

$$\frac{1}{\upsilon(y)} \odot [\mathcal{A}(y) \ominus_h \mathcal{A}(\rho(y))]^\alpha = \frac{1}{\upsilon(y)} \odot [a_\alpha(y) - a_\alpha(\rho(y)), b^\alpha(y) - b^\alpha(\rho(y))]$$
$$= \left[\frac{a_\alpha(y) - a_\alpha(\rho(y))}{\upsilon(y)}, \frac{b^\alpha(y) - b^\alpha(\rho(y))}{\upsilon(y)}\right]$$
$$= [a_\alpha^\nabla(y), b_\alpha^\nabla(y)]$$

or

$$\frac{-1}{\upsilon(y)} \odot [\mathcal{A}(\rho(y)) \ominus_h \mathcal{A}(y)]^\alpha = \frac{-1}{\upsilon(y)} \odot [a^\alpha(\rho(y)) - a^\alpha(y), b^\alpha(\rho(y)) - b^\alpha(y)]$$
$$= [\frac{b^\alpha(\rho(y)) - b^\alpha(y)}{-\upsilon(y)}, \frac{a^\alpha(\rho(y)) - a^\alpha(y)}{-\upsilon(y)}]$$
$$= \left[\frac{b^\alpha(y) - b^\alpha(\rho(y))}{\upsilon(y)}, \frac{a^\alpha(y) - a^\alpha(\rho(y))}{\upsilon(y)}\right]$$
$$= [b_\alpha^\nabla(y), a_\alpha^\nabla(y)]$$

Example 3.1 *Let $\mathcal{A}: \mathcal{T}^{[a,b]} \to \mathscr{F}_{\Re}$ be a f-function defined by $\mathcal{A}(y) = (2,4,6) \odot y$, where the level-sets are*

$$[\mathcal{A}(y)]^\alpha = [2 + 2\alpha, 6 - 2\alpha] \odot y, for \;\; all \quad \alpha \in [0,1].$$

$$[\mathcal{A}(y)]^\alpha = \begin{cases} (6 - 2\alpha)y, if y < 0 \\ (1 + 2\alpha)y, if y \geq 0, \end{cases}$$

and

$$[b(y)]^\alpha = \begin{cases} (1 + 2\alpha)y, if y < 0 \\ (6 - 2\alpha)y, if y \geq 0. \end{cases}$$

If $\mathcal{T}^{[a,b]} = \Re$, then $[a(y)]^\alpha, [b(y)]^\alpha$ are ∇-derivable at $y = 0$, In fact, $\nabla a_-^\alpha(0) = \nabla h_+^\alpha(0)$ and $\nabla a_+^\alpha(0) = \nabla h_-^\alpha(0)$, but

$\lim_{h\to 0^+} a^\alpha(y) \neq \lim_{h\to 0^+} b^\alpha(y)$. Now, it is necessary to show that at leftdense-point s

$$[a^\nabla(0)]_+^\alpha = \lim_{b\to 0^+} \frac{[a(h+0) \ominus_h a(0)]^\alpha}{h}$$
$$= \lim_{h\to 0^+} \frac{1}{h} [\min\{(1 + 2\alpha)h, (6 - 2\alpha)h\}, \max\{(1 + 2\alpha)h, (6 - 2\alpha)h\}]$$
$$= [1 + 2\alpha, 6 - 2\alpha]$$

and

$$[a^\nabla(0)]_-^\alpha = \lim_{h\to 0^+} \frac{[a(0-h) \ominus_h a(0)]^\alpha}{-h}$$

$$= \lim_{h\to 0^+} \frac{-1}{h}[\min\{(6-2\alpha),(1+2\alpha)\},\max\{(6-2\alpha),(1+2\alpha)\}](-h)$$

$$= [1+2\alpha, 6-2\alpha]$$

Therefore, $\mathcal{A}$ is ∇^g derivable, but the end-point functions are not ∇-derivable and $\mathcal{A}$ satisfies the condition of Theorem 3.1(1(b)) and $[\mathcal{A}^\nabla(y)]^\alpha = [1+2\alpha, 6-2\alpha]$.

If $\mathcal{T} = \mathcal{Z}$, then $\rho(y) = y-1$ and $v(y) = 1$, also $\nabla\mathcal{A}(y) = \mathcal{A}(y) - \mathcal{A}(y-1)$. For $y = 0$ the end-point functions are ∇-derivable, then $[a^\nabla(0)]^\alpha = [1+2\alpha, 6-2\alpha], [b^\nabla(0)]^\alpha = [6-2\alpha, 1+2\alpha]$ are ∇-derivable. At L-scattered point s, we have

$$[\mathcal{A}^\nabla(0)]^\alpha = \frac{\mathcal{A}^\alpha(\rho(0))\ominus\mathcal{A}^\alpha(0)}{-v(0)}$$

$$= \frac{\mathcal{A}^\alpha(-1)\ominus\mathcal{A}^\alpha(0)}{-1}$$

$$= \frac{-1}{1}\odot[1+2\alpha, 6-2\alpha](-1)\ominus_h[0,0]$$

$$= [1+2\alpha, 6-2\alpha].$$

Therefore, $\mathcal{A}$ is GH_3 ∇-derivable at L-scattered point $y = 0$.

Now, consider $\mathcal{T} = [-2,-1] \cup 1$, i.e., $\mathcal{T}$ is left-scattered at $y = 1$, and $\rho(y) = -1; v(y) = 2$. Thus,

$$[a^\nabla(1)] = \frac{[a^\alpha(1) - a^\alpha(-1)}{2}$$

$$= \frac{1+2\alpha-(6-2\alpha)(-1)}{2} = \frac{7+0\alpha}{2} = \frac{7}{2}$$

and

$$[b^\nabla(1)] = \frac{[b^\alpha(1) - b^\alpha(-1)}{2}$$

$$= \frac{6-2\alpha-(1+2\alpha)(-1)}{2} = \frac{7+0\alpha}{2} = \frac{7}{2}$$

For L-scattered at $y = 1$, we have

$$[\mathcal{A}^\nabla(1)]^\alpha = \frac{[\mathcal{A}(1)\ominus_h\mathcal{A}(\rho(1))}{v(1)} = \frac{\mathcal{A}\ominus_h\mathcal{A}(-1)}{2}$$

$$= \frac{1}{2}[1+2\alpha, 6-2\alpha] - [6-2\alpha(-1), 1+2\alpha(-1)) = \frac{7+0\alpha}{2} = \frac{7}{2}.$$

4 Local Existence-uniqueness and Characterization Theorems

This section aims to prove the local existence, uniqueness of the FNDEs on $\mathcal{T}$. Using this theorem, we establish a characterization theorem, which converts FNDEs to system of NDEs on $\mathcal{T}$.

Definition 4.1 *A function* $\mathcal{A}: \mathcal{T} \times \mathscr{J}_{\mathfrak{R}} \to \mathscr{J}_{\mathfrak{R}}$ *is* leftdense-continuous, if $\mathscr{B}$ defined by $\mathscr{B}(y) = \mathcal{A}(y, v(y))$ is leftdense-continuous where $v: \mathcal{T} \to \mathscr{J}_{\mathfrak{R}}$ is a continuous function

- bounded on $S \subset \mathcal{T} \times \mathscr{J}_{\mathfrak{R}}$, if ∃ a constant $M > 0$ ∋

$$D_H(\mathcal{A}(y,v), \hat{0}) \leq M, for\ all \quad (y,v) \in S.$$

• Lipschitz-continuous on a set $S \subset \mathcal{T} \times \mathscr{J}_{\Re}$, if ∃ a constant $L > 0$ such that

$$D_H(\mathcal{A}(y, v_1), \mathcal{A}(y, v_2)) \leq L D_H(v_1, v_2), for\ \ all \quad (y, v_1), (y, v_2) \in S.$$

Theorem 4.1 *Let $y_0 \in \mathcal{T}$, $t_0 \in \mathscr{J}_{\Re}$ and $l, n > 0$ with $inf(\mathcal{T}) \leq y_0 - l$ and $sup(\mathcal{T}) \geq y_0 + l$. Denote $\mathcal{T}_l = \mathcal{T}^{[y_0 - l, y_0 + l]}$ and $B_q = \{D_H(v, v_0) \leq n, \quad for\ \ all \quad v \in \mathscr{J}_{\Re}\}$.*

Assume that $\mathcal{A}: \mathcal{T}_l \times B_n \to \mathscr{J}_{\Re}$ is leftdense-continuous, bounded, and Lipschitz continuous. Then the FNIVP

$$v^{\nabla} = \mathcal{A}(y, v), v(y_0) = v_0 \tag{7}$$

has exactly one GH_1 or GH_4 ∇-differentiable solution on $\mathcal{T}_\eta = \mathcal{T}^{[y_0 - \eta, y_0 + \eta]}$, where $\eta = \min\{l, \frac{n}{M}, \xi\}$, with $\xi < \frac{1}{L}$. Moreover, if there exists $\beta > 0$ such that $y \in \mathcal{T}_\alpha$, the Hukuhara-difference of $v_0 \ominus_h (-1) \odot \int_{y_0}^{y} \mathcal{A}(\xi, v(\xi)) \nabla \xi$ exists, then the FNIVP (7) has exactly one GH_2 or GH_3 ∇-differentiable solution on $\mathcal{T}^{[y_0 - \eta, y_0 + \eta]}$, where $\eta = \min\{l, \frac{n}{M}, \xi, \beta\}$. If y_0 is Left-scattered and $\eta < v(y_0)$, then the solution of FNIVP (7) exists on $\mathcal{T}^{[\rho(y_0), y_0 + \eta]}$.

Proof. Let the space $\mathcal{C}(S, I)$ be all continuous functions $\mathcal{A}: S \to T$. Put $Q_1 = \mathcal{Q}(\mathscr{J}_l, \mathscr{J}_{\Re})$ and define an operator $I_1: Q_1 \to Q_1$ by

$$I_1(v(y)) = v_o \oplus \int_{y_0}^{y} \mathcal{A}(\xi, v(\xi)) \nabla \xi, \quad I_1(v)(y_0) = v_0,$$

where $v \in \mathcal{Q}(\mathscr{J}_l, \mathscr{J}_{\Re})$. Consider $r_0 = \min(\alpha, l)$, and

$Q_2 = \mathcal{Q}(\mathscr{J}_{r_0}, \mathscr{J}_{\Re})$ and define an operator $I_2: C_2 \to C_2$ by

$$I_2(v(y)) = v_0 \ominus_h (-1) \odot \int_{y_0}^{y} \mathcal{A}(\xi, v(\xi)) \nabla \xi, \quad I_2(v)(y_0) = v_0,$$

where $v \in \mathcal{Q}(\mathscr{J}_{r_0}, \mathscr{J}_{\Re})$.

$$\begin{aligned} D_H(I_1(v(y)), v_0) &= D_H\left(\int_{y_0}^{s} \mathcal{A}(\xi, v(\xi)) \nabla \xi, \hat{0}\right) \\ &\leq \int_{y_0}^{s} D_H(\mathcal{A}(\xi, v(\xi)) \nabla \xi, \hat{0}) \\ &\leq |s - y_0| M, for\ \ all \quad s \in \mathcal{T}_l. \end{aligned}$$

Similarly,

$$D_H(I_2(v(y)), v_0) \leq |s - y_0| M, \forall \in \mathcal{T}_{y_0},$$

where

$$M = \sup_{(v,s) \in \mathcal{T}_l \times B_n} D_H(\mathcal{A}(s, v(y)), \hat{0}).$$

Let $\tilde{Q}_1 = Q(\mathcal{T}_{l_1}, B_n)$, $l_1 = \min\{l, \frac{n}{M}\}$, and letting $I_1: \tilde{Q}_1 \to Q(\mathcal{T}_{l_1}, \mathscr{J}_{\Re})$, then we have $D_H(I_1(v(y)), v_o) \leq n$, $y \in \mathcal{T}_{l_1}$.

Again, $\tilde{Q}_2 = C(\mathcal{T}_{l_2}, B_n)$, take $l_2 = \min\{l, \frac{n}{M}, \beta\}$, and letting $I_2: \tilde{Q}_2 \to Q(\mathcal{T}_{l_2}, \mathscr{J}_{\mathfrak{R}})$, then we have $D_H(I_2(v(y), v_o) \leq n$, $y \in \mathcal{T}_{l_2}$. Thus, for $v \in \tilde{Q}_1$ and $v \in \tilde{Q}_2$, we have $I_1(v) \in \tilde{Q}_1$,$I_2(v) \in \tilde{Q}_2$. We observe that both $\tilde{Q}_1, \tilde{Q}_2$ are having uniform distance and are omplete metric spaces.

Now, we prove that I_1, I_2 are contraction mappings. For $v_1, v_2 \in \tilde{Q}_1$,

Consider

$$D_H[I_1(v_1(y)), I_1(v_2(y))] = D_H\left(\int_{y_0}^{y} \mathcal{A}(\xi, v_1(\xi))\nabla\xi, \int_{y_0}^{y} \mathcal{A}(\xi, v_2(\xi))\nabla\xi, \right)$$
$$\leq L|y - y_0| D_H(v_1, v_2), \forall \quad y \in \mathcal{T}_{l_1}.$$

Similarly, for $v_1 v_2 \in \tilde{Q}_2$, we have

$$D_H[I_1(v_1(y)), I_1(v_2(y))] = D_H\left[\int_{y_0}^{y} \mathcal{A}(\xi, v_1(\xi))\nabla\xi, \int_{y_0}^{y} \mathcal{A}(\xi, v_2(\xi))\nabla\xi, \right]$$
$$\leq L|y - y_0| D_H(v_1, v_2), \forall \quad s \in \mathcal{T}_{l_2}.$$

Choose $m_1 = \min\{l_1, \epsilon\}$ and $m_2 = \min\{l_2, \epsilon\}$, with $\epsilon < \frac{1}{L}$ and $I_1: M_1 \to M_1$ and $I_2: M_2 \to M_2$, with $M_1 = Q(\mathcal{T}_{m_1}, B_q)$, $M_2 = Q(\mathcal{T}_{m_2}, B_n)$. Thus, I_1, I_2 are contraction mapping on M_1 and M_2 respectively.

From Banach's fixed point theorem, there exists exactly one fixed point $v_1 \in M_1$ for the contraction I_1 and $v_2 \in m_2$ for the contraction I_2, then for $I_1(v_1) = v_1$ and $I_2(v_2) = v_2$.

Therefore, v_1 has exactly one GH_1 or GH_4 ∇-differentiable solution to the corresponding integral form equation $v_1(y) = v_0 \oplus \int_{y_0}^{y} \mathcal{A}(\xi, v(\xi))\nabla\xi$and v_2 has exactly one GH_2 or GH_3 ∇-differentiable solution to the corresponding integral form equation

$$v_2(y) = v_0 \ominus_h (-1) \odot \int_{y_0}^{y} \mathcal{A}(\xi, v(\xi))\nabla\xi.$$

If y_0 is Left-scattered and u is leftdense-continuous at y_0, then from Theorem 2.1. in [16], we have $v(y) \ominus_h v(\rho(y)) = v(y) \odot \mathcal{A}^{\nabla g}(y)$ i.e., $v_1(y) = v(y) = v(\rho(y)) \oplus v(y) \odot \mathcal{A}(s, v(y))$.

$$(or)$$

From Theorem 3.4. in [17], we have $v(\rho(y)) \ominus_h v(y) = (-1)v(y) \odot \mathcal{A}^{\nabla g}(y)$ i.e., $v_2(y) = v(\rho(y)) \ominus_h (-1)v(y) \odot \mathcal{A}(s, v(y))$. Thus, the solutions are uniquely determined.

Definition 4.2 *The family* $\{\mathcal{A}^{\alpha}(y, v_-, v_+), \quad for \ \ all \quad \alpha \in [0,1]\}$ is leftdense-equicontinuous function. If, for every $v: \mathcal{T} \to \mathscr{J}_{\mathfrak{R}}$ is continuous, $\mathcal{A}^{\alpha}(y, v_-(y), v_+(y))$ is leftdense-continuous $\forall\ 0 \leq \alpha \leq 1$.

The family $\mathcal{A}^{\alpha}(y, v_-, v_+)$, for all $0 \leq \alpha \leq 1$ is uniformly leftdense-equicontinuous function. If, for every $v: \mathcal{T} \to \mathscr{J}_{\mathfrak{R}}$ is continuous, $\mathcal{A}^{\alpha}(y, v_-(y), v_+(y))$ is uniformly leftdense-continuous for all $0 \leq \alpha \leq 1$.

Theorem 4.2 *Let $y_0 \in \mathcal{T}$ and $v_0 \in \mathscr{J}_{\mathfrak{R}}$, with $sup(\mathcal{T}) \geq y_0 + l$, $inf(\mathcal{T}) \leq y_0 - l$. For all $l, n > 0$, denote $\mathcal{T}_l = \mathcal{T}^{[y_0 - l, y_0 + l]}$, and $B_n = \{D_H(v, v_0) \leq n \quad \forall \quad v \in \mathscr{J}_{\mathfrak{R}}\}$. Suppose $\mathcal{A}: \mathcal{T}_l \times B_n \to \mathscr{J}_{\mathfrak{R}}$ is bounded such that*

$$[\mathcal{A}(y,v)]^{\alpha}=[a_{-}^{\alpha}(y,v_{-}^{\alpha},v_{+}^{\alpha}),a_{+}^{\alpha}(y,v_{-}^{\alpha},v_{+}^{\alpha})],\quad for\ all\quad 0\le\alpha\le 1.$$

If $\{a_{\pm}^{\alpha}(y,v_{-}^{\alpha},v_{+}^{\alpha})\}$ is the family of uniformly bounded, leftdense-equicontinuous for (y,v) in any bounded set, uniformly Lipschitz with $M>0$ such that

$$|a_{\pm}^{\alpha}(y,v_{-}^{\alpha},v_{+}^{\alpha})-a_{\pm}^{\alpha}(y,w_{-}^{\alpha},w_{+}^{\alpha})|\le M\max\{|v_{-}^{\alpha}-w_{-}^{\alpha}|,|v_{+}^{\alpha}-w_{+}^{\alpha}|\},$$

for all $0\le\alpha\le 1\ and\quad (y,v),(y,w)\in\mathcal{T}_l\times B_n$. Also if the end-points functions v_{-},v_{+} are ∇-derivable, then the FNDETs

$$v^{\nabla}(y)=\mathcal{A}(y,v(y)),v(y_0)=v_0 \tag{8}$$

is equal to the union of the two families of system of NDEs

$$\begin{cases}(v_{-}^{\alpha})^{\nabla}=a_{-}^{\alpha}(y,v_{-}^{\alpha},v_{+}^{\alpha}), & v_{-}^{\alpha}(y_0)=(v_0)_{-}^{\alpha},\\(v_{+}^{\alpha})^{\nabla}=a_{+}^{\alpha}(y,v_{-}^{\alpha},v_{+}^{\alpha}), & v_{+}^{\alpha}(y_0)=(v_0)_{+}^{\alpha},\end{cases} \tag{9}$$

$$\begin{cases}(v_{-}^{\alpha})^{\nabla}=a_{+}^{\alpha}(y,v_{-}^{\alpha},v_{+}^{\alpha}), & v_{-}^{\alpha}(y_0)=(v_0)_{-}^{\alpha},\\(v_{+}^{\alpha})^{\nabla}=a_{-}^{\alpha}(y,v_{-}^{\alpha},v_{+}^{\alpha}), & v_{+}^{\alpha}(y_0)=(v_0)_{+}^{\alpha}.\end{cases} \tag{10}$$

Proof. The uniformly leftdense-equicontinuity of $a_{\pm}^{\alpha}$ guarantees the leftdense-continuity of $\mathcal{A}$, while the Lipschitz property in the statement of the theorem guarantees that $\mathcal{A}$ satisfies the Lipschitz property as follows

$$\begin{aligned}&D_H(\mathcal{A}(y,v),(y,w))\\&=\sup_{0\le\alpha\le1}\max\{|a_{-}^{\alpha}(y,v_{-}^{\alpha},v_{+}^{\alpha})-a_{-}^{\alpha}(y,w_{-}^{\alpha},w_{+}^{\alpha})|,|a_{+}^{\alpha}(y,v_{-}^{\alpha},v_{+}^{\alpha})-a_{+}^{\alpha}(y,w_{-}^{\alpha},w_{+}^{\alpha})|\}\\&\le M\sup_{0\le\alpha\le1}\max\{|v_{-}^{\alpha}-w_{-}^{\alpha}|,|v_{+}^{\alpha}-w_{+}^{\alpha})|\}\\&=MD_H(v,w)\quad\forall\quad(y,v),(y,w)\in\mathcal{T}_l\times B_n.\end{aligned}$$

Then by the leftdense-continuity, the boundedness property and from above Lipschitz condition also from Theorem 4.1 and guarantees that, the FNDETs (8) has exactly one GH_4 or GH_1 ∇-differentiable (or) GH_4 or GH_3 ∇-differentiable solution which are v_1, v_2 respectively.

Therefore, from Theorem 3.1, $[v_{-}^{\alpha},v_{+}^{\alpha}]$, and $[v_{+}^{\alpha},v_{-}^{\alpha}]$ are the corresponding level-sets of v_1, v_2, with ∇-differentiable end-points. The level-sets of $v_1^{\nabla g}$ w.r.t to GH_1 or GH_4 ∇-differentiability is $[\nabla v_{-}^{\alpha},\nabla v_{+}^{\alpha}]$ and the level-set of $v_2^{\nabla g}$ w.r.t to GH_2 or GH_3 ∇-differentiablility is $[\nabla v_{+}^{\alpha},\nabla v_{-}^{\alpha}]$.

Thus, by the equality of fuzzy numbers, we guaranteed that $[v_1]^{\alpha}=[v_{-}^{\alpha},v_{+}^{\alpha}]$ is a solution of the family of NDEs (9) and $[v_2]^{\alpha}=[v_{+}^{\alpha},v_{-}^{\alpha}]$, is a solution of family of NDEs (10) on $\mathcal{T}$, for all $0\le\alpha\le 1$.

Conversely, for each $\alpha\in[0,1]$, $[v_{-},v_{+}]$ is a solution to the system of NDEs (9), and from the hypothesis, we have unique GH_1 or GH_4 ∇-differentiable solution. Hence, for all $0\le\alpha\le 1$, $[\tilde{v}^{\nabla g}]^{\alpha}=[\nabla v_{-}^{\alpha},\nabla v_{+}^{\alpha}]$, are the level-sets of the f-functions and their endpoints itself a solution to (9).

Similarly, $[\tilde{v}^{\nabla g}]^{\alpha}=[\nabla v_{+}^{\alpha},\nabla v_{-}^{\alpha}]$, the level-set of f-functions and their endpoints is itself a solution to (10). Since, $[\tilde{v}_1]^{\alpha}=[v_{-}^{\alpha},v_{+}^{\alpha}]$ is the unique solution of the system (9) and $[\tilde{v}_2]^{\alpha}=[v_{+}^{\alpha},v_{-}^{\alpha}]$ is the unique solutions of the system (10).

Finally, we conclude that the two families of systems of NDEs (9) and (10) on $\mathcal{T}$ are the same as the solutions of the FNIVP (8) on $\mathcal{T}$.

Example 4.1 *Let*

$$v^{\nabla}(y) = -3v(y) \oplus (2,4,6) \odot s, \quad v(y_0) = v_0. \tag{11}$$

be the FNDEs on $\mathcal{T}$. Corresponding to GH_2 or GH_3-derivative, the Equation (11), can be transformed to the following system of NDEs on $\mathcal{T}$

$$\begin{cases} (v_-^{\alpha})^{\nabla}(y) = -3v_-^{\alpha}(y) \oplus (6-2\alpha) \odot y, v_-^{\alpha}(y_0) = v_0^{\alpha} \\ (v_+^{\alpha})^{\nabla}(y) = -3v_+^{\alpha}(y) \oplus (2+2\alpha) \odot y, v_+^{\alpha}(y_0) = v_5^{\alpha} \, where [v_o]^{\alpha} = [v_0^{\alpha}, v_5^{\alpha}]. \\ for \;\; all \quad 0 \le \alpha \le 1, \end{cases} \tag{12}$$

Since $1 - p(y)(v(y)) \neq 0 \quad \forall s \quad \in \mathcal{T}_k$, for any time scales $\mathcal{T}$ the system (12) is regressive and from Remark 3.43. in [8], the solution of (12) is of the form

$$\begin{cases} v_-^{\alpha}(y) = \hat{e}_{-3}(y, y_0) \odot \left[v_0^{\alpha} \oplus \int_{y_0}^{s} \hat{e}_{-3}(y_0, \rho(\xi)) \odot a_1^{\alpha}(\xi) \nabla \xi \right], \\ v_+^{\alpha}(y) = \hat{e}_{-3}(y, y_0) \odot \left[v_0^{\alpha} \oplus \int_{y_0}^{s} \hat{e}_{-3}(y_0, \rho(\xi)) \odot a_2^{\alpha}(\xi) \nabla \xi \right], \end{cases}$$

where $a_1^{\alpha}(y) = (6-2\alpha) \odot s$ and $a_2^{\alpha}(y) = (2\alpha + 2)s$.

If $\mathcal{T} = \Re$ and $y_0 = 0$, then

$$\begin{cases} v_-^{\alpha}(y) = \hat{e}^{-3y} \odot \left[v_0^{\alpha} \oplus \int_{y_0}^{y} \hat{e}^{-3s} \odot a_1^{\alpha}(\xi) \nabla \xi \right] \\ v_+^{\alpha}(y) = \hat{e}^{-3y} \odot \left[v_0^{\alpha} \oplus \int_{y_0}^{y} \hat{e}^{-3s} \odot a_2^{\alpha}(\xi) \nabla \xi \right] \end{cases}$$

is the solution of (12).

If $\mathcal{T} = \mathbb{Z}$ and $y_0 = 0$, then

$$\begin{cases} v_-^{\alpha}(y) = \left(\frac{1}{4}\right)^{y} \odot \left[v_0^{\alpha} \oplus \frac{1}{4} \int_{y_0}^{y} a_1^{\alpha}(\xi) \nabla \xi \right] \\ v_+^{\alpha}(y) = \left(\frac{1}{4}\right)^{y} \odot \left[v_5^{\alpha} \oplus \frac{1}{4} \int_{y_0}^{y} a_2^{\alpha}(\xi) \nabla \xi \right] \\ \end{cases}$$

is the solution of (12).

By similar reasoning as above we get the solution for the NDEs (13) on $\mathcal{T}$ and is of the form Again, corresponding to GH_1 or GH_4 derivative, the FNDEs (11) on $\mathcal{T}$ can be converted to the following system of NDEs on$\mathcal{T}$

$$\begin{cases} (v_-^{\alpha})^{\nabla}(y) = -3v_+^{\alpha}(y) \oplus (2+2\alpha) \odot s, v_-^{\alpha}(y_0) = v_0^{\alpha} \\ (v_+^{\alpha})^{\nabla}(y) = -3v_-^{\alpha}(y) \oplus (6-2\alpha) \odot s, v_+^{\alpha}(y_0) = v_5^{\alpha} \, where [v_o]^{\alpha} = [v_0^{\alpha}, v_5^{\alpha}]. \\ for \;\; all \quad 0 \le \alpha \le 1, \end{cases} \tag{13}$$

By comparative thinking as above, we get the solution for the NDEs (13) on $\mathcal{T}$ and is of the form

$$\begin{cases} v_-^\alpha(y) = \hat{e}_{-3}(y, y_0) \odot \left[v_0^\alpha \oplus \int_{y_0}^{s} \hat{e}_{-3}(y_0, \rho(\xi)) \odot a_1^\alpha(\xi) \nabla\xi \right], \\ v_+^\alpha(y) = \hat{e}_{-3}(y, y_0) \odot \left[v_5^\alpha \oplus \int_{y_0}^{s} \hat{e}_{-3}(y_0, \rho(\xi)) \odot a_2^\alpha(\xi) \nabla\xi \right], \end{cases}$$

where $a_1^\alpha(y) = (2\alpha + 2) \odot s$ and $a_2^\alpha(y) = (6 - 2\alpha) \odot s$.

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