Advances in Nonlinear Variational Inequalities
ISSN: 1092-910X
Vol 28 No. 3s (2025)

Exploring Transformations and Properties of Secondary J Binormal Matrices: Closure Challenges and Skew-Symmetric Conversions

K.Jaikumar[1],K.Saravanan[2]*

[1,2]*Department of Mathematics, A.V.C.College (Autonomous), (Affiliated to Bharathidasan University, Trichy), Mannampandal, Tamilnadu, India.

*Corresponding Author Email: sarant100@gmail.com

Article History:

Received: *24-09-2024*

Revised: *17-11-2024*

Accepted: *27-11-2024*

Abstract:

This paper investigates the conditions for a secondary J binormal matrix to transition into a secondary J normal matrix and examines the analogous transformation for a secondary Jbnbinormal matrix to become a secondary J normal matrix. We also analyze the distinctive properties associated with secondary binormal is J_{2bn}matrices. Moreover, we address a crucial issue: the lack of closure under addition for secondary J_{2bn}binormal matrices. Furthermore, we investigate the essential requirements and conditions that are both necessary and sufficient for transforming a secondary J skew-symmetric matrix into a secondary J_{2bn} binormal matrix.

Keywords: Normal matrix, Bi-normal matrix, secondary binormal matrix

---

1. Introduction

Matrices, both in the real and complex domains, have held a central position in the exploration of linear algebra for many decades, representing a fundamental algebraic category. Initially investigated by the French mathematician Cayley in 1860, matrices are characterized by their organization around the main diagonal [1]. Over time, matrices have become indispensable tools in various branches of Applied Mathematics, Physics, and Engineering. Particularly noteworthy are specialized matrix types such as symmetric, skew symmetric, hermitian, skew hermitian, unitary, and normal matrices. Generalised normal matrices are important for the study of generalised inverses and the spectral theory of rectangular matrices [2-4].

Toeplitz [6] first proposed the idea of a normal matrix including components from the complex field in 1918. He determined the necessary and sufficient criteria under which a matrix over complex numbers may be regarded as normal [7, 8]. Beyond Hermitian matrices, the idea of matrix normalcy also includes the feature of commuting with its conjugate transpose. Consequently, the category of normal matrices bears significance within the wider domain of matrix analysis [9, 10].

Any square matrix A over a field can be transformed into its transpose A^T through a similarity transformation. This suggests that a symmetric matrix B exists, such that $A^T = B^{-1}AB$the product of B and A is equal to the transpose of A [11]. To further investigate this idea, consider the class of matrices that this connection holds when paired with a fixed symmetric matrix. Examine the permutation matrix 'V,' in which the units are situated in the secondary diagonal. In this case, the similarity transformation $n \times n$yields matrices that are balanced with esteem towards the inferiors loping. This work leads to the study of secondary transposition, or transposing an arbitrary medium A with respect to the lesser sloping. This served as inspiration for Anna Lee [12], who developed the concepts of secondary orthogonal (s-orthogonal), secondary skew symmetric (s-skew symmetric), and secondary symmetric (s-symmetric) matrices and derived their characteristics.

The author has demonstrated a connection between the conventional transpose matrix A and the secondary transpose matrix A^S, denoted by $A^S = VA^TV$, anywhere V represents the variation a unit is situated in the secondary diagonal of the matrix [13]. The existence of a secondary symmetric matrix B, which $A^S = B^{-1}AB$is a positive semi-definite matrix divided into four blocks, has also been shown by the author. Numerous studies and surveys have been conducted in the literature because of the importance of these matrices in matrix analysis and quantum theory [14].

The collection of $n \times n$complex one conditions is written as$\Psi^{n\times n}$, and the individuality average of proper scope is represented as Ψ. Let's define the interplanetary of$n \times n$ complex mediums as$U(n)$. Suppose that A is a Hermitian matrix and that is $\lambda(A)_{max}$its biggest eigenvalue. $A \leq 0$if A stands undesirable semi-definite, we use the notation $A \geq 0 (A \succ 0)$to show that it remnants likewise constructive semi-definite or definite [15]. A matrix X's spectral norm is represented as$\|X\|_{sp}$.

In this work, we choose M to be the optimistics emi-definite lump matrix with the given structure:

$$M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \Psi^{2n\times 2n}, \ (1)$$

In the sequel$A, B, X \in \Psi^{n\times n}$, Positive Partial Transposition (PPT) is miscalled if, M,

$$M' = \begin{bmatrix} A & X^* \\ X & B \end{bmatrix}, \ (2)$$

has an additional confident semi-definite. A average finished the planetary$\|\bullet\|$ of conditions is unitarily invariant if $\|U \times V\| = \|X\|$and$U, V \in \Psi(n)$ only if$X \in \Psi^{n\times n}$.

2. Literature Review

This section provides precise approximations using low-rank matrices by deriving explicit constraints for the remarkable values of positive definite Hankel conditions. Additionally, the relationship between normal and binormal matrices is explored, revealing specific eigenvalue characteristics associated with badly conditioned normal matrices.

Using rational functions, Beckermann et al. [16] addressed an extreme situation by putting explicit limits on the extraordinary values of matrices. Specifically, they showed that a constant C multiplied by the reciprocal of ρ raised to the power of k, divided by the logarithm of n, and finally multiplied by the 2-norm of H_n limited the kth singular value of a positive definite Hankel matrix of size n (referred to as H_n) in the real domain. In this case, the preset constants C and ρ have values greater than 0 and more than 1, respectively. The symbol |H_n|_2 represents the spectral norm. This conclusion implies that a matrix of rank O(log n log(1/ε)) may accurately represent a real n × n positive definite Hankel matrix with epsilon|H_n|_2 (where 0 < ε < 1).

Ikramov et al. [17] introduced the concept of normal and binormal matrices, demonstrating that a binormal matrix can be derived from a normal matrix through right multiplication by a suitable unitary matrix. Specifically, considering a normal matrix N with a large condition number $\mathrm{cond}_2 N$ concerning inversion, they established the existence of a binormal matrix B, derived from N, with individual eigenvalue condition numbers of order $(\mathrm{cond}_2 N)^{1/2}$.

In another work, Ikramov et al. [18] presented square matrix is characterized as binormal when its rows and columns are pairwise orthogonal, and it exhibits the property that the products of its conjugate transpose ($AA*$) and transpose conjugate ($A*A$) matrices commute. Additionally, a matrix is termed unitoid if it can be transformed into a diagonal form through a congruence with a Hermitian matrix. In Thompson et al.'s study [19], binormal, complex symmetric operators were presented and examined. Their connections to the Duggal and Aluthge transformations were examined, and some of their related characteristics were explored.

Meenambika et al. [20] extended concepts and ideas from skew binormal operators to [nsnBN] operators acting on a Hilbert space H. They proved results based on self-adjoint and [nsnBN] operators, establishing independence between [nsnBN] operators and n-isometry operators. Kaiser et al. [21] delved into the normality properties of composition matrices, emphasizing their significance in linear algebra due to their connection with diagonalization. Normal solutions were discussed, followed by more complex binormal solutions, and finally criteria for the composition matrix to commute with a general matrix.

The motivation underlying the research presented in the articles stems from the quest to understand and address key challenges in linear algebra, particularly in the context of matrix properties and their applications. As stated by Beckermann et al. [16], was to clearly define the boundaries of positive definite Hankel matrices' singular values in order to provide insightful insights into the creation of effective approximations for these matrices. Ikramov et al. [17] sought to explore the relationships between normal and binormal matrices, focusing on conditions and properties, especially in cases of poorly conditioned normal matrices. The work of Ikramov et al. [18] aimed at investigating square matrices featuring orthogonal rows and columns, with a specific focus on identifying the circumstances that lead a nonsingular binormal matrix to transform into a unitoid. Thompson et al. [19] were motivated to study binormal, complex symmetric operators, exploring connections to transformative operations and uncovering additional properties. Meenambika et al. [20] extended concepts to [nsnBN] operators, establishing results based on self-adjoint properties and emphasizing the independence of [nsnBN] operators and n-isometry operators as distinct classes. Lastly, Kaiser et al. [21] delved into the normality properties of composition matrices, highlighting their importance in linear algebra and their connection to diagonalization. Overall, the common thread across these articles lies in the researchers' motivation to deepen our understanding of matrix structures and behaviors, with implications for both theoretical advancements and practical applications in various domains.

3. Preliminaries

Definition 3.1

$P_b = P_1 \cup P_2$ [$\cup$is not only an process a representation] Bi-matrix

For example

$$\text{Let}P_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, P_2 = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \text{Then} P_b = \left[\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cup \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \right].$$

Definition 3.2

A Matrix $P \in R_{i \times i}$is Normal matrix $PP^* = P^*P$.

Definition 3.3

A Bi-Matrix $P_b \in R_{i \times i}$is Normal Bi-matrix$P_b P_b^* = P_b^* P_b$.

$$(P_1 \cup P_2)(P_1 \cup P_2)^* = (P_1 \cup P_2)^*(P_1 \cup P_2)$$

$$P_1 P_1^* \cup P_2 P_2^* = P_1^* P_1 \cup P_2^* P_2.$$

Example 1.4

$$\text{Let}P_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, P_1^* = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, P_2^* = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \text{Then}$$

Now to check Normal Bi- matrix$P_1 P_1^* \cup P_2 P_2^* = P_1^* P_1 \cup P_2^* P_2$.

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Definition 3.4

Let $x_1, x_2, \ldots, x_i$be an orthonormaleigenvector basis of this matrix and let $\alpha_1, \alpha_2, \ldots, \alpha_i$be the corresponding eigenvalues.Expanding a vector βin this basis, we have [22],

$$\beta = \textstyle\sum_{n=1}^{i} \xi_n x_n,$$

$$\|Pi\|^2 = \left\| \textstyle\sum_{n=1}^{x} \alpha_n \xi_n x_n \right\|^2 = \textstyle\sum_{n=1}^{x} |\alpha_n|^2 \left|\xi_n\right|^2,$$

$$\left\|P^2 i\right\|^2 = \left\| \textstyle\sum_{n=1}^{x} \alpha_n^2 \xi_n x_n \right\|^2 = \textstyle\sum_{n=1}^{x} |\alpha_n|^4 \left|\xi_n\right|^2,$$

And

$$\|Pi\|^2 = \sum_{n=1}^{x} |\alpha_n|^2 \left|\xi_n\right|^2 \leq \left(\sum_{n=1}^{x} |\alpha_n|^4 \left|\xi_n\right|^2 \right) \frac{1}{2} \bullet \left(\sum_{n=1}^{x} \left|\xi_n\right|^2 \right)^{\frac{1}{2}} = \left\|P^2 i\right\| \bullet \|i\|$$

In the preceding connection, we employed the Cauchy–Schwarz–Bunyakovskii inequality.

Though there are relevant finite-dimensional modifications of the normalcy idea, they do not exist in the cases described above. One such extension employing square matrices is the subject of this discussion. A matrix P is termed binormal when the associated matrices PP^* and P^*P exhibit commutation. It is evident that all normal matrices P, where PP^* and P^*P are identical, fall under the category of binormal matrices. However, there are additionally binormal matrix that are not normal. The Jordan block is the simplest illustration.

$$J_2' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

This is diagonal and so commuted in relations of the matrices$J_2' {J'}_2^{x}$and ${J'}_2^{x} J_2'$.

It should be noted that the literature has a variety of definitions of binormality. A binormal matrix, for example, is described in [23] as a matrix of even order 2x composed of four normal x×x blocks that commute pairwise. In our definition, such a matrix is likewise binormal; however, the latter does not depend on the order being even and does not associate binormality with a matrix's block structure.)

3.1. Properties of binormal matrices

As per [8], the set of binormal matrices of order i is represented by$(\Psi_{bn})_i$, or just by(Ψ_{bn}). We start with a brief summary of characteristics that $(\Psi_{bn})$are comparable to the set of normal matrices in some way.

Proposition 1. Allow $P \in (\Psi_{bn})$. Then, the following claims are true for each $\lambda \in \Omega$and every unitary matrixU_t:

(i)$\lambda P \in (\Psi_{bn})$

(ii)$U_t^* P U_t \in (\Psi_{bn})$

(iii)$P^*, P^n, \bar{P} \in (\Psi_{bn})$

(iv) If P is a matrix that is not singular, therefore $P^{-1} \in (\Psi_{bn})$.

Still, there are some clear differences between the two groups. Each positive fundamental power of a normal matrix, for example, is likewise normal. Conversely, however, the structure of the matrix,

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

is unique because

$$PP^n = diag(1,2,2), P^n P = diag(2,2,1).$$

Meanwhile,$Q = P^2 \notin (\Psi_{bn})$since

$$\{(\Psi_{bn}\Psi_{bn}^n)(\Psi_{bn}^n\Psi_{bn})\}_{12} = |2, \{(\Psi_{bn}^n\Psi_{bn})(\Psi_{bn}\Psi_{bn}^n)\}_{12} = 3.$$

Moreover, the total of typical matrix that commute is also normal. Conversely, take into account the binormal matrix's total.

$$P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

as well as the standard matrices N_2:

$$Q = P + N_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In fact, matrix B is not binormal.

$$(\Psi_{bn}\Psi_{bn}^{n})(\Psi_{bn}^{n}\Psi_{bn}) = \begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix},$$
$$(\Psi_{bn}^{n}\Psi_{bn})(\Psi_{bn}\Psi_{bn}^{n}) = \begin{pmatrix}3 & 2\\ 2 & 4\end{pmatrix}.$$

Let P represent any random matrix inΨ_{bn}. Is matrix P normal. The location of the P field of values in relation to the point k = 0 affects the solution somewhat. One may acquire a comprehensive response by examining P's behaviour when its primary diagonal is shifted. First, we mention the preliminary theorem of M. Embry and S. Campbell (see [24]) for related conclusions.

Theorem 3.1. Let $R$and S be ordinary matrix that commute. Let us suppose that there is a matrix of numbers P that

$$PR = SP \tag{1}$$

and that the field of values $G(P)$of this matrix has an exterior point at k = 0. R therefore equals S.

Proof. Let$x_1, x_2, \ldots., x_i$ be an orthonormal foundation made up of the flowing matrix R and S's shared eigenvectors. All three matrices, R, S, and P, should be transformed to this basis:

$R \to \wedge = U_t^* R U_t = diag(\alpha_1, \ldots, \alpha_i)$,

$$S \to L = U_t^* S U_t = diag(\beta_1, \ldots, \beta_i),$$

$P \to Q = U_t^* P U_t$.

Equation (1) therefore read as

$Q \wedge = LQ$.

or, the form of entries,

$$q_{xy}(\alpha_x - \beta_y) = 0 x, y = 1, 2, \ldots., i \tag{2}$$

For any x in the index, the inequality $\alpha_x \neq \beta_x$is impossible. In fact, if (2) were any other case, it would imply that$q_{xx} = (Qe_x, e_x) = 0$, that is$0 \in W(Q) = W(P)$, which defies the theorem's assumptions. Consequently$\wedge = L$, R=S.

Proposition 3.2. Allow $P \in (\Psi_{bn})$and allow$0 \notin W(P)$. P is a regular matrix therefore.

Proof. Put in place $R = P^*P$and$S = PP^*$. All of the matrices R, S, and P meet the requirements.

Theorem 3.1. Therefore, $P^*P = PP^*$.

Proposition 2 is clarified as follows in [3].

Proposition 3.3. Additionally, let's $P \in (\Psi_{bn})$suppose that the matrix

$$P_r = \frac{1}{2}(P + P^*) \tag{3}$$

is in the case of positives semidefinite. P is a regular matrix then.

Proof. The matrix (3) is part of the Toeplitz or Hermitian decomposition of P, as it is known:

$$P = Q + xT$$

$$Q = P_r, T = P_{im} = \frac{1}{2x}(P - P^*).$$

The relations provide a connection between the matrices$R = P^*P$ and$S = PP^*$are Q and T.

$$P^*P = Q^2 + T^2 + x(QT - TQ) \tag{4}$$

$$PP^* = Q^2 + T^2 + x(TQ - QT) \tag{5}$$

The process of proving Theorem 3.1 involves us going to an orthonormal basis $\{x\}$made up of the common eigenvectors of the Hermitian matrices R and S that commute.

$$\begin{aligned} R &\rightarrow \wedge = diag(\alpha_1, \alpha_2, \ldots., \alpha_i), \\ S &\rightarrow L = diag(\beta_1, \beta_2, \ldots., \beta_i), \end{aligned} \tag{6}$$

$\bar{P}, \bar{Q}, \bar{T}$stand for the transformed matrices P, Q, and T, respectively. Next, we have

$$\bar{p}_{xy}(\alpha_x - \beta_y) = 0 x, y = 1,2, \ldots., i$$

Let us assume that$\alpha_x \neq \beta_x$ for a given index x. We can set x = 1 without losing generality.

Then$\bar{p}_{11} = 0$, from $\bar{q}_{11} = \bar{t}_{11} = 0$. The matrix $\bar{Q}$is deemed to be positive semidefinite. Therefore, it follows that the equality $\bar{q}_{11} = 0$indicates that there are zero entries in this matrix's first row and first column. Consequently, this suggests that the matrix's $\bar{Q}\bar{T} - \bar{T}\bar{Q}$diagonal entry (1,1) is zero. According to the premise, relations (4) and (5) now give$\{x\}$,

$$\alpha_1 = \{\bar{T}^2\}_{11}, \beta_1 = \{\bar{T}^2\}_{11},$$

Such that is$\alpha_1 = \beta_1$,which defies the first assumption. This implies that $P^*P = PP^*$the equality $\alpha_x = \beta_x$is true for all values of x.

Remark. The more general condition $0 \in \partial W(P)$may be used in substitute of the criterion of positive semidefiniteness of the matrix $P_r$by using item (n) of Proposition 3.1. This variant of statement 3.3 implies that a matrix$P \in (\Psi_{bn})$ may only be nonnormal when z = 0 is an interior point of the field of values W(P), in conjunction with the statement that came before it.

All of the shifted matrice$P + \alpha N$s is normal if P is a normal matrix. It was found by Campbell (see [25]) that non-normal matrices$P \in (\Psi_{bn})$ respond to diagonal shifts completely differently.

4. Proposed Secondary J$_2$ Binormal$J_2\Delta_{bn}$ Matrices

We examine the characteristics of bi-normal and J$_2$ bi-normal matrix as a square matrix that has a similar structure to a regular Jordan block but with a different eigenvalue on the main diagonal and ones on the super diagonal in this section.

Definition 4.1

A Bi-Matrix $J_{bn} \in J_{n*n}$is J bi-normal matrix $J_{bn}J_{bn}^*J_{bn}^*J_{bn} = J_{bn}^*J_{bn}J_{bn}J_{bn}^*$

$$[J_1J_1^*J_1^*J_1] \cup [J_2J_2^*J_2^*J_2] = [J_1^*J_1J_1J_1^*] \cup [J_2^*J_2J_2J_2^*].$$

Example 4.1

$$J_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, J_1^* = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, J_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, J_2^* = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

Now to check binormal matrix $J_{bn}J_{bn}^*J_{bn}^*J_{bn} = J_{bn}^*J_{bn}J_{bn}J_{bn}^*$

$$[J_1J_1^*J_1^*J_1] \cup [J_2J_2^*J_2^*J_2] = [J_1^*J_1J_1J_1^*] \cup [J_2^*J_2J_2J_2^*]$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{bmatrix} \cup \begin{bmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{bmatrix} \cup \begin{bmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{bmatrix}$$

Definition 4.2

A J Bi- normal Matrix $J_{bn}^* \in J_{n*n}$is J_2 bi-normal matrix $J_{2bn}J_{bn}J_{bn}^*J_{bn}^*J_{bn} = J_{bn}^*J_{bn}J_{bn}J_{bn}^*J_{2bn}$

$$\left[J_{2bn}^1J_{bn}^{1*}J_{bn}^{1*}J_{bn}^1\right] \cup \left[J_{2bn}^2J_{bn}^{2*}J_{bn}^{2*}J_{bn}^2J_{2bn}^2\right] = \left[J_{bn}^{1*}J_{bn}^1J_{bn}^1J_{bn}^{1*}\right] \cup \left[J_{bn}^{2*}J_{bn}^2J_{bn}^2J_{bn}^{2*}J_{2bn}^2\right].$$

Example 4.1

$$J_{bn}^1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, J_{bn}^2 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, J_{bn}^{1*} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, J_{bn}^{2*} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, J_{2bn}^1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} J_{2bn}^2 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Now to check binormal matrix

$$\left[J_{2bn}^1J_{bn}^{1*}J_{bn}^{1*}J_{bn}^1\right] \cup \left[J_{2bn}^2J_{bn}^{2*}J_{bn}^{2*}J_{bn}^2J_{2bn}^2\right] = \left[J_{bn}^{1*}J_{bn}^1J_{bn}^1J_{bn}^{1*}\right] \cup \left[J_{bn}^{2*}J_{bn}^2J_{bn}^2J_{bn}^{2*}J_{2bn}^2\right]$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Theorem 4.1.

Transpose of a J bi-normal matrix is a J_2bi-normal matrix.

Proof

For a binormal matrix J_{bn},

$$J_{bn}J_{bn}^* = J_{bn}^*J_{bn}$$

That is, J_2bi-normal matrix

$$J_{2bn}J_{bn}J_{bn}^*J_{bn}^*J_{bn} = J_{bn}^*J_{bn}J_{bn}J_{bn}^*J_{2bn}$$

$$J_{bn} = J_{bn}^1 \cup J_{bn}^2$$

Transpose of a J₂bi-normal matrix

$$J_{2bn}^{T}[J_{bn}^{T}][J_{bn}^{T}]^{*}[J_{bn}^{T}]^{*}[J_{bn}^{T}] = [J_{bn}^{T}]^{*}[J_{bn}^{T}][J_{bn}^{T}][J_{bn}^{T}]^{*}J_{2bn}^{T}$$

$$= J_{2bn}^{T}[J_{bn}^{T}][J_{bn}^{T}]^{*}[J_{bn}^{T}]^{*}[J_{bn}^{T}] = \left(J_{2bn}^{1} \cup J_{2bn}^{2}\right)^{T}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)$$

$$= \left(J_{2bn}^{1T} \cup J_{bn}^{2T}\right)\left(J_{bn}^{1T} \cup J_{bn}^{2T}\right)\left[J_{bn}^{1T} \cup J_{bn}^{2T}\right]^{*}\left[J_{bn}^{1T} \cup J_{bn}^{2T}\right]^{*}\left(J_{bn}^{1T} \cup J_{bn}^{2T}\right)$$

$$= \left(J_{2bn}^{1T} \cup J_{bn}^{2T}\right)\left(J_{bn}^{1T} \cup J_{bn}^{2T}\right)\left[\left(J_{bn}^{1T}\right)^{*} \cup \left(J_{bn}^{2T}\right)^{*}\right]\left[\left(J_{bn}^{1T}\right)^{*} \cup \left(J_{bn}^{2T}\right)^{*}\right]\left(J_{bn}^{1T} \cup J_{bn}^{2T}\right)$$

$$= \left[J_{2bn}^{1T}J_{bn}^{T}\left[J_{bn}^{1T}\right]^{*}\left[J_{bn}^{1T}\right]^{*}J_{bn}^{T}\right] \cup \left[J_{bn}^{2T}\left[J_{bn}^{2T}\right]^{*}\left[J_{bn}^{2T}\right]^{*}J_{bn}^{2T}J_{2bn}^{2T}\right]$$

$$= \left[J_{2bn}^{1T}J_{bn}^{T}\left[J_{bn}^{1*}\right]^{T}\left[J_{bn}^{1*}\right]^{T}J_{bn}^{T}\right] \cup \left[J_{bn}^{2T}\left[J_{bn}^{2*}\right]^{T}\left[J_{bn}^{2*}\right]^{T}J_{bn}^{2T}J_{2bn}^{2T}\right]$$

$$= \left[J_{2bn}^{1T}\left[J_{bn}^{1*}J_{bn}^{T}\right]^{T}\left[J_{bn}^{T}J_{bn}^{1*}\right]^{T}\right] \cup \left[\left[J_{bn}^{2}J_{bn}^{2*}\right]^{T}\left[J_{bn}^{2*}J_{bn}^{2T}\right]^{T}J_{2bn}^{2T}\right]$$

$$= \left[J_{2bn}^{1T}\left[J_{bn}^{1*}\right]^{T}J_{bn}^{T}J_{bn}^{T}\left[J_{bn}^{1*}\right]^{T}\right] \cup \left[\left[J_{bn}^{2*}\right]^{T}J_{bn}^{2}J_{bn}^{2}\left[J_{bn}^{2*}\right]^{T}J_{2bn}^{2T}\right]$$

$$= J_{2bn}^{1T}\left[\left(J_{bn}^{1*}\right)^{T} \cup \left(J_{bn}^{2*}\right)^{T}\right]\left(J_{bn}^{1T} \cup J_{bn}^{2T}\right)\left(J_{bn}^{1T} \cup J_{bn}^{2T}\right)\left(J_{2bn}^{1T} \cup J_{2bn}^{2T}\right)$$

$$= \left(J_{bn}^{1T} \cup J_{bn}^{2T}\right)^{*}\left(J_{bn}^{1T} \cup J_{bn}^{2T}\right)\left(J_{bn}^{1T} \cup J_{bn}^{2T}\right)\left(J_{2bn}^{1T} \cup J_{2bn}^{2T}\right)$$

$$= \left[J_{bn}^{1} \cup J_{bn}^{2}\right]^{*}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)^{T}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)^{T}\left[J_{bn}^{1} \cup J_{bn}^{2}\right]^{*}\left(J_{2bn}^{1T} \cup J_{2bn}^{2T}\right)$$

$$= [J_{bn}^{T}]^{*}[J_{bn}^{T}][J_{bn}^{T}][J_{bn}^{T}]^{*}J_{2bn}^{T}$$

Hence is a J_2 bi-normal matrix

Theorem 4.2. Let $J_{bn}^{*} \in J_{n*n}$and n satisfies the following conditions

(i) $J_{bn} = J_{2bn}^{*}$,

(ii) $J_{bn}J_{2bn}^{*} = J_{2bn}^{*}J_{bn}$,

then $J_{bn}$is an J_2 binormal matrix.

Proof.

Since $J_{bn} \in J_{2bn}^{n-1} \Rightarrow J_{2bn}^{*}J_{bn}^{n-1} = J_{bn}^{n-1} - J_{2bn}^{*}andJ_{bn} = J_{2bn}^{n-1}$, so,

$$J_{2bn}^{*}J_{bn}^{n-1}J_{bn}^{n-1}J_{2bn}^{*} = J_{bn}^{n-1}J_{2bn}^{*}J_{2bn}^{*}J_{bn}^{n-1}$$

$$J_{bn}J_{2bn}^{*}J_{bn}^{n-1}J_{bn}^{n-1}J_{bn}J_{2bn}^{*} = J_{bn}J_{bn}^{n-1}J_{2bn}^{*}J_{2bn}^{*}J_{bn}^{n-1}J_{bn}$$

$$J_{bn}J_{2bn}^{*}J_{bn}^{n-1}J_{bn}^{n-1}J_{bn}^{n-1}J_{bn}J_{2bn}^{*}J_{bn} = J_{bn}J_{2bn}^{*}J_{2bn}^{*}J_{bn}$$

From (2), we have $J_{2bn}^{*}J_{bn}^{n}J_{bn}^{n}J_{2bn}^{*} = J_{bn}^{n}J_{2bn}^{*}J_{2bn}^{*}J_{bn}^{n}$. Hence, $J_{2bn}^{*}$is an J_2 binormal matrix.

Corollary 4.1. Let $J_{2bn}^{*} \in J_{n*n}$and J_{bn}^{n-1}satisfy the condition $J_{bn} = J_{2bn}^{*}J_{bn}$, then $J_{2bn}^{*}$is an J_2 binormal matrix.

Theorem 4.3. Let $J_{2bn}^{*} \in J_{n*n}$, then $J^{2}$is and J_2 binormal matrix

Theorem 4.4. Let $J_{2bn}^{*} \in J_{n*n}$, then J_{bn}^{n} is and J_2 binormal matrix

Proof.

Since $J_{2bn}^{*} \in J_{n*n} \Rightarrow J_{2bn}^{*} J_{bn} = J_{bn} J_{2bn}^{*}$,so

$$J_{2bn}^{n*} J_{bn}^{n} J_{bn}^{n} J_{2bn}^{n*} = J_{2bn}^{*(n-1)} J_{2bn}^{*} J_{bn}^{n} J_{bn}^{(n-1)} J_{bn}^{n} J_{2bn}^{n*} J_{2bn}^{*(n-1)}$$

$$= J_{2bn}^{*(n-1)} J_{bn}^{n} J_{2bn}^{*} J_{bn}^{(n-2)} J_{bn}^{n} J_{2bn}^{n*} J_{2bn}^{*(n-1)}$$

$$= J_{bn}^{*(n-1)} J_{2bn}^{*(n-1)} J_{2bn}^{*} J_{bn}^{n} J_{bn}^{n} J_{2bn}^{*} J_{2bn}^{*(n-1)} J_{bn}^{(n-1)}$$

$$= J_{bn}^{*(n-1)} J_{2bn}^{*(n-2)} J_{2bn}^{2*} J_{bn}^{n} J_{bn}^{n} J_{2bn}^{2*} J_{2bn}^{*(n-2)} J_{bn}^{(n-1)}$$

$$= J_{bn}^{*(n-1)} J_{2bn}^{*(n-3)} J_{2bn}^{3*} J_{bn}^{3n} J_{2bn}^{3*} J_{bn}^{n} J_{2bn}^{*(n-3)} J_{bn}^{(n-1)}$$

By induction , we have $J_{2bn}^{n*} J_{bn}^{n} J_{bn}^{n} J_{2bn}^{n*} = J_{2bn}^{*(n-1)} J_{2bn}^{*} J_{bn}^{n} J_{bn}^{(n-1)} J_{bn}^{n} J_{2bn}^{n*} J_{2bn}^{*(n-1)}$

HenceJ_{bn}^{n} is an J_2 binormal matrix

Theorem 4.5. Let $J_{bn} \in J_{n*n}$, then $J_{bn} = -J_{2bn}^{*} J_{bn}$. J_{bn} is an J binormal matrix, if and only if$J_{bn} J_{2bn}^{*} J_{2bn}^{*} J_{bn} = J_{bn} J_{2bn}^{*} J_{2bn}^{*} J_{bn}$.

Proof.

Since $J_{bn} \in J_{n*n}$and$J_{bn} = -J_{2bn}^{*} J_{bn}$. we assume than$J_{bn} J_{2bn}^{*} J_{2bn}^{*} J_{bn} = J_{bn} J_{2bn}^{*} J_{2bn}^{*} J_{bn}$t,

Then

$$J_{2bn}^{*} J_{bn} J_{bn} J_{2bn}^{*} = J_{2bn}^{*} J_{bn} J_{bn} J_{2bn}^{*}$$

$$= J_{2bn}^{*}(-J_{bn}) J_{2bn}^{*}(-J_{2bn}^{*}) J_{bn} J_{2bn}^{*}$$

$$= J_{bn} J_{bn} J_{2bn}^{*} J_{2bn}^{*}(-J_{bn}) J_{2bn}^{*}$$

$$= J_{bn} J_{2bn}^{*} J_{2bn}^{*} J_{bn}$$

Conversely, suppose J_2 binormal matrix then

$$J_{2bn}^{*} J_{bn} J_{bn} J_{2bn}^{*} = J_{2bn}^{*} J_{bn} J_{bn} J_{2bn}^{*}$$

$$= J_{2bn}^{*}(-J_{bn}) J_{2bn}^{*}(-J_{2bn}^{*}) J_{bn} J_{2bn}^{*}$$

$$= J_{bn} J_{bn} J_{2bn}^{*} J_{2bn}^{*}(-J_{bn}) J_{2bn}^{*} J_{bn}$$

$$= J_{bn} J_{2bn}^{*} J_{2bn}^{*} J_{bn}$$

Theorem 4.6. Let $J_{bn} \in J_{n*n}$, then $J_{bn} = -J_{bn} J_{2bn}^{*} J_{bn}$. J_{bn}commutes with$J_{bn} J_{2bn}^{*} J_{bn} = J_{2bn}^{*} J_{bn} J_{bn}$, then$J_{2bn}$ is a secondary binormal matrix

Proof.

Since$J_{bn} J_{bn} J_{2bn}^{*} = -J_{2bn}^{*}$.

$$J_{bn} J_{2bn}^{*} J_{bn} = J_{bn} J_{bn} J_{bn} J_{2bn}^{*} J_{bn}$$

$$= (-J_{bn}) J_{2bn}^{*} (-J_{bn}) J_{bn} J_{bn}$$

$$= J_{bn} J_{bn} J_{2bn}^{*} J_{bn} J_{bn}$$

$$= J_{2bn}^{*} J_{bn} J_{bn}$$

Hence, J_{2bn} is a secondary binormal matrix.

Theorem 4.7.

If $J_{2bn}$is a secondary binormal matrix and Skew symmetric matrix $\alpha_{sk}$is a complex number, then

(i)$J_{bn} + \alpha_{sk} J_{2bn}$ is a secondary binormal matrix

(ii)$J_{bn} - \alpha_{sk} J_{2bn}$is a secondary binormal matrix

Proof

Let J_{2bn}be a secondary binormal matrix

Therefore of $J_{2bn} J_{bn} J_{bn}^{*} J_{bn}^{*} J_{bn} = J_{bn}^{*} J_{bn} J_{bn} J_{bn}^{*} J_{2bn}$

Proof of (i)

$$(J_{2bn} + \gamma^{ssm} J_n)(J_{2bn} + \gamma^{ssm} J_n)^{*}$$

$$J_{2bn}(J_{2bn} + \gamma^{ssm} J_n)(J_{2bn} + \gamma^{ssm} J_{bn})^{*}(J_{2bn} + \gamma^{ssm} J_n)^{*}(J_{2bn} + \gamma^{ssm} J_n)$$
$$= (J_{2bn} + \gamma^{ssm} J_n)^{*}(J_{2bn} + \gamma^{ssm} J_{bn})(J_{2bn} + \gamma^{ssm} J_n)(J_{2bn} + \gamma^{ssm} J_n)^{*} J_{2bn}$$

$$J_{2bn}(J_{2bn} + \gamma^{ssm} J_n)(J_{2bn} + \gamma^{ssm} J_n)^{*}(J_{2bn} + \gamma^{ssm} J_n)^{*}(J_{2bn} + \gamma^{ssm} J_n)$$
$$= \left(J_{2bn}^{1} \cup J_{2bn}^{2}\right)\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}(J_n^{1} \cup J_n^{2})\right]\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}(J_n^{1} \cup J_n^{2})\right]^{*}$$
$$\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}(J_n^{1} \cup J_n^{2})\right]\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}(J_n^{1} \cup J_n^{2})\right]$$

$$= \left(J_{2bn}^{1} \cup J_{2bn}^{2}\right)\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}(J_n^{1} \cup J_n^{2})\right]\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}(J_n^{1} \cup J_n^{2})\right]^{*}$$
$$\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}(J_n^{1} \cup J_n^{2})\right]^{*}\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}(J_n^{1} \cup J_n^{2})\right]$$

$$= \left(J_{2bn}^{1} \cup J_{2bn}^{2}\right)\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}(J_n^{1} \cup J_n^{2})\right]\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}(J_n^{1} \cup J_n^{2})\right]^{*}$$
$$\left[\left(J_{bn}^{1} + \gamma^{ssm} J_n^{1}\right) \cup \left(J_{bn}^{2} + \gamma^{ssm} J_n^{2}\right)\right]^{*}\left[\left(J_{bn}^{1} + \gamma^{ssm} J_n^{1}\right) \cup \left(J_{bn}^{2} + \gamma^{ssm} J_n^{2}\right)\right]$$

$$= \left(J_{2bn}^{1} \cup J_{2bn}^{2}\right)\left[\left(J_{bn}^{1} + \gamma^{ssm} J_n^{1}\right) \cup \left(J_{bn}^{2} + \gamma^{ssm} J_n^{2}\right)\right]\left[\left(J_{bn}^{1} + \gamma^{ssm} J_n^{1}\right)^{*} \cup \left(J_{bn}^{2} + \gamma^{ssm} J_n^{2}\right)^{*}\right]$$
$$\left[\left(J_{bn}^{1} + \gamma^{ssm} J_n^{1}\right)^{*} \cup \left(J_{bn}^{2} + \alpha_{sk} J_n^{2}\right)^{*}\right]\left[\left(J_{bn}^{1} + \gamma^{ssm} J_n^{1}\right) \cup \left(J_{bn}^{2} + \gamma^{ssm} J_n^{2}\right)\right]$$

$$= \left(J_{2bn}^{1} \cup J_{2bn}^{2}\right)\left[\left(J_{bn}^{1} + \gamma^{ssm} J_n^{1}\right) \cup \left(J_{bn}^{2} + \gamma^{ssm} J_n^{2}\right)\right]\left[\left(J_{bn}^{*1} + \bar{\gamma}^{ssm} J_n^{1}\right) \cup \left(J_{bn}^{*2} + \bar{\gamma}^{ssm} J_n^{2}\right)\right]$$
$$\left[\left(J_{bn}^{*1} + \bar{\gamma}^{ssm} J_n^{1}\right) \cup \left(J_{bn}^{*2} + \gamma^{ssm} J_n^{2}\right)\right]\left[\left(J_{bn}^{1} + \gamma^{ssm} J_n^{1}\right) \cup \left(J_{bn}^{2} + \gamma^{ssm} J_n^{2}\right)\right]$$

$$= J_{2bn}^{1}\left[J_{bn}^{1} J_{bn}^{*1} + J_{bn}^{1} J_{bn}^{*1} + \bar{\gamma}^{ssm} J_n + \alpha_{sk}\bar{\alpha}_{sk}\right]\left[J_{bn}^{1} J_{bn}^{*1} + \bar{\gamma}^{ssm} J_{bn}^{1} + \gamma^{ssm} J_{bn}^{1} + \gamma^{ssm}\bar{\gamma}^{ssm}\right]$$
$$\cup \left[J_{bn}^{2} J_{bn}^{*2} + J_{bn}^{2} J_{bn}^{*2} + \bar{\gamma}^{ssm} J_n + \alpha_{sk}\bar{\alpha}_{sk}\right]\left[J_{bn}^{2} J_{bn}^{*2} + \bar{\gamma}^{ssm} J_{bn}^{2} + \gamma^{ssm} J_{bn}^{2} + \gamma^{ssm}\bar{\gamma}^{ssm}\right] J_{2bn}$$

$$= \left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \left(\gamma^{ssm} J_{bn}^{1} \cup \gamma^{ssm} J_{bn}^{2}\right)\right]^{*}\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \left(\gamma^{ssm} J_{bn}^{1} \cup \alpha_{sk} J_{bn}^{2}\right)\right]$$
$$\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \left(\gamma^{ssm} J_{bn}^{1} \cup \gamma^{ssm} J_{bn}^{2}\right)\right]\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \left(\gamma^{ssm} J_{bn}^{1} \cup \gamma^{ssm} J_{bn}^{2}\right)\right]^{*}\left(J_{2bn}^{1} \cup J_{2bn}^{2}\right)$$

$$= \left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)\right]^{*}\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)\right]$$
$$\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)\right]\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)\right]^{*}\left(J_{2bn}^{1} \cup J_{2bn}^{2}\right)$$

$$= \left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)\right]^{*}\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)\right]$$

$$\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)\right]\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) + \gamma^{ssm}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)\right]^{*}\left(J_{2bn}^{1} \cup J_{2bn}^{2}\right)$$
$$= (J_{bn} + \gamma^{ssm}J_{n})^{*}(J_{bn} + \gamma^{ssm}J_{n})(J_{bn} + \gamma^{ssm}J_{n})(J_{bn} + \gamma^{ssm}J_{n})^{*}J_{2bn}$$

Hence, $J_{bn} + \gamma^{ssm}J_{2bn}$is secondary $\gamma^{ssm}J_{2bn}$skew symmetric matrix

Proof of (ii)

$$(J_{2bn} - \gamma^{ssm}J_{n})(J_{2bn} - \gamma^{ssm}J_{n})^{*}$$

$$J_{2bn}(J_{2bn} - \gamma^{ssm}J_{n})(J_{2bn} - \gamma^{ssm}J_{bn})^{*}(J_{2bn} - \gamma^{ssm}J_{n})^{*}(J_{2bn} - \gamma^{ssm}J_{n})$$
$$= (J_{2bn} - \gamma^{ssm}J_{n})^{*}(J_{2bn} - \gamma^{ssm}J_{bn})(J_{2bn} - \gamma^{ssm}J_{n})(J_{2bn} - \gamma^{ssm}J_{n})^{*}J_{2bn}$$

$$J_{2bn}(J_{2bn} - \gamma^{ssm}J_{n})(J_{2bn} - \gamma^{ssm}J_{n})^{*}(J_{2bn} - \gamma^{ssm}J_{n})^{*}(J_{2bn} - \gamma^{ssm}J_{n})$$
$$= \left(J_{2bn}^{1} \cup J_{2bn}^{2}\right)\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \gamma^{ssm}(J_{n}^{1} \cup J_{n}^{2})\right]\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \gamma^{ssm}(J_{n}^{1} \cup J_{n}^{2})\right]^{*}$$
$$\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \gamma^{ssm}(J_{n}^{1} \cup J_{n}^{2})\right]\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \gamma^{ssm}(J_{n}^{1} \cup J_{n}^{2})\right]$$

$$= \left(J_{2bn}^{1} \cup J_{2bn}^{2}\right)\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \gamma^{ssm}(J_{n}^{1} \cup J_{n}^{2})\right]\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \gamma^{ssm}(J_{n}^{1} \cup J_{n}^{2})\right]^{*}$$
$$\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \gamma^{ssm}(J_{n}^{1} \cup J_{n}^{2})\right]^{*}\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \gamma^{ssm}(J_{n}^{1} \cup J_{n}^{2})\right]$$

$$= \left(J_{2bn}^{1} \cup J_{2bn}^{2}\right)\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \alpha_{sk}(J_{n}^{1} \cup J_{n}^{2})\right]\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \gamma^{ssm}(J_{n}^{1} \cup J_{n}^{2})\right]^{*}$$
$$\left[\left(J_{bn}^{1} - \gamma^{ssm}J_{n}^{1}\right) \cup \left(J_{bn}^{2} - \gamma^{ssm}J_{n}^{2}\right)\right]^{*}\left[\left(J_{bn}^{1} - \gamma^{ssm}J_{n}^{1}\right) \cup \left(J_{bn}^{2} - \gamma^{ssm}J_{n}^{2}\right)\right]$$

$$= \left(J_{2bn}^{1} \cup J_{2bn}^{2}\right)\left[\left(J_{bn}^{1} - \gamma^{ssm}J_{n}^{1}\right) \cup \left(J_{bn}^{2} - \gamma^{ssm}J_{n}^{2}\right)\right]\left[\left(J_{bn}^{1} - \gamma^{ssm}J_{n}^{1}\right)^{*} \cup \left(J_{bn}^{2} - \gamma^{ssm}J_{n}^{2}\right)^{*}\right]$$
$$\left[\left(J_{bn}^{1} - \gamma^{ssm}J_{n}^{1}\right)^{*} \cup \left(J_{bn}^{2} - \gamma^{ssm}J_{n}^{2}\right)^{*}\right]\left[\left(J_{bn}^{1} - \gamma^{ssm}J_{n}^{1}\right) \cup \left(J_{bn}^{2} - \gamma^{ssm}J_{n}^{2}\right)\right]$$

$$= \left(J_{2bn}^{1} \cup J_{2bn}^{2}\right)\left[\left(J_{bn}^{1} - \gamma^{ssm}J_{n}^{1}\right) \cup \left(J_{bn}^{2} - \gamma^{ssm}J_{n}^{2}\right)\right]\left[\left(J_{bn}^{*1} - \bar{\gamma}^{ssm}J_{n}^{1}\right) \cup \left(J_{bn}^{*2} - \bar{\gamma}^{ssm}J_{n}^{2}\right)\right]$$
$$\left[\left(J_{bn}^{*1} - \bar{\gamma}^{ssm}J_{n}^{1}\right) \cup \left(J_{bn}^{*2} + \bar{\gamma}^{ssm}J_{n}^{2}\right)\right]\left[\left(J_{bn}^{1} - \gamma^{ssm}J_{n}^{1}\right) \cup \left(J_{bn}^{2} - \gamma^{ssm}J_{n}^{2}\right)\right]$$

$$= J_{2bn}^{1}\left[J_{bn}^{1}J_{bn}^{*1} - J_{bn}^{1}J_{bn}^{*1} - \bar{\gamma}^{ssm}J_{n} - \alpha_{sk}\bar{\alpha}_{sk}\right]\left[J_{bn}^{1}J_{bn}^{*1} - \bar{\gamma}^{ssm}J_{bn}^{1} - \gamma^{ssm}J_{bn}^{1} - \gamma^{ssm}\bar{\gamma}^{ssm}\right]$$
$$\cup \left[J_{bn}^{2}J_{bn}^{*2} - J_{bn}^{2}J_{bn}^{*2} - \bar{\gamma}^{ssm}J_{n} - \alpha_{sk}\bar{\alpha}_{sk}\right]\left[J_{bn}^{2}J_{bn}^{*2} - \bar{\gamma}^{ssm}J_{bn}^{2} - \bar{\gamma}^{ssm}J_{bn}^{2} - \gamma^{ssm}\bar{\gamma}^{ssm}\right]J_{2bn}$$

$$= \left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \left(\gamma^{ssm}J_{bn}^{1} \cup \gamma^{ssm}J_{bn}^{2}\right)\right]^{*}\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \left(\gamma^{ssm}J_{bn}^{1} \cup \gamma^{ssm}J_{bn}^{2}\right)\right]$$
$$\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \left(\gamma^{ssm}J_{bn}^{1} \cup \gamma^{ssm}J_{bn}^{2}\right)\right]\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \left(\gamma^{ssm}J_{bn}^{1} \cup \gamma^{ssm}J_{bn}^{2}\right)\right]^{*}\left(J_{2bn}^{1} \cup J_{2bn}^{2}\right)$$

$$= \left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \gamma^{ssm}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)\right]^{*}\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \gamma^{ssm}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)\right]$$
$$\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \gamma^{ssm}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)\right]\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \gamma^{ssm}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)\right]^{*}\left(J_{2bn}^{1} \cup J_{2bn}^{2}\right)$$

$$= \left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \gamma^{ssm}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)\right]^{*}\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \gamma^{ssm}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)\right]$$
$$\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \gamma^{ssm}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)\right]\left[\left(J_{bn}^{1} \cup J_{bn}^{2}\right) - \gamma^{ssm}\left(J_{bn}^{1} \cup J_{bn}^{2}\right)\right]^{*}\left(J_{2bn}^{1} \cup J_{2bn}^{2}\right)$$
$$= (J_{bn} - \gamma^{ssm}J_{n})^{*}(J_{bn} - \gamma^{ssm}J_{n})(J_{bn} - \gamma^{ssm}J_{n})(J_{bn} - \gamma^{ssm}J_{n})^{*}J_{2bn}$$

Hence, $J_{bn} - \gamma^{ssm}J_{2bn}$is secondary $\gamma^{ssm}J_{2bn}$skew symmetric matrix

5. Conclusion

In summary, this study has concentrated on unraveling the intricate interplay among secondary J_{2bn}binormal, J_{bn}binormal, andJ_{n} normal matrices, offering insights into their distinctive properties.

The investigation has notably highlighted the non-closure under addition for secondary J_{2bn}binormal matrices, prompting considerations for mathematical operations involving these entities. Looking ahead, future work in this domain could explore avenues for extending these findings to practical applications, such as in mathematical modeling or algorithm development. Additionally, further research may delve into the development of methodologies to address the non-closure issue, enhancing the utility of secondary J_{2bn}binormal matrices in various computational contexts. This study sets the stage for future endeavors aimed at both theoretical advancements and practical implementations in the broader spectrum of matrix theory.

References

[1] Theocharis, Achilleas D., Dimitra Manou, and Nikos K. Karamanos. "The extracellular matrix as a multitasking player in disease." The FEBS journal 286.15 (2019): 2830-2869.

[2] Morone, Flaviano. "Clustering matrices through optimal permutations." Journal of Physics: Complexity 3.3 (2022): 035007.

[3] Cui, Shuhao, et al. "Towards discriminability and diversity: Batch nuclear-norm maximization under label insufficient situations." Proceedings of the IEEE/CVF conference on computer vision and pattern recognition. (2020).

[4] Qian, Lilong, et al. "A matrix inequality for entanglement distillation problem." Linear Algebra and its Applications 616 (2021): 139-177.

[5] Ikramov, Khakim Dododzhanovich. "Binormal matrices." Zapiski Nauchnykh Seminarov POMI 463 (2017): 132-141.

[6] Holleman, Caleb, Thaddeus McClatchey, and Derek Thompson. "Binormal, Complex Symmetric Operators." arXiv preprint arXiv:1705.04882 (2017).

[7] Rasimi, Krutan. "A NOTE ON A RESULT ABOUT SKEW BINORMAL OPERATORS." Journal of Natural Sciences and Mathematics of UT 5.9-10 (2020): 130-133.

[8] Samanci, Hatice Kuşak, and Hüseyin Kocayiğit. N-Bishop Darboux vector of the spacelike curve with spacelike binormal. Infinite Study, 2019.

[9] Weeks, Grace, Hallie Kaiser, and Katy O'Malley. "Normality Properties of Composition Operators." (2021).

[10] Gu, Caixing, et al. "Binormal Toeplitz operators on the Hardy space." International Journal of Mathematics 30.01 (2019): 1950001.

[11] Babadag, F. "Generalized Quaternions: Further Contributions to a Matrix Approach and Frenet Serret Formulas." International Journal of Innovation in Science and Mathematics 2.3 (2016): 304-306.

[12] Speirs, D. C., et al. "A high-k mm-wave scattering diagnostic for measuring binormal wavenumber electron scale turbulence on MAST-U." 2022 15th UK-Europe-China Workshop on Millimetre-Waves and Terahertz Technologies (UCMMT). IEEE, 2022.

[13] Kaliukh, Yu I., and A. Ye Vusatiuk. "Factorization in problems of control and dynamics of lengthy systems." Cybernetics and Systems Analysis 55 (2019): 274-283.

[14] Yamaguchi, Tomoya, et al. "Real-time rendering of layered materials with anisotropic normal distributions." SIGGRAPH Asia 2019 Technical Briefs. 2019. 87-90.

[15] Speirs, D. C., et al. "A high-k mm-wave scattering diagnostic for measuring binormal electron scale turbulence on MAST-U." (2021): 533-536.

[16] Beckermann, Bernhard, and Alex Townsend. "Bounds on the singular values of matrices with displacement structure." SIAM Review 61.2 (2019): 319-344.

[17] Ikramov, Kh D. "On Normal and Binormal Matrices." Moscow University Computational Mathematics and Cybernetics 47.1 (2023): 19-22.

[18] Ikramov, Kh D. "Matrices with Pairwise Orthogonal Rows and Columns." Journal of Mathematical Sciences 262.1 (2022): 27-31.

[19] Thompson, Derek, Thaddeus McClatchey, and Caleb Holleman. "Binormal, complex symmetric operators." Linear and Multilinear Algebra 69.9 (2021): 1705-1715.

[20] Meenambika, K., et al. "The class of n-skew n-binormal operator." IOP Conference Series: Materials Science and Engineering. Vol. 1145. No. 1. IOP Publishing, 2021.
[21] Kaiser, Hallie, Katy O'Malley, and Grace Weeks. "Normality Properties of Composition Matricies." (2022).
[22] Hedenmalm, Haakan, and Aron Wennman. "Planar orthogonal polynomials and boundary universality in the random normal matrix model." arXiv preprint arXiv:1710.06493 (2017).
[23] Ding, Chao, et al. "Spectral operators of matrices: semismoothness and characterizations of the generalized Jacobian." SIAM Journal on Optimization 30.1 (2020): 630-659.
[24] Wang, Guanhua, and Jeffrey A. Fessler. "Efficient approximation of Jacobian matrices involving a non-uniform fast Fourier transform (NUFFT)." IEEE Transactions on Computational Imaging 9 (2023): 43-54.
[25] Djordjević, Slaviša, Jaewoong Kim, and Jasang Yoon. "Generalized spherical Aluthge transforms and binormality for commuting pairs of operators." Mathematische Nachrichten (2023).