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Forward-Backward System for Fuzzy Doubly Stochastic Differential Equations

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Abstract:

Our main goal in this work is to study a general class of forward-backward fuzzy double stochastic differential equations systems (in short FBFDSDEs) in a precise manner, not relaying the equations on the solution process only, but rather on their law. By verifying the existence and uniqueness of the solution to the forward equation of the general system FBFDSDEs as well as the backward equation under Lipschitz conditions, we prove the existence of the maximum solution of the general system FBFDSDEs under monotonic continuity conditions.

Keywords: Brownian motion, Forward-backward Doubly stochastic differential equations, Conditional expectation, Fuzzy random variable, Stochastic fuzzy differential equations, Maximal solution.

MSC2020-Mathematics Subject Classification System: 60, 65.

1. Introduction

In this work, we propose a new class of forward-backward fuzzy stochastic differential equations systems (FBFSDEs), which is called the forward-backward fuzzy double stochastic differential equations systems (FBFDSDEs). We consider the FBDSDEs as follows: for $\xi \in \Gamma^2(\Omega, F_0, P; \mathbb{R})$, $t \in [0, T]$,

$$\begin{cases} X(t)=\xi+\int_0^t f(s,X(s),\Upsilon(s),Z(s))ds+\int_0^t \bar{f}(s,X(s),Z(s))dB(s)+\int_0^t Z(s)dW(s) \\ \Upsilon(t)=\Lambda(X(T))+\int_t^T g(s,X(s),\Upsilon(s),\Psi(s))ds+\int_t^T \bar{g}(s,\Upsilon(s),\Psi(s))dB(s)+\int_t^T \Psi(s)dW(s), \end{cases} \tag{1.1}$$

where W and B are two mutually independent standard Brownian motions with the processes X, Υ, Z and Ψ defined on $\mathbb{R}^b, \mathbb{R}^d, \mathbb{R}^{b\times c}$ and $\mathbb{R}^{d\times c}$ respectively. Also, the function $f, g, \bar{f}, \bar{g}$ and Λ are defined on $\mathbb{R}^b\mathbb{R}^d\mathbb{R}^{b\times c}, \mathbb{R}^{d\times c}$ and $\mathbb{R}^d$ respectively.

Bao et al. [1] presented a study on forward-backward doubly stochastic differential equations and how to find numerical solutions to them, where errors are analyzed and the convergence rate is obtained by proposing numerical algorithms. Qingfeng and Yufeng [2] discussed the process of connecting two forward-backward doubly stochastic differential equations through the idea of a bridge and found that it has the same unique solution property. Therefore, building the appropriate bridge leads to obtaining several classes of equations that can be solved uniquely. Zhu and Shi [3], under same assumptions of monotonicity, proposed a model of mean-field forward-backward doubly stochastic differential equations and proved the existence and uniqueness of measurable solutions using the continuity property. Zhu et al. [4] presented a study on a model of forward-backward doubly stochastic differential equations associated with Brownian motion and the Poisson process, proving the existence and uniqueness of measurable solutions and distinguishing between them based on the parameters. Bao et al. [5] proposed a model for the connection between forward-backward doubly stochastic differential equations the problem of optimal filtering and finding a solution to the forward-backward doubly stochastic differential equations under Markov diffusion process. AbdulRahman [6] presented an important study on forward-backward doubly stochastic differential equations associated with Poisson jumps and emphasized the existence and uniqueness of the solution to this type of equations with the continuity of time. Φksendal and Sulem [7] proposed a different model for the maximum principle, which allows this to give a clear picture of the optimal control in forward-backward stochastic differential equations with random jumps, which leads to reducing risks through the use of g-expectations. Ji and Wei [8] presented a rigorous study on forward-backward stochastic differential equations through optimal central in describing the random system. The focus of this study was on improving utility in the financial market. Zhang and Shi [9] focused on discussing the comprehensive model of forward-backward doubly stochastic differential equations to obtain the maximum principle for controlling such a model of equations under certain conditions and assumptions.

2. Preliminaries

Suppose that $A(\mathbb{R}^b)$ is a family of all nonempty, convex and compact subsets $\mathbb{R}^b$. Let $H(\mathbb{R}^b)$ be fuzzy set space of $\mathbb{R}^b$. For every $\eta \in [0,1]$, the set of functions $l: \mathbb{R}^b \to [0,1]$ such that $[l]^\eta \in A(\mathbb{R}^b)$. Therefore, $[l]^\eta = \{\alpha \in \mathbb{R}^b : l(\alpha) \geq \eta, \eta \in [0,1]\} = [l_L^\eta, l_U^\eta]$, where $l_L^\eta = \inf_{\alpha \in \mathbb{R}^b}\{\alpha \in [l]^\eta\}$ and $l_U^\eta = \sup_{\alpha \in \mathbb{R}^b}\{\alpha \in [l]^\eta\}$ and $l^o = \mathrm{cl}\{\alpha \in \mathbb{R}^b : l(\alpha) > 0\}$. Suppose that (Ω, F, P) is a complete probability space. Let $M(\Omega, F; A(\mathbb{R}^b))$ be the family of F-measurable multifunctions, i.e. the mappings $J: \Omega \to A(\mathbb{R}^b)$ such that $\{\omega \in \Omega: J(\omega) \cap V \neq \emptyset\} \in F$ for every closed set $V \subset \mathbb{R}^b$. We define $r \in H(\mathbb{R}^b)$ as $r = 1_{\{0\}}$, where for $e \in \mathbb{R}^b$ then $1_{\{e\}}(q) = 1$ if $e = q$ and $1_{\{e\}}(q) = 0$ if $e \neq q$ [11].

Definition 2.1. [10] Let (Ω, F, P) be a complete probability space. The fuzzy random variable is a mapping $X: \Omega \to H(\mathbb{R}^b)$, if for all $\eta \in [0,1]$, $[X]^\eta: \Omega \to A(\mathbb{R}^b)$ is an F-measurable multifunction.

Definition 2.2. [10] Let (Ω, F, P) be a complete probability space. The fuzzy stochastic process is a mapping $X: [0, T] \times \Omega \to H(\mathbb{R}^b)$, if for every $t \in [0, T]$, $X(t, .) = X(t): \Omega \to H(\mathbb{R}^b)$ is a fuzzy random variable.

Definition 2.3. [10] Let (Ω, F, P) be a complete probability space. The fuzzy stochastic process X is b_∞-continuous, if the mapping $X(., \omega): [0, T] \to H(\mathbb{R}^b)$ are b_∞-continuous functions.

Definition 2.4. [12] A fuzzy stochastic process X is said to be $\{F_t\}$-adapted, $t \in [0, T]$, if for every $\eta \in [0,1]$, the multifunction $[X(t)]^\eta: \Omega \to A(\mathbb{R}^b)$ is $\{F_t\}_{t \in [0,T]}$-measurable.

Definition 2.5. [12] A fuzzy stochastic process X is a fuzzy Gaussian process if and only if for every $\eta \in [0,1]$, the family $\{X^\eta(t)\}_{t \in [0,T]}$ is an interval-valued Gaussian process.

Theorem 2.6. [13] The approximation solution $\{X^n(t), \Upsilon^n(t)\}$ converges to the exact solution $\{X(t), \Upsilon(t)\}$ of backward fuzzy stochastic differential equations such that

$$\lim_{n\to\infty} E|X(t)-X^n(t)|^2 = 0 \text{ and } \lim_{n\to\infty} E\int_0^T |\Upsilon(t)-\Upsilon^n(t)|^2 dt = 0.$$

3. Basic assumptions

Lorem In this section, we provide some necessary assumptions and concepts used in the sequel. Suppose that $\{W(t); 0 \le t \le T\}$ and $\{B(t); 0 \le t \le T\}$ are two mutually independent standard b-dimensional Brownian motions defined on the complete probability spaces (Ω_1, F_1, P_1) and (Ω_2, F_2, P_2) respectively, and for all finite time horizon $T < \infty$. Let N be the class of P-null sets of F. We consider $\Omega = \Omega_1 \times \Omega_2, F = F_1 \times F_2$ and $P = P_1 \times P_2$. For each $t \in [0,T]$, we define $F_t = F_t^W \vee F_{t,T}^B \vee N$, for any process η_t then $F_t^\eta = F_{0,t}^\eta, F_{s,t}^\eta = \sigma\{\eta_r - \eta_s; s \le r \le t\}$. Not that the family of σ-fields $\{F_t, t \in [0,T]\}$ is neither increasing nor decreasing and it is not a filtration. For an Euclidean space D, we denote $G^2(0,T;D)$ is the set of jointly measurable processes $\{X(t),, t \in [0,T]\}$ taking value in D, for each $t \in [0,T]$ then $X(t)$ is F_t-measurable such that $E[\int_0^T |X(t)|^2 dt] < \infty$. Let μ and v be the probability measures, we define the 2-distance between μ and v with the Wasserstein matric as follows:

$$W_2(\mu, v) = \inf_{J\in\Theta(\mu,v)}\{(\int |X-\Upsilon|^2 dJ(X,\Upsilon))^{\frac{1}{2}}: \iint_A J(X,\Upsilon)d\Upsilon dX = \mu, \iint_A J(X,\Upsilon)dXd\Upsilon = v\},$$

where $\Theta(\mu, v)$ is the set of all joint distributions J. The space $W_2(\mathbb{R}^b)$ is adopted with 2-Wasserstein metric such that, for every $\mu \in W_2(\mathbb{R}^b)$, there is a random variable $\zeta \in \Gamma^2(\Omega, F, P; \mathbb{R}^b)$ such that $P_\zeta = \mu$. For every $t \in [0,T]$, let us consider the following spaces of stochastic processes:

Let $G_1^2(\Omega, F, P; \mathbb{R}^b)$ be the space of F-measurable random variable X such that $||X||_{G_1^2} = (E|X(t)|^2)^{\frac{1}{2}} < \infty$.

Let $G_2^2(\Omega, F, P; \mathbb{R}^d)$ be the continuous space $\{F_s\}_{t\le s\le T}$-adapted process X such that $||X||_{G_2^2} = [E(\sup_{t\le s\le T}|X(s)|^2)]^{\frac{1}{2}} < \infty$.

Let $G_3^2(\Omega, F, P; \mathbb{R}^{b\times d})$ be the space of $\{F_s\}_{t\le s\le T}$-progressively measurable processes X such that $||X||_{G_3^2} = [E\int_0^T |X(s)|^2 ds]^{\frac{1}{2}} < \infty$.

Let $MSC = G_4^2([0,T], G_1^2(\Omega, F, P; \mathbb{R}^b))$ be the class of all mean square continuous second order stochastic processes. We use the Euclidean norms in $\mathbb{R}^b$ and $\mathbb{R}^d$, we define $|Z| = \{\text{trans}(ZZ^T)\}^{\frac{1}{2}}$, where $Z \in \mathbb{R}^{b\times d}$. Thus $\mathbb{R}^{b\times d}$ is a Hilbert space. let

$$f: \Omega \times [0,T] \times W_2(\mathbb{R}^b) \times \mathbb{R}^b \times \mathbb{R}^d \times \mathbb{R}^{b\times c} \to \mathbb{R}^b,$$

$$g: \Omega \times [0,T] \times W_2(\mathbb{R}^d) \times \mathbb{R}^b \times \mathbb{R}^d \times \mathbb{R}^{d\times c} \to \mathbb{R}^d,$$

$$\bar{f}: \Omega \times [0,T] \times \mathbb{R}^b \times \mathbb{R}^{b\times c} \to \mathbb{R}^{b\times c},$$

$$\bar{g}: \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d\times c} \to \mathbb{R}^{d\times c},$$

$$\Lambda: \Omega \times W_2(\mathbb{R}^b) \times \mathbb{R}^b \to \mathbb{R}^b$$

be jointly measurable and satisfy the following assumptions:

A1. For every $t \in [0,T]$, $(X,Z,\Upsilon,\Psi) \in \mathbb{R}^b \times \mathbb{R}^{b\times c} \times \mathbb{R}^d \times \mathbb{R}^{d\times c}$, $(\grave{X},\grave{Z},\grave{\Upsilon},\grave{\Psi}) \in \mathbb{R}^b \times \mathbb{R}^{b\times c} \times \mathbb{R}^d \times \mathbb{R}^{d\times c}$ and $K_1,K_2,K_3,K_4 > 0$ such that

$$|f(t,\mu,X,\Upsilon,Z) - f(t,\grave{\mu},\grave{X},\grave{\Upsilon},\grave{Z})| \leq K_1(W_2(\mu,\grave{\mu}) + |X-\grave{X}| + |\Upsilon-\grave{\Upsilon}| + |Z-\grave{Z}|),$$

$$|g(t,\mu,X,\Upsilon,\Psi) - g(t,\grave{\mu},\grave{X},\grave{\Upsilon},\grave{\Psi})| \leq K_2(W_2(\mu,\grave{\mu}) + |X-\grave{X}| + |\Upsilon-\grave{\Upsilon}| + |\Psi-\grave{\Psi}|),$$

$$|\bar{f}(t,X,Z) - \bar{f}(t,\grave{X},\grave{Z})|^2 \leq K_3(|X-\grave{X}|^2 + |Z-\grave{Z}|^2),$$

$$|\bar{g}(t,\Upsilon,\Psi) - \bar{g}(t,\grave{\Upsilon},\grave{\Psi})|^2 \leq K_4(|\Upsilon-\grave{\Upsilon}|^2 + |\Psi-\grave{\Psi}|^2).$$

A2. For every $t \in [0,T]$, $(X,Z,\Upsilon,\Psi) \in \mathbb{R}^b \times \mathbb{R}^{b\times c} \times \mathbb{R}^d \times \mathbb{R}^{d\times c}$ and $N_1,N_2,N_3,N_4,N_5 > 0$ such that

$$|f(t,P(X,Z),X,\Upsilon,Z)| \leq N_1(W_2(X,Z) + |X| + |\Upsilon| + |Z|),$$

$$|g(t,P(\Upsilon,\Psi),X,\Upsilon,\Psi)| \leq N_2(W_2(\Upsilon,\Psi) + |X| + |\Upsilon| + |\Psi|),$$

$$|\bar{f}(t,,X,Z)|^2 \leq N_3(1 + |X|^2 + |Z|^2),$$

$$|\bar{g}(t,,\Upsilon,\Psi)|^2 \leq N_4(1 + |\Upsilon|^2 + |\Psi|^2),$$

$$|\Lambda(X)|^2 \leq N_5(1 + |X|^2).$$

A3.

$$E(\int_0^T |f(s,\zeta_0,0,0,0)|^2\, ds) < \infty,$$

$$E(\int_0^T |g(s,\zeta_0,0,0,0)|^2\, ds) < \infty,$$

$$E(\int_0^T |\bar{f}(s,,0,0)|^2\, ds) < \infty,$$

$$E(\int_0^T |\bar{g}(s,,0,0)|^2\, ds) < \infty.$$

A4. For any $X_1, X_2 \in \Gamma^2(\Omega,F,P;\mathbb{R})$ and $(s,x,y,z) \in [0,T] \times \mathbb{R}^b \times \mathbb{R}^d \times \mathbb{R}^{b\times c}$, there exists a constant $K_5 > 0$ such that

$$f(s,P(X_1),x,y,z) - f(s,P(X_2),x,y,z) \leq K_5(E[|X_1 - X_2|^2])^{\frac{1}{2}}.$$

A5. For any $\Upsilon_1,\Upsilon_2 \in \Gamma^2(\Omega,F,P;\mathbb{R})$ and $(s,x,y,q) \in [0,T] \times \mathbb{R}^b \times \mathbb{R}^d \times \mathbb{R}^{d\times c}$, there exists a constant $K_6 > 0$ such that

$$g(s,P(\Upsilon_1),x,y,q) - g(s,P(\Upsilon_2),x,y,q) \leq K_5(E[|\Upsilon_2 - \Upsilon_2|^2])^{\frac{1}{2}}.$$

4. Setting of the problem

In this section, let us consider system of forward-backward fuzzy stochastic differential equations as follows

$$\begin{cases} dX(t) = \xi + f\big(t,P(X(t),Z(t)),X(t),\Upsilon(t),Z(t)\big)dt + \bar{f}\big(t,X(t),Z(t)\big)dB(t) + Z(t)dW(t), \\ d\Upsilon(t) = \Lambda(X(T)) + g\big(t,P(\Upsilon(t),\Psi(t)),X(t),\Upsilon(t),\Psi(t)\big)dt + \bar{g}\big(t,\Upsilon(t),\Psi(t)\big)dB(t) + \Psi(t)dW(t), \end{cases} \quad (4.1)$$

where $0 \leq t \leq T$ the processes X, Υ, Z and Ψ take values in $\mathbb{R}^b, \mathbb{R}^d, \mathbb{R}^{b\times c}$ and $\mathbb{R}^{d\times c}$ respectively. The functions $f, g, \bar{f}, \bar{g}$ and Λ are defined as follows

$$f: \Omega \times [0,T] \times W_2(\mathbb{R}^b) \times \mathbb{R}^b \times \mathbb{R}^d \times \mathbb{R}^{b\times c} \times H(\mathbb{R}^b) \to H(\mathbb{R}^b),$$

$$g: \Omega \times [0,T] \times W_2(\mathbb{R}^d) \times \mathbb{R}^b \times \mathbb{R}^d \times \mathbb{R}^{d\times c} \times H(\mathbb{R}^d) \to H(\mathbb{R}^d),$$

$$\bar{f}: \Omega \times [0,T] \times \mathbb{R}^b \times \mathbb{R}^{b\times c} \times H(\mathbb{R}^b) \to \mathbb{R}^{b\times c},$$

$$\bar{g}: \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{d\times c} \times H(\mathbb{R}^d) \to \mathbb{R}^{d\times c},$$

$$\xi: \Omega \to H(\mathbb{R}^b)$$

$$\Lambda: \Omega \times W_2(\mathbb{R}^b) \times \mathbb{R}^b \times H(\mathbb{R}^b) \to H(\mathbb{R}^b).$$

By integrable form of FBFDSDEs

$$\begin{cases} X(t)=\xi+\int_0^t f\big(s,P(X(s),Z(t)),X(s),\Upsilon(s),Z(s)\big)ds+\int_0^t \bar{f}(s,X(s),Z(s))dB(s)+\int_0^t Z(s)dW(s) \\ \Upsilon(t)=\Lambda(X(T))+\int_t^T g\big(s,P(\Upsilon(s),\Psi(s)),X(s),\Upsilon(s),\Psi(s)\big)ds+\int_t^T \bar{g}(s,\Upsilon(s),\Psi(s))dB(s)+\int_t^T \Psi(s)dW(s), \end{cases} \quad (4.2)$$

where $0 \leq t \leq T$. Now, we construct a sequence of FBFDSDEs, for $n \geq 1$, $\xi \in \Gamma^2(\Omega, F_0, P; \mathbb{R})$

$$\begin{cases} X^n(t)=\xi+\int_0^t f\big(s,P(X^n(s),Z^n(t)),X^n(s),\Upsilon^n(s),Z^n(s)\big)ds+\int_0^t \bar{f}(s,X^n(s),Z^n(s))dB(s)+\int_0^t Z^n(s)dW(s) \\ \Upsilon^n(t)=\Lambda(X^n(T))+\int_t^T g\big(s,P(\Upsilon^n(s),\Psi^n(s)),X^n(s),\Upsilon^n(s),\Psi^n(s)\big)ds+\int_t^T \bar{g}(s,\Upsilon^n(s),\Psi^n(s))dB(s)+\int_t^T \Psi^n(s)dW(s), \end{cases} \quad (4.3)$$

where $0 \leq t \leq T$, let $0 = t_0 < t_1 < \cdots < t_n = T$, $n \geq 1$ be a partition of interval $[0,T]$, we denote $\delta = \Delta t_{i+1} = t_{i+1} - t_i = \frac{T}{n}$, $1 \leq i \leq n$, $\Delta B(t_{i+1}) = B(t_{i+1}) - B(t_i)$, $\Delta W(t_{i+1}) = W(t_{i+1}) - W(t_i)$ and $\Delta t = \max \Delta t_{i+1}$ for $i = 0,1,\dots,n-1$, $n \geq 1$. Therefore, we consider

$$\begin{cases} X(t)=\xi+\int_0^t f\Big(s,P\big(X_i^n(s),Z_i^n(t)\big),X_i^n(s),\Upsilon_i^n(s),Z_i^n(s)\Big)ds+\int_0^t \bar{f}\big(s,X_i^n(s),Z_i^n(s)\big)dB(s)+\int_0^t Z_i^n(s)dW(s) \\ \Upsilon(t)=\Lambda(X(T))+\int_t^T g\Big(s,P\big(\Upsilon_i^n(s),\Psi_i^n(s)\big),X_i^n(s),\Upsilon_i^n(s),\Psi_i^n(s)\Big)ds+\int_t^T \bar{g}\big(s,\Upsilon_i^n(s),\Psi_i^n(s)\big)dB(s)+\int_t^T \Psi_i^n(s)dW(s), \end{cases} \quad (4.4)$$

where $0 \leq t \leq T$, $i = 1,\dots,n$. In the following lemmas, we can prove the unique solution of $(X^n, Z^n, \Upsilon^n, \Psi^n)$ of equation (4.3) and we prove that $(X^n, Z^n, \Upsilon^n, \Psi^n)$ is monotonic and its limit verifies equation (4.2). Therefore, the existence and uniqueness of the solution for (4.2) can be obtained.

Lemma 4.1 Under the assumptions (A1-A3), the following FBFDSDEs has an unique solution

$$(X, Z, \Upsilon, \Psi) \in G_2^2(\Omega, F, P; \mathbb{R}^b) \times G_3^2(\Omega, F, P; \mathbb{R}^{b\times c}) \times G_2^2(\Omega, F, P; \mathbb{R}^d) \times G_3^2(\Omega, F, P; \mathbb{R}^{d\times c}),$$

and there exists a constant $M > 0$ such that

$$||X|| + ||Z|| + ||\Upsilon|| + ||\Psi|| \leq M.$$

Proof. We consider forward FDSDE

$$X(t) = \xi + \int_0^t f\Big(s, P\big(X(s), Z(t)\big), X(s), \Upsilon(s), Z(s)\Big)\, ds + \int_0^t \bar{f}\big(s, X(s), Z(s)\big)dB(s) + \int_0^t Z(s)dW(s) \quad (4.5)$$

where $t \in [0,T]$. Let us define the mapping

$$I(x,z) = (X,Z): G_2^2([0.T], \mathbb{R}^b) \times G_3^2([0,T], \mathbb{R}^{b\times c}) \to G_2^2([0.T], \mathbb{R}^b) \times G_3^2([0,T], \mathbb{R}^{b\times c}).$$

For arbitrary $\left(x^i, z^i\right) \in G_2^2([0.T], \mathbb{R}^b) \times G_3^2([0,T], \mathbb{R}^{b\times c})$, we have $\left(X^i, Z^i\right) = I\left(x^i, z^i\right)$, $i = 1,2$. Let $(\hat{X}, \hat{Z}) = (X^1 - X^2, Z^1 - Z^2)$. By applying Itô's formula to $|X(t)|^2$, yields

$$E\left[\left|\hat{X}(t)\right|^2\right] + E\left[\int_0^t \left|\hat{X}(s)\right|^2 ds\right] + \left[\int_o^t |Z(s)|^2 ds\right] \le 2E\left[\int_0^t \hat{X}(s)\hat{f}(s)\, ds\right] + E\left[\int_o^t \left|\hat{\hat{f}}(s)\right|^2 ds\right],$$

where

$$\hat{f}(s) = f\left(s, P\left(X^1(s), Z^1(t)\right), X^1(s), Y^1(s), Z^1(s)\right) - f\left(s, P\left(X^2(s), Z^2(t)\right), X^2(s), Y^2(s), Z^2(s)\right),$$

$$\hat{\hat{f}}(s) = \bar{f}\left(s,, X^1(s), Z^1(s)\right) - \bar{f}\left(s,, X^2(s), Z^2(s)\right)$$

From inequality $2ab \le \frac{1}{c}a^2 + cb^2$, $c > 0$, there exist constants $c_1, c_2, c_3, c_4 > 0$ such that

$$2E\left[\int_0^t \hat{X}(s)\hat{f}(s)\, ds\right] \le 2K_1 E\left[\int_0^t |\hat{X}(s)| W_2(P\left(X^1(s), Z^1(s)), P\left(X^2(s), Z^2(s)\right)\right) ds\right] + 2K_1 E\left[\int_0^t \left|\hat{X}(s)\right|^2 ds\right] + 2K_1 E\left[\int_0^t \left|\hat{X}(s)\right|\left|\hat{Y}(s)\right| ds\right] + 2K_1 E\left[\int_0^t \left|\hat{X}(s)\right|\left|\hat{Z}(s)\right| ds\right] \le \frac{K_1}{c_1} E\left[\int_0^t \left|\hat{X}(s)\right|^2 ds\right] + K_1 c_1 E\left[\int_0^t |\hat{x}(s)|^2 ds\right] + \frac{K_1}{c_2} E\left[\int_0^t \left|\hat{X}(s)\right|^2 ds\right] + K_1 c_2 E\left[\int_0^t |\hat{z}(s)|^2 ds\right] + 2K_1 E\left[\int_0^t \left|\hat{X}(s)\right|^2 ds\right] + \frac{K_1}{c_3} E\left[\int_0^t \left|\hat{X}(s)\right|^2 ds\right] + K_1 c_3 E\left[\int_0^t \left|\hat{Y}(s)\right|^2 ds\right] + \frac{K_1}{c_4} E\left[\int_0^t \left|\hat{X}(s)\right|^2 ds\right] + K_1 c_4 E\left[\int_0^t \left|\hat{Z}(s)\right|^2 ds\right] = \left(\frac{K_1}{c_1} + \frac{K_1}{c_2} + 2K_1 + \frac{K_1}{c_3} + \frac{K_1}{c_4}\right) E\left[\int_0^t \left|\hat{X}(s)\right|^2 ds\right] + K_1 c_3 E\left[\int_0^t \left|\hat{Y}(s)\right|^2 ds\right] + K_1 c_4 E\left[\int_0^t \left|\hat{Z}(s)\right|^2 ds\right] + K_1 c_1 E\left[\int_0^t |\hat{x}(s)|^2 ds\right] + K_1 c_2 E\left[\int_0^t |\hat{z}(s)|^2 ds\right]$$

and

$$E\left[\int_o^t \left|\hat{\hat{f}}(s)\right|^2 ds\right] \le K_3 E\left[\int_0^t \left|\hat{X}(s)\right|^2 ds\right] + K_3 E\left[\int_0^t \left|\hat{Z}(s)\right|^2 ds\right].$$

Let $\beta = \frac{K_1}{c_1} + \frac{K_1}{c_2} + 2K_1 + \frac{K_1}{c_3} + \frac{K_1}{c_4} + K_3$, $\alpha = K_1 c_4 + K_3$, we have

$$(1-\beta)E\left[\int_0^t \left|\hat{X}(s)\right|^2 ds\right] + (1-\alpha)E\left[\int_0^t \left|\hat{Z}(s)\right|^2 ds\right] \le K_1 c_3 E\left[\int_0^t \left|\hat{Y}(s)\right|^2 ds\right] + K_1 c_1 E\left[\int_0^t |\hat{x}(s)|^2 ds\right] + K_1 c_2 E\left[\int_0^t |\hat{z}(s)|^2 ds\right].$$

Now, let us define a norm which is equivalent to space $G_3^2(\Omega, F, P; \mathbb{R}^{b\times d})$

$$\left||x, z|\right| = (E\left[\int_0^T (|x(s)|^2 + |z(s)|^2)\, ds\right])^{\frac{1}{2}} \text{ and } E\left[\int_0^t \left|\hat{Y}(s)\right|^2 ds\right]^{\frac{1}{2}} < \infty.$$

Therefore, the mapping I a contraction map on $G_2^2([0.T], \mathbb{R}^b) \times G_3^2([0,T], \mathbb{R}^{b\times c})$. By the contraction mapping theorem, there exists a unique fixed point $(X, Z) \in G_2^2([0.T], \mathbb{R}^b) \times G_3^2([0,T], \mathbb{R}^{b\times c})$ such that $I(X, Z) = (X, Z)$. Thus, there is a unique solution of equation (4.5).

Now, we must prove that there exists a constant $M_1 > 0$ such that

$$||X||_{G_2^2([0.T],\mathbb{R}^b)} + ||Z||_{G_3^2([0,T],\mathbb{R}^{b\times c})} \le M_1. \quad (4.6)$$

We set

$$|X(t)|^2+\int_0^t|X(s)|^2\,ds+\int_0^t|Z(s)|^2\,ds=|\xi|^2+2\int_0^t X(s)f\Big(s,P\big(X(s),Z(t)\big),X(s),\Upsilon(s),Z(s)\Big)\,ds+ 2\int_0^t X(s)\bar{f}\big(s,X(s),Z(s)\big)dB(s)+2\int_0^t X(s)Z(s)dW(s)+\int_0^t\left|\bar{f}\big(s,X(s),Z(s)\big)\right|^2 ds.$$

By taking expectation and there exist $K_1, N_3, c_1, c_2, c_3, c_4, c_5 > 0$ such that

$$E|X(t)|^2+E[\int_0^t|X(s)|^2\,ds]+E[\int_0^t|Z(s)|^2\,ds]=E|\xi|^2+ 2E[\int_0^t X(s)f\Big(s,P\big(X(s),Z(t)\big),X(s),\Upsilon(s),Z(s)\Big)\,ds]+E[\int_0^t\left|\bar{f}\big(s,X(s),Z(s)\big)\right|^2 ds]\le E|\xi|^2+ 2K_1E[\int_0^t X(s)f(s,\zeta_0,0{,}0{,}0)ds]+2K_1E\left[\int_0^t|X(s)|\,(E|X(s)|^2)^{\frac{1}{2}}ds\right]+2K_1E\left[\int_0^t|X(s)|\,(E|Z(s)|^2)^{\frac{1}{2}}ds\right]+ 2K_1E\left[\int_0^t|X(s)|^2\,ds\right]+2K_1E\left[\int_0^t|X(s)||\Upsilon(s)|\,ds\right]+2K_1E\left[\int_0^t|X(s)||Z(s)|\,ds\right]+ N_3E[\int_0^t\left|\bar{f}(s,0{,}0)\right|^2 ds]+N_3E\left[\int_0^t|X(s)|^2\,ds\right]+N_3E\left[\int_0^t|Z(s)|^2\,ds\right]\le E|\xi|^2+\frac{2K_1}{c_1}E\left[\int_0^t|X(s)|^2\,ds\right]+ 2K_1c_1E[\int_0^t|f(s,\zeta_0,0{,}0{,}0)|^2ds]+\frac{2K_1}{c_2}E\left[\int_0^t|X(s)|^2\,ds\right]+2K_1c_2E\left[\int_0^t E|X(s)|^2\,ds\right]+ \frac{2K_1}{c_3}E\left[\int_0^t|X(s)|^2\,ds\right]+2K_1c_3E\left[\int_0^t E|Z(s)|^2\,ds\right]+2K_1E\left[\int_0^t|X(s)|^2\,ds\right]+\frac{2K_1}{c_4}E\left[\int_0^t|X(s)|^2\,ds\right]+ 2K_1c_4E\left[\int_0^t|\Upsilon(s)|^2\,ds\right]+\frac{2K_1}{c_5}E\left[\int_0^t|X(s)|^2\,ds\right]+2K_1c_5E\left[\int_0^t E|Z(s)|^2\,ds\right]+N_3E[\int_0^t\left|\bar{f}(s,0{,}0)\right|^2 ds]+ N_3E\left[\int_0^t|X(s)|^2\,ds\right]+N_3E\left[\int_0^t|Z(s)|^2\,ds\right]=E|\xi|^2+(\frac{2K_1}{c_1}+\frac{2K_1}{c_2}+2K_1c_2+\frac{2K_1}{c_3}+2K_1+\frac{2K_1}{c_4}+\frac{2K_1}{c_5}+ N_3)E\left[\int_0^t|X(s)|^2\,ds\right]+(2K_1c_3+2K_1c_5+N_3)E\left[\int_0^t|Z(s)|^2\,ds\right]+2K_1c_4E\left[\int_0^t|\Upsilon(s)|^2\,ds\right]+ 2K_1c_1E[\int_0^t|f(s,\zeta_0,0{,}0{,}0)|^2ds]+N_3E[\int_0^t\left|\bar{f}(s,0{,}0)\right|^2 ds].$$

More simplified

$$E|X(t)|^2+R_1E[\int_0^t|X(s)|^2\,ds]+R_2E[\int_0^t|Z(s)|^2\,ds]\le E|\xi|^2+2K_1c_4E\left[\int_0^t|\Upsilon(s)|^2\,ds\right]+ 2K_1c_1E[\int_0^t|f(s,\zeta_0,0{,}0{,}0)|^2ds]+N_3E[\int_0^t\left|\bar{f}(s,0{,}0)\right|^2 ds]\le M_1,$$

where M_1 depends on R_1, R_2, $E|\xi|^2$, $E[\int_0^t|f(s,\zeta_0,0{,}0{,}0)|^2ds]$, $E[\int_0^t\left|\bar{f}(s,0{,}0)\right|^2 ds]$ and $E\left[\int_0^t|\Upsilon(s)|^2\,ds\right]$. Applying Burkholder-Davis-Gund inequality and Young's inequality, using the same technique above, for every $t\in[0,T]$ such that

$$E[\sup_{r\in[0,t]}|X(r)|^2\le T\{E|\xi|^2+2K_1c_4E\left[\int_0^t|\Upsilon(s)|^2\,ds\right]+2K_1c_1E[\int_0^t|f(s,\zeta_0,0{,}0{,}0)|^2ds]+ N_3E[\int_0^t\left|\bar{f}(s,0{,}0)\right|^2 ds]\le M_1.$$

This means that inequality (4.6) is fulfilled. Now, we consider backward FDSDE

$$\Upsilon(t)=\Lambda(X(T))+\int_t^T g\big(s,P(\Upsilon(s),\Psi(s)),X(s),\Upsilon(s),\Psi(s)\big)ds+\int_t^T \bar{g}\big(s,\Upsilon(s),\Psi(s)\big)dB(s)+ \int_t^T \Psi(s)dW(s), \qquad (4.7)$$

where $0\le t\le T$. We first conclude from the proof of the existence and uniqueness of the solution (X,Z) for forward FDSDE (4.5). Once we know (X,Z), the backward FDSDE becomes a classical equation. Therefore, using the same proof technique of forward FDSDE, we can prove that the backward FDSDE has an unique solution (Υ,Ψ), then there exists $M_2 > 0$ such that

$$||\Upsilon||_{G_2^2([0.T],\mathbb{R}^d)} + ||\Psi||_{G_3^2([0,T],\mathbb{R}^{d\times c})} \leq M_2 \qquad (4.8)$$

Let $M = M_1 + M_2$ and by combining (4.6) and (4.8), we have

$$||X||_{G_2^2([0.T],\mathbb{R}^b)} + ||Z||_{G_3^2([0,T],\mathbb{R}^{b\times c})} + ||\Upsilon||_{G_2^2([0.T],\mathbb{R}^d)} + ||\Psi||_{G_3^2([0,T],\mathbb{R}^{d\times c})} \leq M.$$

5. Maximal solution

Definition 5.1. [5] Let $l(t)$ be a solution of stochastic differential equation. $l(t)$ is a maximal solution if every solution $X(t)$ such that $E(X^2(t)) \leq E(l^2(t))$.

Definition 5.2. [5] Let $X, \Upsilon \in G_2^2([0.T],\mathbb{R}^b)$ such that $||X(t)||^2 < ||\Upsilon(t)||^2$. A function $f: G_2^2([0.T],\Omega,F,P,\mathbb{R}^b) \to G_2^2([0.T],\Omega,F,P,\mathbb{R}^b)$, is stochastically increasing if $||f(t,X(t))||^2 < ||f(t,\Upsilon(t))||^2$.

Definition 5.3. [5] Let $X, \Upsilon \in G_2^2([0.T],\mathbb{R}^b)$ such that $||X(t)||^2 < ||\Upsilon(t)||^2$. A function $f: G_2^2([0.T],\Omega,F,P,\mathbb{R}^b) \to G_2^2([0.T],\Omega,F,P,\mathbb{R}^b)$, is stochastically decreasing if $||f(t,X(t))||^2 > ||f(t,\Upsilon(t))||^2$.

Lemma 5.4. Let $f^i = f^i(t,\mu,X,\Upsilon,Z)$, $= 1,2$ be two functions satisfying (A1, i). Let (X^1,Z^1) and (X^2,Z^2) be the solutions of forward equation of FBFDSDEs. If $\xi^1 \leq \xi^2$, $f^1 \leq f^2$ then $X^1(t) \leq X^2(t)$ for every $t \in [o,T]$.

Proof. Without loss of generality, let $b = c = 1$ and f^1 satisfy A1 and A4. We define $\hat{\xi} = \xi^1 - \xi^2$ then $\hat{\xi} = 0$, $(\hat{X}(t),\hat{Z}(t)) = (X^1(t) - X^2(t), Z^1(t) - Z^2(t))$, $t \in [o,T]$. From Itô's formula and the inequality $2ab \leq \frac{1}{c}a^2 + cb^2, c > 0$, there is

$$\begin{aligned}
&E[(\hat{X}(t))^2] + E[\int_0^t I_{\{\hat{X}(s)\geq 0\}}|\hat{Z}(s)|^2\,ds] \\
&\quad = E[(\hat{\xi})^2] \\
&\quad + 2E[\int_0^t \hat{X}(s)(f^1\left(s, P\left(X^1(s)\right), X^1(s), \Upsilon^1(s), Z^1(s)\right) \\
&\quad - f^2\left(s, P\left(X^2(s)\right), X^2(s), \Upsilon^2(s), Z^2(s)\right))ds] \\
&\quad + E[\int_0^t I_{\{\hat{X}(s)\geq 0\}}|\bar{f}\left(s,, X^1(s), Z^1(s)\right) - \bar{f}\left(s,, X^2(s), Z^2(s)\right)|^2 ds] \\
&\quad \leq E[(\hat{\xi})^2] \\
&\quad + E[\int_0^t \hat{X}(s)(f^1\left(s, P\left(X^1(s)\right), X^1(s), \Upsilon^1(s), Z^1(s)\right) \\
&\quad - f^1\left(s, P\left(X^2(s)\right), X^1(s), \Upsilon^1(s), Z^1(s)\right) + f^1\left(s, P\left(X^2(s)\right), X^1(s), \Upsilon^1(s), Z^1(s)\right) \\
&\quad - f^1\left(s, P\left(X^2(s)\right), X^2(s), \Upsilon^2(s), Z^2(s)\right) + f^1\left(s, P\left(X^2(s)\right), X^2(s), \Upsilon^2(s), Z^2(s)\right) \\
&\quad - f^2\left(s, P\left(X^2(s)\right), X^2(s), \Upsilon^2(s), Z^2(s)\right))ds] \\
&\quad + E[\int_0^t I_{\{\hat{X}(s)\geq 0\}}|\bar{f}\left(s,, X^1(s), Z^1(s)\right) - \bar{f}\left(s,, X^2(s), Z^2(s)\right)|^2 ds] \\
&\quad \leq E[\int_0^t (\hat{X}(s)((2K_5(E|\hat{X}(s)|^2)^{\frac{1}{2}}) + 2K_1|\hat{X}(s)| + 2K_1|\hat{\Upsilon}(s)| + 2K_1|\hat{Z}(s)|))\,ds] \\
&\quad + K_3E[\int_0^t I_{\{\hat{X}(s)\geq 0\}}(|X(s)|^2 + |Z(s)|^2)ds] \\
&\quad \leq E[\int_0^t (2K_5(\hat{X}(s))^2 + 2K_1(\hat{X}(s))^2 + \frac{K_1}{c}(\hat{X}(s))^2 + 2K_1|\hat{\Upsilon}(s)|^2 + \frac{K_1}{c}(\hat{X}(s))^2 \\
&\quad + 2K_1|\hat{Z}(s)|^2)ds] + K_3E[\int_0^t |\hat{X}(s)|^2\,ds] + K_3E[\int_0^t I_{\{\hat{X}(s)\geq 0\}}|\hat{Z}(s)|^2\,ds] \\
&\quad \leq (2K_5 + 2K_1 + \frac{K_1}{c} + \frac{K_1}{c} + K_3)E[\int_0^t (\hat{X}(s))^2\,ds] + K_1cE[\int_0^t |\hat{\Upsilon}(s)|^2\,ds] + (K_1c \\
&\quad + K_3)E[\int_0^t I_{\{\hat{X}(s)\geq 0\}}|\hat{Z}(s)|^2\,ds] \\
&\quad \leq M(2K_5 + 2K_1 + \frac{2K_1}{c} + K_3)E[\int_0^t (\hat{X}(s))^2\,ds] + (K_1c + K_3)E[\int_0^t I_{\{\hat{X}(s)\geq 0\}}|\hat{Z}(s)|^2\,ds]
\end{aligned}$$

where $M > 0$. Since $E[\int_0^t |\widehat{\Upsilon}(s)|^2 ds] < \infty$. From Gronwall's inequality, we have $E[(\widehat{X}(s))^2] = 0, t \in [o,T]$, therefore, we have $X^1(t) \leq X^2(t)$, P. a.s.

Lemma 5.5. Let $g^i = g^i(t, \mu, X, \Upsilon, \Psi), = 1,2$ be two functions satisfying (A1, ii). Let (Υ^1, Ψ^1) and (Υ^2, Ψ^2) be the solutions of backward equation of FBFDSDEs. If $X^1(T) \leq X^2(T)$, $g^1 \leq g^2$ then $\Upsilon^1(t) \leq \Upsilon^2(t)$ for every $t \in [o,T]$.

Proof. Without loss of generalized, let $d = c = 1$ and g^1 satisfy A1 and A5. We define $(\widehat{\Upsilon}(t), \widehat{\Psi}(t)) = (\Upsilon^1(t) - \Upsilon^2(t), \Psi^1(t) - \Psi^2(t))$, $t \in [o,T]$. From Itô's formula and the inequality $2ab \leq \frac{1}{c}a^2 + cb^2, c > 0$, there is

$$E[(\widehat{\Upsilon}(t))^2] + E[\int_t^T I_{\{\widehat{\Upsilon}(s)\geq 0\}}|\widehat{\Psi}(s)|^2 ds] \leq 2E[\int_t^T \widehat{\Upsilon}(s)(g^1\left(s, P\left(\Upsilon^1(s)\right), X^1(s), \Upsilon^1(s), \Psi^1(s)\right) - g^2\left(s, P\left(\Upsilon^2(s)\right), X^2(s), \Upsilon^2(s), \Psi^2(s)\right))ds] + E[\int_t^T I_{\{\widehat{\Upsilon}(s)\geq 0\}}|\bar{g}\left(s,, X^1(s), \Psi^1(s)\right) - \bar{g}\left(s,, X^2(s), \Psi^2(s)\right)|^2 ds] \leq E[\int_t^T \widehat{\Upsilon}(s)(g^1\left(s, P\left(\Upsilon^1(s)\right), X^1(s), \Upsilon^1(s), \Psi^1(s)\right) - g^1\left(s, P\left(\Upsilon^2(s)\right), X^1(s), \Upsilon^1(s), \Psi^1(s)\right) + g^1\left(s, P\left(\Upsilon^2(s)\right), X^1(s), \Upsilon^1(s), \Psi^1(s)\right) - g^1\left(s, P\left(\Upsilon^2(s)\right), X^2(s), \Upsilon^2(s), \Psi^2(s)\right) + g^1\left(s, P\left(\Upsilon^2(s)\right), X^2(s), \Upsilon^2(s), \Psi^2(s)\right) - g^2\left(s, P\left(\Upsilon^2(s)\right), X^2(s), \Upsilon^2(s), \Psi^2(s)\right))ds] + E[\int_t^T I_{\{\widehat{\Upsilon}(s)\geq 0\}}|\bar{g}\left(s,, \Upsilon^1(s), \Psi^1(s)\right) - \bar{g}\left(s,, \Upsilon^2(s), \Psi^2(s)\right)|^2 ds] \leq 2E[\int_t^T (\widehat{\Upsilon}(s)((K_6(E|\widehat{\Upsilon}(s)|^2)^{\frac{1}{2}}) + K_2|\widehat{X}(s)| + K_2|\widehat{\Upsilon}(s)| + K_2|\widehat{\Psi}(s)|)) ds] + K_4 E[\int_t^T |\widehat{\Upsilon}(s)|^2 ds] + K_4 E[\int_t^T I_{\{\widehat{\Upsilon}(s)\geq 0\}}|\widehat{\Psi}(s)|^2 ds] \leq E[\int_t^T (2K_6(\widehat{\Upsilon}(s))^2 + \frac{K_2}{c}|\widehat{\Upsilon}(s)|^2 + K_2 c|\widehat{X}(s)|^2 + 2K_2 c|\widehat{\Upsilon}(s)|^2 + \frac{K_2}{c}|\widehat{\Upsilon}(s)|^2 + K_2 c|\widehat{\Psi}(s)|^2)ds] + K_4 E[\int_t^T |\widehat{\Upsilon}(s)|^2 ds] + K_4 E[\int_t^T I_{\{\widehat{\Upsilon}(s)\geq 0\}}|\widehat{\Psi}(s)|^2 ds] \leq (2K_6 + \frac{K_2}{c} + 2K_2 + \frac{K_2}{c} + K_4)E[\int_t^T |\widehat{\Upsilon}(s)|^2 ds] + K_2 c E[\int_t^T |\widehat{X}(s)|^2 ds] + (K_2 c + K_4)E[\int_t^T I_{\{\widehat{\Upsilon}(s)\geq 0\}}|\widehat{\Psi}(s)|^2 ds] \leq N(2K_6 + \frac{2K_2}{c} + 2K_2 + K_4)E[\int_t^T |\widehat{\Upsilon}(s)|^2 ds] + (2K_2 + K_4)E[\int_t^T I_{\{\widehat{\Upsilon}(s)\geq 0\}}|\widehat{\Psi}(s)|^2 ds],$$

where $N > 0$. Since $E[\int_t^T |\widehat{X}(s)|^2 ds] < \infty$. From Gronwall's inequality, we have $E[(\widehat{\Upsilon}(t))^2] = 0, t \in [o,T]$. Therefore, we have $\Upsilon^1(t) \leq \Upsilon^2(t)$, P.a.s.

Theorem 5.6. Suppose that $f, \bar{f}, g$ and $\bar{g}$ are stochastically increasing function. Under the assumptions (A1-A5), the general system of FBFDSDEs has a maximal solution (X, Z, Υ, Ψ).

Proof. Let $\theta > 0$, we consider the forward equation of FBFDSDEs system as follows

$$X_\theta(t) = X(0) + \int_0^t f_\theta\left(s, P\left(X_\theta^n(s), Z_\theta^n(t)\right), X_\theta^n(s), \Upsilon_\theta^n(s), Z_\theta^n(s)\right) ds + \int_0^t \bar{f}_\theta\left(s, X_\theta^n(s), Z_\theta^n(s)\right) dB(s) + \int_0^t Z_\theta^n(s) dW(s).$$

Let's put

$$f_\theta\left(s, P\left(X_\theta^n(s), Z_\theta^n(t)\right), X_\theta^n(s), \Upsilon_\theta^n(s), Z_\theta^n(s)\right) = f\left(s, P\left(X_\theta^n(s), Z_\theta^n(t)\right), X_\theta^n(s), \Upsilon_\theta^n(s), Z_\theta^n(s)\right) + \theta,$$

$$\bar{f}_\theta\left(s, X_\theta^n(s), Z_\theta^n(s)\right) = \bar{f}\left(s, X_\theta^n(s), Z_\theta^n(s)\right) + \theta.$$

Therefore, if θ_1 and θ_2 are chosen such that $0 < \theta_2 < \theta_1 < \theta$, we deduce

$$X_{\theta_1}(t) = X(0) + \int_0^t (f\left(s, P\left(X_{\theta_1}^n(s), Z_{\theta_1}^n(t)\right), X_{\theta_1}^n(s), \Upsilon_{\theta_1}^n(s), Z_{\theta_1}^n(s)\right) + \theta_1)ds + \int_0^t (\bar{f}\left(s, X_{\theta_1}^n(s), Z_{\theta_1}^n(s)\right) + \theta_1)dB(s) + \int_0^t Z_{\theta_1}^n(s)dW(s).$$

By Lemma 5.4, we have

$$||X_{\theta_2}(t)||^2 = ||X(0) + \int_0^t (f\left(s, P\left(X_{\theta_2}^n(s), Z_{\theta_2}^n(t)\right), X_{\theta_2}^n(s), \Upsilon_{\theta_2}^n(s), Z_{\theta_2}^n(s)\right) + \theta_2)ds + \int_0^t (\bar{f}\left(s, X_{\theta_2}^n(s), Z_{\theta_2}^n(s)\right) + \theta_2)dB(s) + \int_0^t Z_{\theta_2}^n(s)dW(s)\,||^2 \leq ||X_{\theta_1}(t) + \int_0^t (f\left(s, P\left(X_{\theta_1}^n(s), Z_{\theta_1}^n(t)\right), X_{\theta_1}^n(s), \Upsilon_{\theta_1}^n(s), Z_{\theta_1}^n(s)\right) + \theta_1)ds + \int_0^t (\bar{f}\left(s, X_{\theta_1}^n(s), Z_{\theta_1}^n(s)\right) + \theta_1)dB(s) + \int_0^t Z_{\theta_2}^n(s)dW(s) - [\int_0^t (f\left(s, P\left(X_{\theta_2}^n(s), Z_{\theta_2}^n(t)\right), X_{\theta_2}^n(s), \Upsilon_{\theta_2}^n(s), Z_{\theta_2}^n(s)\right) + \theta_2)ds + \int_0^t (\bar{f}\left(s, X_{\theta_2}^n(s), Z_{\theta_2}^n(s)\right) + \theta_2)dB(s) + \int_0^t Z_{\theta_1}^n(s)dW(s)]||^2 \leq ||X_{\theta_1}(t)||^2 + \int_0^t ||f\left(s, P\left(X_{\theta_1}^n(s), Z_{\theta_1}^n(t)\right), X_{\theta_1}^n(s), \Upsilon_{\theta_1}^n(s), Z_{\theta_1}^n(s)\right) - f\left(s, P\left(X_{\theta_2}^n(s), Z_{\theta_2}^n(t)\right), X_{\theta_2}^n(s), \Upsilon_{\theta_2}^n(s), Z_{\theta_2}^n(s)\right)||^2 ds + \int_0^t ||\bar{f}\left(s, X_{\theta_1}^n(s), Z_{\theta_1}^n(s)\right) - \bar{f}\left(s, X_{\theta_2}^n(s), Z_{\theta_2}^n(s)\right)||^2 ds + \int_0^t ||Z_{\theta_2}^n(s) - Z_{\theta_1}^n(s)||^2\, ds + 2\int_0^t |\theta_1 - \theta_2|\, ds.$$

Because θ_1 and θ_2 are chosen, it follows that $|\theta_1 - \theta_2| \to 0$. Also, because f and $\bar{f}$ are stochastically increasing functions, we have

$$||f\left(s, P\left(X_{\theta_1}^n(s), Z_{\theta_1}^n(t)\right), X_{\theta_1}^n(s), \Upsilon_{\theta_1}^n(s), Z_{\theta_1}^n(s)\right) - f\left(s, P\left(X_{\theta_2}^n(s), Z_{\theta_2}^n(t)\right), X_{\theta_2}^n(s), \Upsilon_{\theta_2}^n(s), Z_{\theta_2}^n(s)\right)||^2 \to 0,$$

$$||\bar{f}\left(s, X_{\theta_1}^n(s), Z_{\theta_1}^n(s)\right) - \bar{f}\left(s, X_{\theta_2}^n(s), Z_{\theta_2}^n(s)\right)||^2 \to 0.$$

Following the same technique as the proof of the theorem 2.6 [13], we have

$\int_0^t ||Z_{\theta_2}^n(s) - Z_{\theta_1}^n(s)||^2\, ds \to 0$ as $n \to \infty$

Hence, $||X_{\theta_1}(t)||^2 \leq ||X_{\theta_2}(t)||^2$. For $\theta_n \leq \theta_{n-1} \leq \ldots \leq \theta_2 \leq \theta_1 \leq \theta$, we have

$||X_{\theta_n}(t)||^2 \leq ||X_{\theta_{n-1}}(t)||^2 \leq \ldots \leq ||X_{\theta_2}(t)||^2 \leq ||X_{\theta_1}(t)||^2 \leq ||X_{\theta}(t)||^2$.

Now, let $\gamma > 0$, we consider the backward equation of FBFDSDEs system as follows

$$\Upsilon_\gamma(t) = \Upsilon(T) + \int_t^T g_\gamma\left(s, P\left(\Upsilon_\gamma^n(s), \Psi_\gamma^n(s)\right), X_\gamma^n(s), \Upsilon_\gamma^n(s), \Psi_\gamma^n(s)\right) ds + \int_t^T \overline{g_\gamma}\left(s, \Upsilon_\gamma^n(s), \Psi_\gamma^n(s)\right) dB(s) + \int_t^T \Psi_\gamma^n(s)dW(s).$$

For every chosen small real number γ_1 and γ_2 such that $0 < \gamma_2 < \gamma_1 < \gamma$, we deduce

$$\Upsilon_{\gamma_1}(t) = \Upsilon(T) + \int_t^T (g\left(s, P\left(\Upsilon_{\gamma_1}^n(s), \Psi_{\gamma_1}^n(s)\right), X_{\gamma_1}^n(s), \Upsilon_{\gamma_1}^n(s), \Psi_{\gamma_1}^n(s)\right) + \gamma_1)ds + \int_t^T (\bar{g}\left(s, \Upsilon_{\gamma_1}^n(s), \Psi_{\gamma_1}^n(s)\right) + \gamma_1)dB(s) + \int_t^T \Psi_{\gamma_1}^n(s)dW(s).$$

By lemma 5.5, we have

$$||\Upsilon_{\gamma_2}(t)||^2 = ||\Upsilon(T) + \int_t^T (g\left(s, P\left(\Upsilon_{\gamma_2}^n(s), \Psi_{\gamma_2}^n(s)\right), X_{\gamma_2}^n(s), \Upsilon_{\gamma_2}^n(s), \Psi_{\gamma_2}^n(s)\right) + \gamma_2)ds + \int_t^T (\bar{g}\left(s, \Upsilon_{\gamma_2}^n(s), \Psi_{\gamma_2}^n(s)\right) + \gamma_2)dB(s) + \int_t^T \Psi_{\gamma_2}^n(s)dW(s)\,||^2 \le ||\Upsilon_{\gamma_1}(t) + \int_t^T (g\left(s, P\left(\Upsilon_{\gamma_1}^n(s), \Psi_{\gamma_1}^n(s)\right), X_{\gamma_1}^n(s), \Upsilon_{\gamma_1}^n(s), \Psi_{\gamma_1}^n(s)\right) + \gamma_1)ds + \int_t^T (\bar{g}\left(s, \Upsilon_{\gamma_1}^n(s), \Psi_{\gamma_1}^n(s)\right) + \gamma_1)dB(s) + \int_t^T \Psi_{\gamma_2}^n(s)dW(s) - [\int_t^T (g\left(s, P\left(\Upsilon_{\gamma_2}^n(s), \Psi_{\gamma_2}^n(s)\right), X_{\gamma_2}^n(s), \Upsilon_{\gamma_2}^n(s), \Psi_{\gamma_2}^n(s)\right) + \gamma_2)ds + \int_t^T (\bar{g}\left(s, \Upsilon_{\gamma_2}^n(s), \Psi_{\gamma_2}^n(s)\right) + \gamma_2)dB(s) + \int_t^T \Psi_{\gamma_1}^n(s)dW(s)]||^2 \le ||\Upsilon_{\gamma_1}(t)||^2 + \int_t^T ||g\left(s, P\left(\Upsilon_{\gamma_1}^n(s), \Psi_{\gamma_1}^n(s)\right), X_{\gamma_1}^n(s), \Upsilon_{\gamma_1}^n(s), \Psi_{\gamma_1}^n(s)\right) - g\left(s, P\left(\Upsilon_{\gamma_2}^n(s), \Psi_{\gamma_2}^n(s)\right), X_{\gamma_2}^n(s), \Upsilon_{\gamma_2}^n(s), \Psi_{\gamma_2}^n(s)\right)||^2 ds + \int_t^T ||\bar{g}\left(s, \Upsilon_{\gamma_1}^n(s), \Psi_{\gamma_1}^n(s)\right) - \bar{g}\left(s, \Upsilon_{\gamma_2}^n(s), \Psi_{\gamma_2}^n(s)\right)||^2 ds + \int_t^T ||\Psi_{\gamma_2}^n(s) - \Psi_{\gamma_1}^n(s)||^2\, ds + 2\int_t^T |\gamma_1 - \gamma_2|\, ds.$$

Because γ_1 and γ_2 are chosen, it follows that $|\gamma_1 - \gamma_2| \to 0$. Also, because g and $\bar{g}$ are stochastically increasing functions, we have

$$||g\left(s, P\left(\Upsilon_{\gamma_1}^n(s), \Psi_{\gamma_1}^n(s)\right), X_{\gamma_1}^n(s), \Upsilon_{\gamma_1}^n(s), \Psi_{\gamma_1}^n(s)\right) - g\left(s, P\left(\Upsilon_{\gamma_2}^n(s), \Psi_{\gamma_2}^n(s)\right), X_{\gamma_2}^n(s), \Upsilon_{\gamma_2}^n(s), \Psi_{\gamma_2}^n(s)\right)||^2 \to 0,$$

$$||\bar{g}\left(s, \Upsilon_{\gamma_1}^n(s), \Psi_{\gamma_1}^n(s)\right) - \bar{g}\left(s, \Upsilon_{\gamma_2}^n(s), \Psi_{\gamma_2}^n(s)\right)||^2 \to 0.$$

By theorem 2.6 [13], we have $\int_t^T ||\Psi_{\gamma_2}^n(s) - \Psi_{\gamma_1}^n(s)||^2\, ds \to 0$, as $n \to \infty$. Hence $||\Upsilon_{\gamma_1}(t)||^2 \le ||\Upsilon_{\gamma_2}(t)||^2$. For $\gamma_n \le \gamma_{n-1} \le \ldots \le \gamma_2 \le \gamma_1 \le \gamma$, we deduce

$$||\Upsilon_{\gamma_n}(t)||^2 \le ||\Upsilon_{\gamma_{n-1}}(t)||^2 \le \ldots \le ||\Upsilon_{\gamma_2}(t)||^2 \le ||\Upsilon_{\gamma_1}(t)||^2 \le ||\Upsilon_{\gamma}(t)||^2.$$

Therefore, there exists a decreasing sequence θ_n such that $\theta \to 0$ as $n \to \infty$ with $\lim_{n\to\infty} X_{\theta_n}(t) \in MSC$. Also, there exists a decreasing sequence γ_n such that $\gamma \to 0$ as $n \to \infty$ with $\lim_{n\to\infty} \Upsilon_{\gamma_n}(t) \in MSC$. Since $f, \bar{f}, g$ and $\bar{g}$ are continuous functions by applying Lebesgue convergence theorem, we have $a(t) = \lim_{n\to\infty} X_{\theta_n}(t)$ and $b(t) = \lim_{n\to\infty} \Upsilon_{\gamma_n}(t)$. Therefore, $(a(t), Z(t), b(t), \Psi(t))$ is a solution of FBFDSDEs (4.2). Now, we show that the solution $(a(t), Z(t), b(t), \Psi(t))$ of FBFDSDEs is a maximal solution. Suppose that $X(t)$ is any solution of forward of FBFDSDEs system. Therefore, $||X_\theta(t) - X(t)||^2 = \theta$, then $||X_\theta(t)||^2 - ||X(t)||^2 \ge \theta$, we have $||X_\theta(t)||^2 \ge ||X(t)||^2$ as $\theta \to 0$. Since the maximal solution is unique, we have that $X_\theta(t)$ tends to $a(t)$ as $\theta \to 0$. Suppose that $\Upsilon(t)$ is any solution of backward of FBFDSDEs system. Therefore, $||\Upsilon_\gamma(t) - \Upsilon(t)||^2 = \gamma$, then $||\Upsilon_\gamma(t)||^2 - ||\Upsilon(t)||^2 \ge \gamma$, we have

$||\Upsilon_\gamma(t)||^2 \geq ||\Upsilon(t)||^2$ as $\gamma \to 0$. Since the maximal solution is unique, we have $\Upsilon_\gamma(t)$ tends to $b(t)$ as $\gamma \to 0$.

Theorem 5.7. Suppose that $f, \bar{f}, g$ and $\bar{g}$ are stochastically increasing function. Under the assumptions (A1-A5), the general system of FBFDSDEs has a minimal solution.

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