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Approximate Solution of an Epidemic SIR Model via Laplace Transform and Q-Homotopy Analysis Method

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Abstract:

In this paper, we study an ad-hoc epidemic SIR model governed by the Caputo derivative. We use the Laplace transform and the q- Homotopy analysis method (Lq-HAM) to obtain approximate solutions to the system. We f inally present some numerical simulations using Mathematica to illustrate the qualitative results. Our approach can be used even in the context of construc tive fractional models and other abstract fractional differential systems.

Keywords: SIR model, caputo derivative, Laplace Transform.

1. Introduction:

Several phenomena can be modeled by non-linear fractional differential equa tions (FDE)[1, 2, 8, 19, 20, 26, 37]. In epidemiology for instance [38], FDE is used to study and analyze efficiently the impact of diseases on populations. Other fields include electrochemistry by [22], electrical circuits in [6], Single pro cessing in [35] and probability [5]. In the recent decade, analytical and numerical technique have been created to obtain approximate solutions to the fractional differential equations(FDEs), such as the adomian decomposition technique (ADM) in [9, 27], variation iteration technique (VIM) in [36], homotopy perturbation technique (HPM) in [39], and the homotopy analysis technique (HAM) [4, 7, 11, 14]. Recently, a new analytic method named q-homotopy analysis method (q-HAM) was introduced in [12, 13]. The "q-HAM" has numerous applications in a wide range of problems [16, 17, 18, 19, 20, 23]. The purpose of this paper is to apply the Lq-HAM, a method that combines q HAM and the Laplace transform to provide an approximate solution for the SIR model, widely used with the most popular for many traditional diseases. Section 2. introduces some essential concepts from fractional calculus and an important lemma utilized in this study. In Section 3. the fundamental principles of Lq-HAM as applied to SIR model are presented. Section 4. includes numerical obtained using Mathematica, showcasing the mth order series solution of Lq-HAM and il lustrating the approximate solutions of S(t), I(t) and R(t) for different values of α which represents the fractional derivative order. This section also details each equation in the system for varying α values of with $n = 1$ and $h = -1$ followed by the approximation solution for the SIR model for the same previous values of [α, n, and h], Section 5. summarizes the significance of each figures in Section 4. represents and concludes the paper. At the end of this paper, we have compiled all relevant information to facilitate a review of this work, including important and thoroughly researched references.

2. Preliminaries and Notation

In this section, we state some necessary concepts of fractional calculus that will help us to achieve the aim of this paper [30, 31]

Definition 2.1.

A real function v(t), t > 0 is said to be in space C_a(a ∈ R)if there exists a real number p > a, such that v(t) = $t^p v_I$(t) , where v_I(t) ∈ C(0,∞) , and it is said to be in the space C_a^m if and only if $v^m \in C_a$,m ∈ N.

Definition 2.2. :

The Riemann-Liouville fractional integral operator Iα of order α ≥ 0 for a func tion f(t) is defined as:

$${}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds,$$
$${}_aI_t^0 f(t) = f(t) \qquad (\alpha > 0)$$

Some of the basic properties of operator $I^\propto$, which are required here, are introduced.

For $f \in C_a, a \geq -1, \alpha, \beta \geq 0, \gamma \geq -1$

(i) $I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t),$

(ii) $I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t),$

(iii) $I^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Definition 2.3 :

The Fractional derivative of order α in the sense of Caputo is :

$${}^cD_{a+}^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{1+\alpha-n}} ds,$$

where f_n is the nth derivative of function f; with α ∈ (n−1, n) and n ∈ N such as n = [α]+1, if [α] = 0 then any derivative of order α in the sense of Caputo will be

written in the following (left-sided and right-sided respectively)

$${}^cD_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(t-s)^\alpha} ds,$$

and $${}^cD_{b-}^\alpha f(x) = \frac{-1}{\Gamma(1-\alpha)} \int_t^b \frac{f'(s)}{(s-t)^\alpha} ds.$$

Some of the most important properties are needed here are as follows:

i) ${}^cD_{a+}^\alpha I^\alpha f(x) = f(x)$

ii) $I^{\alpha c}D_{a+}^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^{(n)}(f(0)) \frac{x^k}{k!}, \quad x > 0$

ii) ${}^cD_{a+}^\alpha x^\sigma = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\alpha+1)} x^{\sigma-\alpha} \qquad \text{for } n-1 < \alpha \leq n, \quad n \in \mathbb{N}$

Lemma 2.4. [15]

If $n-1 < \alpha \leq n$, $n \in N$ then the Laplace transforms of the fractional derivative

$D_t^\alpha f(t) is \quad \mathscr{L}[D_t^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} f^{(k)}(0) s^{\alpha-k-1} \quad t>0,$ where F(s) is the Laplace transform of f(t).

3. Lq-HAM for SIR model

SIR is an epidemiological model that compute the number of individuals

infected with a contagious illness in a closed population over time:

$$\begin{cases} S'(t) &=& \Lambda - \mu S(t) - \frac{\theta\beta S(t)I(t)}{N(t)}, \\ I'(t) &=& \frac{\theta\beta S(t)I(t)}{N(t)} - \gamma I(t) - \mu I(t), \\ R'(t) &=& \gamma I(t) - \mu R(t) \end{cases}$$

In this system the population is divided into three categories S(t), I(t) and R(t) respectively represent the number of susceptible (who haven' t been infected yet) become infected exposed, infections (currently sick and transmitting the disease) and healed or recovered individuals (who have an outcome after being infected) at time t and N(t) is the total population size which is N=S+I+R.

Each parameter in the system represents the following:

- Λ = the recruitment rate of the population.
- μ = the natural mortality rate.
- γ =the infection related mortality rate.
- β =the strength of infection force.
- θ =the recovery rate of infected.

Let consider the following SIR model which we call ad-hoc model, using Caputo

derivative as:

$$(1) \qquad \begin{cases} {}^cD_t^\alpha S(t) &=& \Lambda - \mu S(t) - \frac{\theta\beta S(t)I(t)}{N(t)} \\ {}^cD_t^\alpha I(t) &=& \frac{\theta\beta S(t)I(t)}{N(t)} - (\gamma+\mu)I(t), \\ {}^cD_t^\alpha R(t) &=& \gamma I(t) - \mu R(t) \end{cases}$$

Where D_t^α is the Caputo fractional derivative on the order $\alpha \in (0, 1)$) In system

(1), the total population size is N = S+I+R, considering the initial approximation

for system (1) is 0. It' s subject to the initial condition

$S(0) = S_0(t), I(0) = I_0(t), R(0) = R_0(t)$

We will apply the Laplace transform to both sides of the SIR system (1)

$$\begin{cases} \mathscr{L}[{}^cD_t^\alpha S(t)] &=& \mathscr{L}[\Lambda - \mu S(t) - \frac{\theta\beta S(t)I(t)}{N(t)}], \\ \mathscr{L}[{}^cD_t^\alpha I(t)] &=& \mathscr{L}[\frac{\theta\beta S(t)I(t)}{N(t)} - (\gamma+\mu)I(t)], \\ \mathscr{L}[{}^cD_t^\alpha R(t)] &=& \mathscr{L}[\gamma I(t) - \mu R(t)] \end{cases}$$

Using the linearity of the Laplace transform, we obtain :

$$(2)\quad \begin{cases} \mathscr{L}[{}^{c}D_{t}^{\alpha}S(t)] &= \mathscr{L}[\Lambda]-\mu\mathscr{L}[S(t)]-\theta\beta\mathscr{L}[\frac{S(t)I(t)}{S(t)+I(t)+R(t)}] \\ \mathscr{L}[{}^{c}D_{t}^{\alpha}I(t)] &= \theta\beta\mathscr{L}[\frac{S(t)I(t)}{S(t)+I(t)+R(t)}]-(\mu+\gamma)\mathscr{L}[I(t)] \\ \mathscr{L}[{}^{c}D_{t}^{\alpha}R(t)] &= \gamma\mathscr{L}[I(t)]-\mu\mathscr{L}[R(t)] \end{cases}$$

By using lemma (2.4), we obtain

$$\begin{cases} \mathscr{L}[S(t)] &= \frac{S_0(t)}{s}+\frac{\Lambda}{s^{\alpha+1}}-\frac{1}{s^{\alpha}}[\mu\mathscr{L}[S(t)]+\theta\beta\mathscr{L}[\frac{S(t)I(t)}{S(t)+I(t)+R(t)}]] \\ \mathscr{L}[I(t)] &= \frac{I_0(t)}{s}+\frac{1}{s^{\alpha}}[\theta\beta\mathscr{L}[\frac{S(t)I(t)}{S(t)+I(t)+R(t)}]-(\mu+\gamma)\mathscr{L}[I(t)]] \\ \mathscr{L}[R(t)] &= \frac{R_0(t)}{s}+\frac{1}{s^{\alpha}}[\gamma\mathscr{L}[I(t)]-\mu\mathscr{L}[R(t)]] \end{cases}$$

According to (2), the zero-order deformation equation can be given by

(3)

$$\begin{cases} (1-nq)L[\phi_1(t;q)-S_0(t)] = qh\left[D_t^{\alpha}(\phi_1(t;q))-\Lambda+\mu\phi_1(t;q)+\frac{\Theta\beta\phi_1(t;q)\phi_2 2(t;q)}{\phi_1(t;q)+\phi_2(t;q)+\phi_3(t;q)}\right] \\ (1-nq)L[\phi_2(t;q)-I_0(t)] = qh\left[D_t^{\alpha}(\phi_2(t;q))-\frac{\Theta\beta\phi_1(t;q)\phi_2(t;q)}{\phi_1(t;q)+\phi_2(t;q)+\phi_3(t;q)}+\gamma\phi_2(t;q)+\mu\phi_2(t;q)\right] \\ (1-nq)L[\phi_3(t;q)-R_0(t)] = qh\left[D_t^{\alpha}(\phi_3(t;q))-\gamma\phi_2(t;q)+\mu\phi_3(t;q)\right] \end{cases}$$

where S_0, I_0 and R_0 are initial approximations. where n ≥ 1 , 0 ≤ q ≤ $\frac{1}{n}$ denotes the embedded parameter, h ≠ 0 is an auxiliary parameter. It is obvious that when q = 0 and q = $\frac{1}{n}$, equation (3) becomes

$$\begin{cases} \Phi_1(t,0)=S_0(t), & \Phi_1(t,\frac{1}{n})=S(t) \\ \Phi_2(t,0)=I_0(t), & \Phi_2(t,\frac{1}{n})=I(t) \\ \Phi_3(t,0)=R_0(t), & \Phi_3(t,\frac{1}{n})=R(t) \end{cases}$$

Thus, as q increases from 0 to $\frac{1}{n}$, the solution ϕ_i (t; q), i = 1, 2, 3 varies from the

initials S_0, I_0 and R_0 to the solutions S(t), I(t), R(t). By expanding ϕ_i(t; q), i = 1, 2, 3

in Taylor series with respect to q, we get

$$(4)\quad \begin{cases} \Phi_1(t,q)=S_0(t)+\sum_{m=1}^{\infty}S_m(t)q^m, \\ \Phi_2(t,q)=I_0(t)+\sum_{m=1}^{\infty}I_m(t)q^m, \\ \Phi_3(t,q)=R_0(t)+\sum_{m=1}^{\infty}R_m(t)q^m, \end{cases}$$

where

$$\begin{cases} S_m(t)=\frac{1}{m!}\frac{\partial^m\phi_1(t,q)}{\partial q^m}|_{q=0} \\ I_m(t)=\frac{1}{m!}\frac{\partial^m\phi_2(t,q)}{\partial q^m}|_{q=0} \\ R_m(t)=\frac{1}{m!}\frac{\partial^m\phi_3(t,q)}{\partial q^m}|_{q=0} \end{cases}$$

Assume that h, $S_0(t)$, I_0(t) and $R_0(t)$ are chosen such that the series (4) converges at q = $\frac{1}{n}$ then under these conditions the series solutions give

(5)
$$\begin{cases} \Phi_1(t,q) = S_0(t) + \sum_{m=1}^{\infty} S_m(t)(\frac{1}{n})^m, \\ \Phi_2(t,q) = I_0(t) + \sum_{m=1}^{\infty} I_m(t)(\frac{1}{n})^m, \\ \Phi_3(t,q) = R_0(t) + \sum_{m=1}^{\infty} R_m(t)(\frac{1}{n})^m, \end{cases}$$

Defining the vectors

$$\vec{S_r(t)} = \{S_0(t), S_1(t), S_2(t)...S_r(t)\},$$
$$\vec{I_r(t)} = \{I_0(t), I_1(t), I_2(t)...I_r(t)\},$$
$$\vec{R_r(t)} = \{R_0(t), R_1(t), R_2(t)...R_r(t)\}$$

Differentiating equation (3), (m) times with respect to q then setting q = 0 and finally dividing them by m! yields the so-called (mth) order deformation equations

(6)
$$\begin{cases} S_m(t) = z_m S_{m-1}(t) + h\mathscr{L}^{-1}[\mathscr{L}[S_{m-1}(t)] + \frac{1}{s^\alpha}[\mu\mathscr{L}[S_{m-1}(t)] + \\ \qquad \theta\beta\mathscr{L}[\frac{\sum_{i=0}^{m-1} S_i(t) I_{m-1-i}(t)}{S_{m-1}(t)+I_{m-1}(t)+R_{m-1}(t)}]] - (1-\frac{1}{n}z_m)(\frac{s_0(t)}{s} + \frac{\Lambda}{s^{\alpha+1}})] \\ I_m(t) = z_m I_{m-1}(t) + h\mathscr{L}^{-1}[\mathscr{L}[I_{m-1}(t)] - \frac{1}{s^\alpha}[\theta\beta\mathscr{L}[\frac{\sum_{i=0}^{m-1} S_i(t) I_{m-1-i}(t)}{S_{m-1}(t)+I_{m-1}(t)+R_{m-1}(t)} \\ \qquad -(\gamma+\mu)\mathscr{L}[I_{m-1}(t)]]] - (1-\frac{1}{n}z_m)(\frac{I_0(t)}{s})], \\ R_m(t) = z_m R_{m-1}(t) + h\mathscr{L}^{-1}[\mathscr{L}[R_{m-1}(t)] - \frac{1}{s^\alpha}[\gamma\mathscr{L}[I_{m-1}(t) - \mu\mathscr{L}[R_{m-1}(t)]]] \\ \qquad -(1-\frac{1}{n}z_m)(\frac{R_0(t)}{s})] \end{cases}$$

(7)
$$z_m = \begin{cases} 0 & m \le 1 \\ n & m > 1 \end{cases}$$

4 Numerical Results

In this section, we apply the Lq-HAM to the SIR model (1), and utilize Mathematica to study and analyze the effects of varying the order of fractional derivative on solution of this epidemiological system. We Consider the SIR model (1) and the parameters values listed in Table 1 and using the initial approximations $S_0(t) = 0.07$, $I_0(t)=0.06$ and $R_0(t) = 0.01$.

TABLE 1. Parameters in SIR fractional differential model

parameters	Value	Definition
Λ	0.5	Recruitment rate of the population
μ	0.05	strength of infection
θ	0.6	Natural mortality rate
β	0.4	Recovery rate of infection
γ	0.001	Infection-related mortality rate

Use the analysis in the previous section regarding Eq (6), to derive the following results.

$S_1(t) = -\frac{0.4893ht^\alpha}{\Gamma(\alpha+1)}$

$I_1(t) = -\frac{0.00414ht^\alpha}{\Gamma(\alpha+1)}$

$R_1(t) = \frac{0.00044ht^\alpha}{\Gamma(\alpha+1)}$

$S_2(t) = hn(-\frac{0.4893t^\alpha}{\Gamma(\alpha+1)}) + h(\frac{(0.01443-0.4893h)t^\alpha}{\Gamma(\alpha+1)} - \frac{0.024465ht^{2\alpha}}{\Gamma(2\alpha+1)})$

$I_2(t) = hn(-\frac{0.00414t^\alpha}{\Gamma(\alpha+1)}) + h(\frac{(-0.01443-0.00414h)t^\alpha}{\Gamma(\alpha+1)} - \frac{0.00021ht^{2\alpha}}{\Gamma(2\alpha+1)})$

$R_2(t) = hn(\frac{0.00044t^\alpha}{\Gamma(\alpha+1)}) + h(\frac{0.00044ht^\alpha}{\Gamma(\alpha+1)} + \frac{0.00026ht^{2\alpha}}{\Gamma(2\alpha+1)})$

$\vdots$

Then the (mth) order series solution of Lq-HAM is as follows

$$S^{[m]}(t) = \sum_{i=0}^{m} S_i(t)(\frac{1}{n})^i$$

$$I^{[m]}(t) = \sum_{i=0}^{m} I_i(t)(\frac{1}{n})^i$$

$$R^{[m]}(t) = \sum_{i=0}^{m} R_i(t)(\frac{1}{n})^i$$

Fractional differential equations are an effective tool for studying diseases and their effects on populations. Their capacity to simulate complicated interactions and incorporate memory effects provides a richer understanding of disease dynamics compared to traditional methods. As research in this area progresses, it is expected that FDE's will become more relevant in epidemiology and public health decisionmaking.

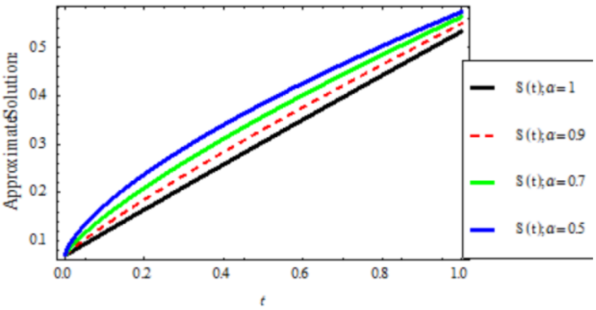

FIGURE 1. The approximation solution of S(t) for different values of α with $n = 1$, $h = -1$

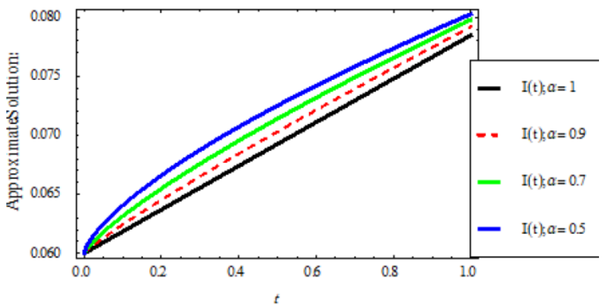

FIGURE 2. The approximation solution of I(t) for different values of α with $n = 1$, $h = -1$

Figures (4-7) show the approximation solutions of S(t),I(t) and R(t) for α = 1, 0.9, 0.8, 0.7 respectively with n = 1 and h = −

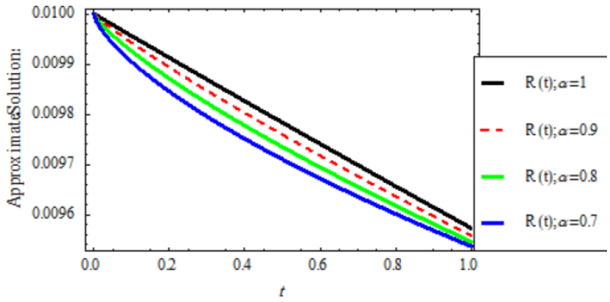

FIGURE 3. The approximation solution of R(t) for different values of α with $n = 1$, $h = -1$

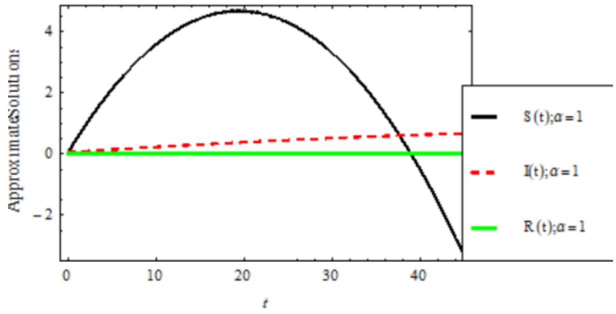

FIGURE 4. Show the approximation solution of S(t),I(t)and R(t) for $\alpha = 1$ with $n = 1, h = -1$

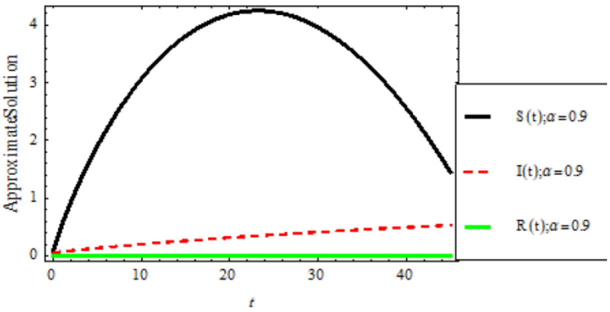

FIGURE 5. Show the approximation solution of S(t),I(t)and R(t) for $\alpha = 0.9$ with $n = 1, h = -1$

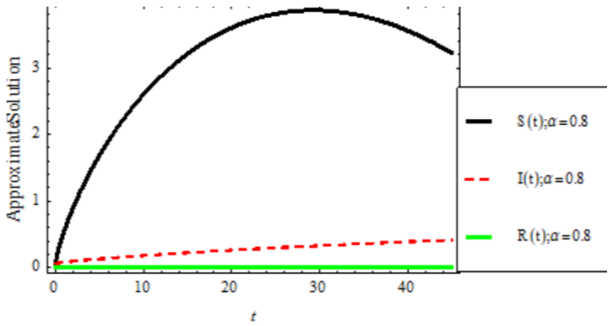

FIGURE 6. Show the approximation solution of S(t),I(t)and R(t) for $\alpha = 0.8$ with $n = 1, h = -1$

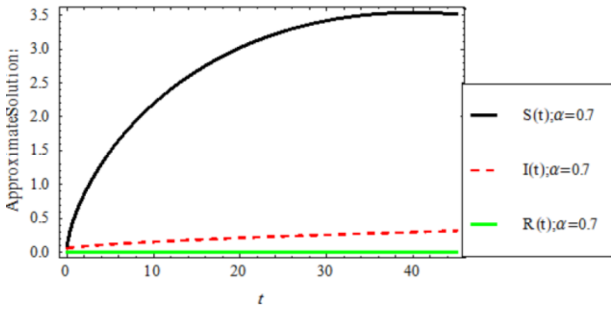

FIGURE 7. Show the approximation solution of S(t),I(t)and R(t) for $\alpha = 0.7$ with $n = 1, h = -1$

4. Discussion and Conclusion

In this section, we analyze the numerical result and simulate it, we utilized $\Lambda = 0.5$, $\mu = 0.05$ (low rate of infection), a natural death rate ($\theta = 0.6$) an average rate, recovery rate $\beta = 0.4$ and a low death rate because of the disease to study the impact of fractional derivative effects on the population dynamics. Figures (1-3) demonstrate the influence of varying orders of fractional derivatives(α =0.7, 0.8, 0.5, 1) for each of S(t), I(t) and R(t) respectively and considering $h = -1$ and $n = 1$, the observations from these three figures indicate that varying the order of the fractional derivative leads to a solution that is more accurate and approaches the exact result more closely and it shows that the susceptible population function is increasing with values equal to the infected population function then the decreasing values of the recovered population. Values can be adjusted to reflect the biological condition of any disease that can be analyzed using this model.

Figures (4-7), using the same values with $\alpha = 1$, figure 4 depicts the approximate solution to the system SIR where the function S(t) initially increases as the disease progresses, reaching the peak before gradually declining until it intersects with the other two functions I(t) and R (t). The subsequent figures (5,6, and 7) for α = 0.9, 0.8, 0.7 respectively show a similar asymptotic shape to the first figure but with more precise and clear values . This demonstrates that the fractional derivatives effectively represent the most accurate cases of value correctness. Our approach can be used even in the context of constructive fractional models [25] and many other abstract fractional differential systems.

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