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Oscillation Criteria for a Certain Class of Cantilever Beam Equations with Nonlinear Damping Term

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Abstract:

The main aim of this paper is to establish some new oscillation criteria for a certain class of cantilever beam equations with the clamped-free end boundary conditions. We will establish the sufficient conditions for the oscillation by using the generalized Riccati technique. Our main tool of this paper is to generalize the Philo's criteria with three variables in the oscillation results. Our approach is using Jenson's inequality to reduce the problem to ordinary differential inequality and satisfy the clamped-free end boundary condition. A solution of u is oscillatory if it has arbitrary large number of zeros, otherwise it is nonoscillatory. Some illustrative examples are given to explain our effectiveness of new results. Analytical methods are now used together with the cantilever beam methods and constitute a significant component in modern vibration analysis.These solutions provide accurate answers if the experiments include real-world problems, are too expensive, or cannot be solved analytically. Beams with a permanent support at one end and no support at the other are called cantilever (or clamped-free) beams. Engineers and architects usually have to build structures that wear the structure and limit radiated noise and substantial displacement amplitudes by designing structures that react minimally to applied loading.

Keywords: cantilever beam, oscillation, damping term, nonlinear.

1. Introduction

Since ancient times, beams have been utilized to reinforce structures such as a certain types of bridges, framed buildings, thin engineering projects, robotic and arms, nantechnology, airplane wings stabilizers [1], [3], [5], [18], [20], [21]. Beams with a permanent support at one end and no support at the other are called cantilever (or clamped-free) beams. The beam is believed to be much longer than it is wide. For long, thin beams, this model is appropriate. The existence and uniqueness of nonlocal strong solutions were studied [2], [11], [12].

The subject of oscillation and nonoscillation of beam equations has been discussed by several authors, [4], [6], [7], [9],[14-16], [19], [22-24] and the references therein.

Analytical methods are now used together with the cantilever beam methods and constitute a significant component in modern vibration analysis. If the experiments involve real-world issues, are too costly, or cannot be solved using analytical methods these solutions provide accurate answers [8], [10], [13], [17I]. Engineers and architects typically have to design structures that react little to applied loading in order to reduce radiated noise, high stress, significant displacement amplitudes, and fatigue in the structure.

Motivated by the above discussion, we initiate the oscillation of cantilever beam equations of the for

$$\frac{\partial}{\partial t}\left(r(t)g\left(\frac{\partial w(x,t)}{\partial t}\right)\right)+\vartheta g\left(\frac{\partial w(x,t)}{\partial t}\right)+\frac{EI}{A}\frac{\partial^4 w(x,t)}{\partial x^4}+c\big(x,t,w(x,t)\big) = f(x,t),$$

$$(\mathrm{x},\mathrm{t}) \in \Omega \times \mathbb{R}_+ = \mathrm{G}, \qquad (1)$$

where, Ω = (0, L), $\mathbb{R}_+$ = (0,∞). Then E is the modulus of elasticity, I is the moment of inertia of the cross-section, A - Cross section area, $w(x,t)$ is beam deflection at the axial location x and time t.

We assume the following conditions,

(A_1) $r(t) \in \mathbb{C}^1(\mathbb{R}_+,\mathbb{R}_+)$, $\int_0^\infty g^{-1}\left(\frac{1}{r(s)}\right)ds = \infty$,

(A_2) $c\big(x,t,w(x,t)\big) \in \mathbb{C}(\bar{G},\mathbb{R})$ *and* $c(x,t,\xi) \geq b\phi(\xi)$, where b is a constant,

(A_3) $f \in \mathbb{C}(\bar{G},\mathbb{R}_+)$ is load distribution, ∋ $\int_\Omega f(x,t)\psi(x)dx \leq 0$.

(A_4) $\phi \in \mathbb{C}(\mathbb{R},\mathbb{R})$ is convex in $\mathbb{R}_+$ with $\xi\phi(\xi) > 0$, $\frac{\phi(\xi)}{\xi} \leq k$ for $\xi \neq 0, k$ is constant.

(A_5) $g^{-1} \in \mathbb{C}(\mathbb{R},\mathbb{R})$ is continuous function with $\xi g^{-1}(\xi) > 0$, there exists nonnegative constant μ ∋ $\xi g^{-1}(\xi) \leq \mu$.

The nature of this term $g\left(\frac{\partial w(x,t)}{\partial t}\right)$ is usually determined by external nonlinear damping mechanisms.

A solution of (1) is oscillatory if it has arbitrary large number of zeros on $\Omega \times (0,\infty)$ for any $t > 0$, otherwise it is nonoscillatory, (1) is said to be oscillatory if all its solutions are oscillatory.

2. Oscillation results with clamped-free ends

We examine the oscillation of (1) with clamped-free ends in this section. Our approach is using Jenson's inequality to reduce the problem to ordinary differential inequality and satisfy the condition

$$\frac{\partial^2 \mathrm{w}(0,\mathrm{t})}{\partial \mathrm{x}^2} = \frac{\partial^2 \mathrm{w}(\mathrm{L},\mathrm{t})}{\partial \mathrm{x}^2} = \mathrm{w}(0,\mathrm{t}) = w(L,t) = 0. \qquad \text{(B)}$$

Theorem 2.1. *Suppose that* $\boldsymbol{\psi^4(x)} \geq \boldsymbol{a\psi(x)}$, $\boldsymbol{x}$ *in* $\boldsymbol{\Omega}$ *for some* $\boldsymbol{a} \geq \mathbf{0}$. *Then the equations (1) and (B) are oscillatory in* $\boldsymbol{G}$ *if the differential inequality,*

$$(\boldsymbol{r(t)g(W'(t))})' + \boldsymbol{\gamma g}\left(\boldsymbol{W'(t)}\right) + \frac{\boldsymbol{EIa}}{\boldsymbol{A}}\boldsymbol{W(t)} + \boldsymbol{b\phi(W(t))} \leq \mathbf{0}, \qquad (2)$$

has no solution which is positive on $[t_0,\infty)$, *for any* $t_0 > 0$.

Proof:

Let w is solution of (1), (B). First assume that $w > 0$. Multiply by

$\psi(x) = sin\left(\frac{2\pi}{L}\right)x$, integrating over Ω.

$$\int_{\Omega} \frac{\partial}{\partial t}\left(r(t)g\left(\frac{\partial w}{\partial t}\right)\right)\psi(x)dx + \int_{\Omega} \gamma g\left(\frac{\partial w}{\partial t}\right)\psi(x)dx + \int_{\Omega} \frac{EIa}{A}\frac{\partial^4 w}{\partial x^4}\psi(x)dx$$

$$+\int_{\Omega} c(x,t\,w)\psi(x)dx = \int_{\Omega} f(\psi(x)dx. \qquad (3)$$

Applying Jensen's inequality,

$$\int_{\Omega} \frac{\partial}{\partial t}\left(r(t)g\left(\frac{\partial w}{\partial t}\right)\right)\psi(x)dx = (r(t)g(W'(t))'. \qquad (4)$$

Again apply Jensen's inequality,

$$\int_{\Omega}\gamma g\left(\frac{\partial w}{\partial t}\right)\psi(x)dx = \gamma g\left(W'(t)\right). \qquad (5)$$

Taking integrating by parts and using (B),

$$\int_{\Omega}\frac{\partial^4 w}{\partial x^4}\psi(x)dx \geq aW(t) \qquad (6)$$

Using (A_2), we have

$$\int_{\Omega} c(x,t\,w)\psi(x)dx \geq b\ \phi(W(t)) \qquad (7)$$

Equations (4), (7) are substituted in (3), where $W(t) = \int_{\Omega} w(x,t)\psi(x)dx$,

i.e. $W(t) > 0$ is a solution of (2).

We use the Riccati techniques to establish some new oscillations in the next theorem.

Theorem 2.2. *Consider the conditions* $(\boldsymbol{A_4})$, $(\boldsymbol{A_5})$ *and* $\left(\frac{\boldsymbol{EIa}}{\boldsymbol{A}} + \boldsymbol{bk}\right) \leq \boldsymbol{\lambda} \ni$

$$\lim_{t\to\infty} sup \int_{t_1}^{t}\left(\lambda - \frac{\mu\gamma^2}{4r(s)}\right)ds = \infty. \qquad (8)$$

Then there exists an oscillatory solution of equation.

Proof: Define the Riccati transformation,

$$U(t) = \left(\frac{r(t)g(W'(t))}{W(t)}\right), \qquad t \geq t_0,$$

$$U'(t) \leq -\lambda - \frac{\gamma}{r(t)}U(t) - \frac{1}{\mu r(t)}U^2(t). \qquad (9)$$

Integrating from t_1 to t,

$$-U(t_1) \leq \frac{1}{4}\int_{t_1}^{t}\frac{\mu\gamma^2}{r(s)}ds - \int_{t_1}^{t}\lambda\, ds.$$

Taking $\limsup_{t\to\infty}$, we get

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(\lambda-\frac{\mu\gamma^2}{r(s)}\right)ds \le U(t_1) < \infty,$$

which leads to a contradiction.

In the sequel, a function $\mathbb{J} \coloneqq \mathbb{J}(t,s,l), \mathbb{J} \in \Gamma$, if $\mathbb{J} \in \mathbb{C}(\mathbb{D},\mathbb{R}_+)$ satisfying $\mathbb{J}(t,t,l)=0$, $\mathbb{J}(t,l,l)=0, \mathbb{J}(t,s,l)\neq 0$ for $t>s>l\ge t_0$, where $\mathbb{D}=\{(t,s,l): t_0\le s<t<\infty\}$. Furthermore, $\mathbb{J}$ has continuous derivatives on $\mathbb{D}$, define the function $j=j(t,s,l)$,

$$\frac{\partial \mathbb{J}}{\partial s}=j(t,s,l)\mathbb{J}(t,s,l).$$

If $\mathbb{J}(t,s,l)=\vartheta(s)(t-s)^m(s-l)^n$ for $m,n>\frac{1}{2}$, $\vartheta(t)\in\mathbb{C}(\mathbb{R}_+,\mathbb{R}_+)$. Then we have

$$j(t,s,l)=\frac{\vartheta'(s)}{\vartheta(s)}+\frac{nt-(m+n)s+ml}{(t-s)(s-l)}.$$

By virtue of Theorem 2.2, we get

Theorem 2.3. *Assume that conditions* (A_4), (A_5) *such that*

$$\limsup_{t\to\infty}\int_{l}^{t}\vartheta(s)(t-s)^m(s-l)^n \left(\boldsymbol{\lambda}-\frac{\mu r(s)}{4}\left(\frac{\gamma}{r(s)}-\left(\frac{\vartheta'(s)}{\vartheta(s)}+\frac{nt-(m+n)s+ml}{(t-s)(s-l)}\right)\right)^2\right)ds=\infty. \quad (10)$$

Then there exists an oscillatory solution of equations.

Proof: Multiply by $\mathbb{J}(t,s,l)$ and integrating on bothsides of (9),

$$\int_l^t \mathbb{J}(t,s,l)\,\mathrm{U}'(\mathrm{s})ds \le -\int_l^t \mathbb{J}(t,s,l)\left(\frac{\gamma}{r(s)}U(s)-\frac{1}{\mu r(s)}U^2(s)\right)ds-\lambda\int_l^t \mathbb{J}(t,s,l)\,ds,$$

$$0\le -\int_l^t \mathbb{J}(t,s,l)\left(\lambda-\frac{\mu r(s)}{4}\left(\frac{\gamma}{r(s)}-j(t,s,l\right)^2\right)ds$$

$$\limsup_{t\to\infty}\int_{l}^{t}\vartheta(s)(t-s)^m(s-l)^n \left(\boldsymbol{\lambda}-\frac{\mu r(s)}{4}\left(\frac{\gamma}{r(s)}-\left(\frac{\vartheta'(s)}{\vartheta(s)}+\frac{nt-(m+n)s+ml}{(t-s)(s-l)}\right)\right)^2\right)ds<\infty.$$

Which leads to contradictions.

Let choose $\mathbb{J}(t,s,l)=\sqrt{\mathbb{H}_1(s,l)\mathbb{H}_2(t,s)}$, where $\mathbb{H}_1,\mathbb{H}_2\in\mathbb{C}(\mathbb{R},\mathbb{R})$. Let $\mathbb{H}_1,\mathbb{H}_2$ are conditions as the following

$$\frac{\partial \mathbb{H}_1}{\partial s} = h_1(s.l)\sqrt{\mathbb{H}_1(s,l)}, \qquad \frac{\partial \mathbb{H}_2}{\partial s} = -h_2(t,s)\sqrt{\mathbb{H}_2(t,s)}.$$

We obtain the following theorem from the basic computation.

Theorem 2.4. *Assume that conditions (A_4), (A_5) such that*

$$\lim_{t\to\infty} \sup \int_l^t \mathbb{J}(t,s,l)\left(\lambda - \frac{\mu r(s)}{4}\left(\frac{\gamma}{r(s)} - \frac{1}{2}\left(\frac{h_1}{\sqrt{\mathbb{H}_1}} - \frac{h_2}{\sqrt{\mathbb{H}_2}}\right)\right)^2\right) ds = \infty, \qquad (11)$$

Then there exists an oscillatory solution of equations.

Proof: Multiply by $\mathbb{J}(t,s,l)$ and integrating on bothsides of (9),

$$\int_l^t \mathbb{J}(t,s,l)\, \mathrm{U}'(\mathrm{s}) ds \leq -\int_l^t \mathbb{J}(t,s,l) \left(\frac{\gamma}{r(s)} U(s) - \frac{1}{\mu r(s)} U^2(s)\right) ds - \lambda \int_l^t \mathbb{J}(t,s,l)\, ds,$$

$$0 \leq -\int_l^t \mathbb{J}(t,s,l) \left(\lambda - \frac{\mu r(s)}{4}\left(\frac{\gamma}{r(s)} - \frac{1}{2}\left(\frac{h_1}{\sqrt{\mathbb{H}_1}} - \frac{h_2}{\sqrt{\mathbb{H}_2}}\right)\right)^2\right) ds$$

$$\lim_{t\to\infty} \sup \int_l^t \mathbb{J}(t,s,l)\left(\lambda - \frac{\mu r(s)}{4}\left(\frac{\gamma}{r(s)} - \frac{1}{2}\left(\frac{h_1}{\sqrt{\mathbb{H}_1}} - \frac{h_2}{\sqrt{\mathbb{H}_2}}\right)\right)^2\right) ds < \infty,$$

Which leads to contradictions.

Corollary 2.5 *Suppose that Therorem 2.4's requirements are satisfied and that (2.11) is substituted with*

$$\lim_{t\to\infty} \sup \int_l^t \lambda\sqrt{\mathbb{H}_1\mathbb{H}_2}\, ds = \infty, \quad \text{and}$$

$$\lim_{t\to\infty} \sup \int_l^t \frac{\mu r(s)}{4}\sqrt{\mathbb{H}_1\mathbb{H}_2}\left(\frac{\gamma}{r(s)} - \frac{1}{2}\left(\frac{h_1}{\sqrt{\mathbb{H}_1}} - \frac{h_2}{\sqrt{\mathbb{H}_2}}\right)\right)^2 ds < \infty.$$

Consequently, each oscillatorysolution $w(x,t)$ of equations (1), (B) exists in G.

Example Consider the cantilever beam equations,

$$\frac{\partial}{\partial t}\left(r(t) g\left(\frac{\partial w}{\partial t}\right)\right) + g\left(\frac{\partial w}{\partial t}\right) - 2\left(\frac{L}{\pi}\right)^4 \left(\frac{\partial^4 w}{\partial x^4}\right) + c(x,t,w)$$
$$= (1 + \cos t + \mathrm{t}\sin t) \sin\left(\frac{\pi}{L}\right) x + 4t\cos t, \qquad (x,t) \in G, \qquad (12)$$

with Boundary condition (B).

Here $r = 2$, $g(w) = 1 + w, \gamma = 1, \frac{EI}{A} = -2\left(\frac{L}{\pi}\right)^4$, $\lambda = \left(\frac{L}{\pi}\right)^4 + 4, b = 4$,

$f(x,t) = (1 + \cos t + \mathrm{t}\sin t) \sin\left(\frac{\pi}{L}\right) x + 4t\cos t$ and $\mu = 4$.

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(\lambda-\frac{\mu\gamma^2}{4r(s)}\right)ds=\infty$$

Therefore, Theorem 2.2's requirements are all satisfied. As a result, each and every solution to (12) oscillates. In fact, $w(x,t)=t\,cost\,sin\left(\frac{\pi}{L}\right)x$ is one such solution of (12).

Conclusion

In this paper, our main goal has been to provide new sufficient conditions for the oscillation criteria for a certain class of cantilever beam equations with some boundary conditions.The newly produced result also includes required examples.

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