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Optimizing Subdivisions on Fair Domination in Petersen Graph Structures

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Abstract:

A *fair dominating set (FDS)* J is a dominating set of a graph H, such that all vertices in $V \setminus J$ are dominated by the same number of vertices of J. The *fair domination number (FDN)* is the minimum cardinality of a fair dominating set of H, denoted by $\gamma_{fd}(H)$. The *domination subdivision number (FDSN)* $Sd_{\gamma}(H)$, represents the smallest number of edges in H that needs to be subdivided (with each edge being subdivided at most once) to increase the domination number of the graph. The fair domination subdivision number, $Sd^{+}_{\gamma_{fd}}(H)$ (or $Sd^{-}_{\gamma_{fd}}(H)$) is the minimum number of edge subdivisions needed to be applied to the graph H to increase (or decrease) the fair domination number of the graph. In this paper, we present the fair domination subdivision numbers for Petersen graphs $\mathrm{P}(n,k)$ for $k = 1,2$, revealing the structural impact of edge subdivisions on these graph families.

Introduction: Leonhard Euler's solution to the Seven Bridges of Königsberg problem in 1736 laid the foundation for graph theory, now a major branch of mathematics. Domination in graphs, introduced by Haynes, studies subsets of vertices that can control a graph's behavior. A *dominating set* (DS) ensures every vertex outside the set is adjacent to at least one vertex in it, with number of elements in the smallest such set called the domination number $\gamma(H)$. Caro et al. introduced "*fair domination*," where a "fair dominating set (FDS)" J is a DS of a graph H, such that all vertices in $V \setminus J$ are dominated by the same number of vertices from J. The minimum size of an FDS is the fair domination number $\gamma_{fd}(H)$. Arumugam and Joseph extended domination concepts with the *subdivision number* $Sd_{\gamma}(H)$, the minimum edge subdivisions needed to alter a graph's domination properties. Similarly, the *fair domination subdivision number* $Sd^{+}_{\gamma_{fd}}(H)$ (or $Sd^{-}_{\gamma_{fd}}(H)$) tracks subdivisions to increase or decrease $\gamma_{fd}(H)$. These concepts have significant theoretical and practical applications, especially in network design and algorithm development.

Objectives: The Fair Domination Subdivision Number (FDSN**)** is a graph theory parameter that examines domination properties by assessing how efficiently a set of vertices can exert control over an entire graph. This metric introduces a novel twist by allowing strategic subdivision of specific edges, enabling adjustments to the network structure. Its objective is to highlight the minimum modifications required to enhance the influence of certain nodes while ensuring equitable information spread. In practical applications, FDSN provides valuable insights into social networks and telecommunications,

optimizing resource distribution and ensuring consistent service quality across regions. The fair domination criterion requires that each vertex not in the dominating set be adjacent to exactly one vertex in the set, promoting balanced influence. This unique approach balances control and influence within a graph through structural modifications, making it a useful tool for network design and decision-making. Determining FDSN introduces new challenges and dimensions to traditional graph theory, emphasizing fairness, efficiency, and reliability.

Results: Fair domination, initially introduced by Caro et al., has garnered growing interest from researchers. In this paper, we present new insights into the fair domination subdivision number, and useful new theorems with proof.

Conclusions: The fair domination subdivision number, a key concept in graph theory, has important implications for optimizing network design. By identifying the minimum number of edge subdivisions needed for a fair domination set, this parameter provides insights into the resilience and efficiency of communication networks, social systems, and other graph-modelled structures. Understanding this metric enables the formulation of strategies to bolster network robustness, decrease susceptibility to failures, and enhance resource distribution, ultimately supporting the development of more resilient and equitable network architectures.

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Keywords: Domination, Fair domination, k-fair domination, Subdivision number, Fair domination subdivision number

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1. Introduction

Mathematician Leonhard Euler's contributions to the 18th century are attributed with the development of graph theory. The Seven Bridges of Konigsberg problem, which Euler solved in 1736, is often cited as the starting point of graph theory. Euler's solution laid the foundation for the study of networks and led to the creation of graph theory which is most studied branch of mathematics. The concept of domination in graph theory began to emerge in the mid- 20th century as mathematicians started studying various types of vertex subsets that could influence or control the behaviour of a graph. Haynes [11] introduced one of the most basic and extensively studied concepts in graph theory known as domination in graphs. A *dominating set (DS)* of a graph H is a set J which is a subset of the vertex set $V\ (H)$ such that $\forall\ y\ \in\ V \setminus J$ is adjacent to at least one vertex $x\ \in\ J$. The smallest possible size of a DS in H is called the domination number $\gamma(H)$ and a DS of this minimum size is referred to as a γ -set of H. Extensive research is currently being conducted on various aspects and extensions of domination in graph theory [4, 9, 12]. A "*fair domination*" is one of the interesting parameters in the variety of domination which was introduced by Caro et al [2]. A "fair dominating set (FDS)" J is a DS of a graph H, such that all vertices in $V \setminus J$ are dominated by the same number of vertices from J. The "*fair domination number (FDN)*" is the minimum cardinality of a FDS of H, denoted by $\gamma_{fd}(H)$. It aims to minimize the disparity in the number of vertices dominated by each vertex in the set. A "*m-fair dominating set*" abbreviated mFD-set, is a DS such that $|N(x)\ \cap\ J|\ =\ m, \forall x\ \in\ V \setminus J$. A "*perfect* *dominating set*" in H is a DS such that every vertex not in the set J is adjacent to exactly one vertex in J. Hence 1FD - set is precisely a "*perfect dominating set*" which we can see in the literature [2, 3, 10]. The upper bounds (UB) and the lower bounds (LB) of the γ_{fd} are discussed in [2]. The FDN for various graphs are found in [2] and other general results and bounds of γ_{fd} can be found in [6, 7, 8, 15]. The "*domination subdivision number (DSN)* denoted by $Sd_{\gamma}(H)$, represents the smallest number of edges in graph H that need to be subdivided (with each edge being subdivided at most once) to increase the γ - set of the graph." "S. Arumugam and J. Paulraj Joseph" [1] initially introduced the concept of the $Sd_{\gamma}(H)$ for a graph H. They demonstrated that for any tree T with at least three vertices, $Sd_{\gamma}(H) \leq 3$. The "*fair domination subdivision number (FDSN)*", denoted as $Sd^{+}_{\gamma_{fd}}(H)$ (or $Sd^{-}_{\gamma_{fd}}(H)$) is the minimum number of edge subdivisions needed to be applied to the graph H to increase (or decrease) the γ_{fd} of the graph. Trees are fundamental structures in graph theory and are used in various algorithms and data structures due to their simple yet powerful properties. They have applications in computer science, data analysis, network design, and many other fields.

2. Objectives

The Fair Domination Subdivision Number (FDSN) is a graph theory parameter that examines domination properties by assessing how efficiently a set of vertices can exert control over the entire graph. This concept adds a unique twist, specific edges in the graph can be subdivided, allowing for strategic adjustments in network structure. In social networks or online platforms, this metric offers insights into influence dynamics, highlighting the minimum modifications needed to boost the reach of certain nodes or users while ensuring equitable information spread. The FDSN is especially useful for telecommunications network design, such as fiberoptic or wireless systems, where fair resource distribution and coverage are critical. By identifying the least number of additional connections or infrastructure improvements needed, the FDSN aids in delivering consistent service quality across regions. This parameter is a valuable tool for optimizing system design, resource allocation, and decision-making, promoting fairness, efficiency, and reliability across various practical applications. Domination problems in graph theory focus on ensuring that every vertex is adjacent to at least one vertex in the dominating set. The fair domination criterion introduces an additional layer by requiring that each vertex not in the dominating set be adjacent to exactly one vertex in the set, promoting a more balanced influence. The novelty of the fair domination subdivision number lies in its unique approach to balancing influence and control within a graph through structural modifications. The problem of determining the fair domination subdivision number introduces new mathematical challenges and it gives several new dimensions to traditional graph theory and domination problems, with potential applications in network design, resource allocation. Developing efficient algorithms to find this number for various types of graphs can lead to advancements in computational graph theory.

3. Notations

Let $H = (V, E)$ be a connected, simple graph with order $|V| = n$. We use Harary [11] for graph theoretic notation. For any vertex $v \in V$, the open neighbourhood of v is the set $N_H(v) = \{u \in V : uv \in E\}$ and the closed neighbourhood is the set $N_H[v] = N_H(v) \cup \{v\}$. For a set $J \subseteq V$, the open neighbourhood of J is $N_H(J) = \cup_{v \in J} N_H(v)$, the closed neighbourhood of J is $N_H[J] = N_H(J) \cup J$ and the private neighbourhood $pn(v, J)$ of a vertex $u \in J$ is defined by $pn(v, J) = \{u \in$

$V \setminus J : N_H(u) \cap J = \{v\}\}$. A *path* is a sequence of vertices connected by edges, where all vertices are distinct except possibly the first and last. A *cycle* is a special type of path that begins and ends at the same vertex. A complete graph is a simple, undirected graph in which every possible edge is present and denoted by K_n. "A graph H is k-partite, $k \geq 1$ if it is possible to partition $V(H)$ into k subsets. $V_1, V_2, V_3, \dots, V_k$. (called partite set) such that every element of $E(H)$ joins a vertex of V_i to a vertex of V_j, $i \neq j$. If H is a 1-partite graph of order n, then $H = \overline{K_n}$. For $k = 2$, such graphs are called *bipartite graphs*". "A vertex with a degree of one in a graph is called a *pendant vertex* (a leaf). A vertex v in a graph H is called a *strong support vertex* if it is adjacent to at least two leaves. If a vertex v in a graph H is adjacent to exactly one leaf, then it is called a *weak support vertex*". The distance between the vertices u and v of a graph H is denoted as $d(u,v)$ and the distance between the edges e_1 and e_2 of H is denoted by $d(e_1, e_2)$.

4. Main Results

Fair domination, initially introduced by Caro et al., has garnered growing interest from researchers. In this paper, we present new insights into the fair domination subdivision number, with the theorems cited here playing a critical role in advancing our research.

Theorem 4.1. [5] For Petersen graph $P(n,1)$,

$$\gamma(P(n,1)) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & n \equiv 0,1,3 \ (mod\ 4) \\ \left\lceil \frac{n}{2} \right\rceil + 1, & n \equiv 2 \ (mod\ 4) \end{cases}$$

Theorem 4.2. [5] For Petersen graph $P(n,2), n \geq 5$ of order $2n$, $\gamma(P(n,2)) = \left\lceil \frac{3n}{5} \right\rceil$

Theorem 4.3. [14] For Petersen graph $P(n,1), n \geq 4$ of order $2n$,

$$\gamma_{fd}(P(n,1)) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & n \equiv 0,1, (mod\ 4) \\ \left\lceil \frac{n}{2} \right\rceil + 1, & n \equiv 2,3 \ (mod\ 4) \end{cases}$$

Theorem 4.4. [14] For Petersen graph $P(n,2),\ n \geq 5$ of order $2n$,

$$\gamma_{fd}(P(n,2)) = \begin{cases} \left\lceil \frac{3n}{5} \right\rceil, & n = 7 \\ 2\left\lceil \frac{n}{3} \right\rceil, & Otherwise \end{cases}$$

Theorem 4.5. [13] For any graph H, $\gamma(H) = \gamma_{fd}(H)$ then $Sd^{-}_{\gamma_{fd}}(H) = 0$

Theorem 4.6. [13] For any graph H, $\gamma(H) < \gamma_{fd}(H)$ if and only if $Sd^{-}_{\gamma_{fd}}(H) \geq 1$

Theorem 4.7. For Petersen graph $P(n,1), n \geq 4$ of order $2n$,

$$Sd^{+}_{\gamma_{fd}}(P(n,1)) = 1$$

and

$$Sd^{-}_{\gamma_{fd}}(P(n,1)) = \begin{cases} 2, & n \equiv 3 \pmod 4 \\ 0, & Otherwise \end{cases}$$

Proof: Let $H \cong P(n,1)$ and C' and C'' be inner and outer cycles of H respectively. Let $V(C') = \{u_1, u_2, u_3 \ldots, u_n\}$ and $V(C'') = \{v, v_2, v_3 \ldots, v_n\}$. By theorem (4.1) and theorem (4.3),

$$\gamma(H) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & n \equiv 0,1,3 \pmod 4 \\ \left\lceil \frac{n}{2} \right\rceil + 1, & n \equiv 2 \pmod 4 \end{cases}$$

and

$$\gamma_{fd}(H) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & n \equiv 0,1, \pmod 4 \\ \left\lceil \frac{n}{2} \right\rceil + 1, & n \equiv 2,3 \pmod 4 \end{cases}$$

Let J and J' be the γ and γ_{fd} sets of H respectively and defined as follows,

$$J = \begin{cases} \left\{u_{4k-3}: 1 \le k \le \left\lceil \frac{n}{4} \right\rceil\right\} \cup \left\{v_{4k-1}: 1 \le k \le \left\lceil \frac{n}{4} \right\rceil\right\} \cup \{u_n\}, & n \equiv 2 \pmod 4 \\ \left\{u_{4k-3}: 1 \le k \le \left\lceil \frac{n}{4} \right\rceil\right\} \cup \left\{v_{4k-1}: 1 \le k \le \left\lceil \frac{n}{4} \right\rceil\right\}, & Otherwise \end{cases}$$

$$J' = \begin{cases} \left\{u_{4k-3}: 1 \le k \le \left\lceil \frac{n}{4} \right\rceil\right\} \cup \left\{v_{4k-1}: 1 \le k \le \left\lceil \frac{n}{4} \right\rceil\right\}, & n \equiv 0,1 \pmod 4 \\ \left\{u_{4k-3}: 1 \le k \le \left\lceil \frac{n}{4} \right\rceil\right\} \cup \left\{v_{4k-1}: 1 \le k \le \left\lceil \frac{n}{4} \right\rceil\right\} \cup \{u_n\}, & n \equiv 2 \pmod 4 \\ \left\{u_{4k-3}: 1 \le k \le \left\lceil \frac{n}{4} \right\rceil\right\} \cup \left\{v_{4k-1}: 1 \le k \le \left\lfloor \frac{n}{4} \right\rfloor\right\} \cup \{u_{n-1}\} \cup \{u_n\}, & n \equiv 3 \pmod 4 \end{cases}$$

Let H_1 be the graph obtained by subdividing the edges of $E(H)$ by the subdivision vertices $\{x_1, x_2, x_3 \ldots, x_n\}$ and J_1 be the γ_{fd}-set of H_1. Note that J' is an 1FD-set of H. Now H_1 is obtained by subdividing an edge $u_1u_n \in E(H)$ (or $v_1v_n \in E(H)$) by the subdivision vertex x_1 is dominated by both u_1 and u_n of J'. Hence reconstructing the γ_{fd} - set of H_1 as follows

$$J_1 = J' \cup x_1,$$

$$|J_1| > |J'|$$

This implies that $\gamma_{fd}(H) < \gamma_{fd}(H_1)$. Hence $Sd^{+}_{\gamma_{fd}}(H) = 1$. Now to determine $Sd^{-}_{\gamma_{fd}}(H)$, consider the following cases,

$Case\ (i)\ n \equiv 3 \pmod 4$

In this case, we clearly notice that $\gamma(H) < \gamma_{fd}(H))$ and by the theorem (4.5), $Sd^{-}_{\gamma_{fd}}(H) > 0$. Now by the above argument we see that, $Sd^{-}_{\gamma_{fd}}(H) > 1$. In J, v_n and u_{n-1} are dominated by both u_1 and v_n, where $u_1, v_n \in J$ and all other vertices are dominated by exactly one vertex of J. Let H_1 be obtained by subdividing the edges u_nu_1 and v_nv_1 by the subdivision vertices x_1, x_2 respectively, clearly J is 1FD set of H_1, then $J_1 = J$, $|J| < |J'|$, hence $\gamma_{fd}(H) > \gamma_{fd}(H_1)$. Therefore $Sd^{-}_{\gamma_{fd}}(H) = 2$.

$Case\ (ii)\ n \not\equiv 3\ (mod\ 4)$

In this case $\gamma(H) = \gamma_{fd}(H)$ and by the theorem (4.5), $Sd^{-}_{\gamma_{fd}}(H) = 0$.

Theorem 4.8. For Petersen graph $P(n,2), n \geq 5$ of order $2n$,

$$Sd^{+}_{\gamma_{fd}}(P(n,2)) = 1$$

and

$$Sd^{-}_{\gamma_{fd}}(P(n,2)) = \begin{cases} 2\left\lceil \frac{n}{5} \right\rceil, & n \equiv 0,1,2,3\ (mod\ 5) \\ 2(\left\lceil \frac{n}{5} \right\rceil + 1), & n \equiv 4\ (mod\ 5) \end{cases}$$

Proof: Let $H \cong P(n,2)$ and C' and C'' be inner and outer cycles of H respectively. Let $V(C') = \{u_1, u_2, u_3 \dots, u_n\}$ and $V(C'') = \{v, v_2, v_3 \dots, v_n\}$. By theorems (4.2) and theorem (4.4) we have

$$\gamma(H) = \left\lceil \frac{3n}{5} \right\rceil$$

$$\gamma_{fd}\left(P(n,2)\right) = \begin{cases} \left\lceil \frac{3n}{5} \right\rceil, & n = 7 \\ 2\left\lceil \frac{n}{3} \right\rceil, & Otherwise \end{cases}$$

Let J and J' be the γ and γ_{fd} sets of H respectively and defined as

$$J = \begin{cases} \left\{u_{5k-3}: 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil\right\} \cup \left\{v_{5k-4} \cup v_{5k-2}: 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil\right\}, & n \equiv 0, 2, 4\ (mod\ 5) \\ \left\{u_{5k-3} \cup u_{n-1}: 1 \leq k \leq \left\lfloor \frac{n}{5} \right\rfloor\right\} \cup \left\{v_{5k-4} \cup v_{5k-2}: 1 \leq k \leq \left\lfloor \frac{n}{5} \right\rfloor\right\}, & n \equiv 1\ (mod\ 5) \\ \left\{u_{5k-3}: 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil\right\} \cup \left\{v_{5k-4}: 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil\right\} \cup \left\{v_{5k-2}: 1 \leq k \leq \left\lfloor \frac{n}{5} \right\rfloor\right\}, & n \equiv 3\ (mod\ 5) \end{cases}$$

$$J' = \begin{cases} \left\{u_{3k-2} \cup v_{3k-2}: 1 \leq k \leq \frac{n}{3}\right\}, & n \equiv 0\ (mod\ 3) \\ \left\{u_{3k-2} \cup v_{3k-2}: 1 \leq k \leq \frac{n-4}{3}\right\} \cup \{u_{n-1}, u_{n-2}, u_{n-3}, u_{n-4}\}, & n \equiv 1\ (mod\ 3), n \neq 7 \\ \left\{u_{3k-2} \cup v_{3k-2}: 1 \leq k \leq \frac{n-2}{3}\right\} \cup \{u_{n-1}, u_{n-2}\}, & n \equiv 2\ (mod\ 3) \\ \left\{v_{3k-2}: 1 \leq k \leq \frac{n-4}{3}\right\} \cup \{u_{n-1}, u_{n-2}, u_{n-3}, u_{n-4}\}, & n = 7 \end{cases}$$

Here we observe that $\gamma(H) \leq \gamma_{fd}(H)$ for $n \geq 5$. Here J' is a 1FD-set of H. Let H_1 be obtained by subdividing an edge $u_1u_n \in E(H)$ (or $v_1v_n \in E(H)$) by a subdivision vertex x_1 then u_1 (or v_2) is not dominated by any of the vertices of J'. Hence the fair dominating set of H_1 is defined as

$$J_1 = J' \cup x_1$$

$$|J_1| > |J'|$$

This implies that $\gamma_{fd}(H) < \gamma_{fd}(H_1)$. Hence $Sd^{+}_{\gamma_{fd}}(H) = 1$. To determine the $Sd^{-}_{\gamma_{fd}}(H)$ consider the following cases

$Case\ (i)\ n \equiv 0\ (mod\ 5)$

In this case, we clearly see that $\gamma(H) < \gamma_{fd}(H)$,$\forall\, n \geq 5$ and by theorem (4.6), $Sd^{-}_{\gamma_{fd}}(H) > 0$. From the above argument in $Sd^{+}_{\gamma_{fd}}(H)$ we observe that $Sd^{+}_{\gamma_{fd}}(H) = 1$ and $Sd^{-}_{\gamma_{fd}}(H) > 0$. The γ_{fd} - set is a 1FD - set of H. Consider the γ - set of H,

$$J = \left\{u_{5k-3} : 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil\right\} \cup \left\{v_{5k-4} \cup v_{5k-2} : 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil\right\}$$

Here we observe that $|N(v_{5k-3}) \cap J| = 3,\ \forall\, 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil$ and all the vertices of $v \in V \setminus J$ are dominated by exactly one vertex of J except $v_{5k-3},\ \forall\, 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil$. Let H_1 be a graph obtained by subdividing any two incident edges with $v_{5k-3},\ \forall\, 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil$ by a subdivision vertices x_{k1} and $x_{k2},\ \forall\, 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil$ respectively. On observation totally $2\left\lceil \frac{n}{5} \right\rceil$ number of edges are subdivided to get H_1. Now the FD - set of H_1 is $J_1 = J$. This implies $|J_1| < |J'|$ and $\gamma_{fd}(H_1) < \gamma_{fd}(H)$. Hence $Sd^{-}_{\gamma_{fd}}(H) = 2\left\lceil \frac{n}{5} \right\rceil$.

$Case\ (ii)\ n \equiv 1\ (mod\ 5)$

In this case, we clearly see that $\gamma(H) < \gamma_{fd}(H)$, $\forall\, n \geq 5$ except $n = 6$ and by theorem (4.6), $Sd^{-}_{\gamma_{fd}}(H) > 0$, From the above argument in $Sd^{+}_{\gamma_{fd}}(H)$ we observe that $Sd^{+}_{\gamma_{fd}}(H) = 1$ and $Sd^{-}_{\gamma_{fd}}(H) > 0$. The γ_{fd} - set is a 1FD - set of H. Consider the γ - set of H,

$$J = \left\{u_{5k-3} \cup u_{n-1} : 1 \leq k \leq \left\lfloor \frac{n}{5} \right\rfloor\right\} \cup \left\{v_{5k-4} \cup v_{5k-2} : 1 \leq k \leq \left\lfloor \frac{n}{5} \right\rfloor\right\}$$

Here we observe that $|N(v_{5k-3}) \cap J| = 3,\ \forall\, 1 \leq k \leq \left\lfloor \frac{n}{5} \right\rfloor$ and $|N(v_n) \cap J| = 2$, $|N(v_{n-1}) \cap J| = 2$ and all the vertices of $v \in V \setminus J$ are dominated by exactly one vertex of J except $v_n, v_{n-1}, v_{5k-3},\ \forall\, 1 \leq k \leq \left\lfloor \frac{n}{5} \right\rfloor$. Let H_1 be a graph obtained by subdividing any two incident edges with $v_{5k-3},\ \forall\, 1 \leq k \leq \left\lfloor \frac{n}{5} \right\rfloor$ by a subdivision vertices x_{k1} and $x_{k2},\ \forall\, 1 \leq k \leq \left\lfloor \frac{n}{5} \right\rfloor$ respectively. Note that the total $2\left\lfloor \frac{n}{5} \right\rfloor$ number of edges are subdivided along with the edges $v_{n-2}v_{n-1}$ and $v_{n-1}v_n$ by the subdivision vertices w_1 and w_2 respectively, results all the vertices of $v \in V \setminus J$ are dominated by one vertex of J, implies that J is a 1FD -set of H_1. Hence $J_1 = J$. This implies $|J_1| < |J'|$ and $\gamma_{fd}(H_1) < \gamma_{fd}(H)$. Totally $2\left\lfloor \frac{n}{5} \right\rfloor + 2$ number of edges are subdivided to obtain H_1. Hence $Sd^{-}_{\gamma_{fd}}(H) = 2\left\lceil \frac{n}{5} \right\rceil$. For $n = 6$, we observe that $\gamma(H) = \gamma_{fd}(H)$ by the theorem (4.5), we have $Sd^{-}_{\gamma_{fd}}(H) = 0$.

$Case\ (iii)\ n \equiv 2\ (mod\ 5)$

In this case, we clearly see that $\gamma(H) < \gamma_{fd}(H)$,$\forall\, n \geq 5$ except $n = 7, 12$ and by theorem (4.6), $Sd^{-}_{\gamma_{fd}}(H) > 0,$ From the first part of the proof we observe that $Sd^{+}_{\gamma_{fd}}(H) = 1$ and $Sd^{-}_{\gamma_{fd}}(H) > 0.$ The γ_{fd} - set is a 1FD - set of H. Consider the γ - set of H,

$$J = \left\{u_{5k-3}: 1 \leq k \leq \left\lceil\frac{n}{5}\right\rceil\right\} \cup \left\{v_{5k-4} \cup v_{5k-2}: 1 \leq k \leq \left\lceil\frac{n}{5}\right\rceil\right\}$$

Here we observe that $|N(v_{5k-3}) \cap J| = 3,\ \forall\, 1 \leq k \leq \left\lceil\frac{n}{5}\right\rceil$ and all the vertices of $v \in V \setminus J$ are dominated by exactly one vertex of J except $v_{5k-3,}\ \forall\, 1 \leq k \leq \left\lceil\frac{n}{5}\right\rceil$. Let H_1 be a graph obtained by subdividing any two incident edges with $v_{5k-3,}\ \forall\, 1 \leq k \leq \left\lceil\frac{n}{5}\right\rceil$ by a subdivision vertices x_{k1} and $x_{k2},\ \forall\, 1 \leq k \leq \left\lceil\frac{n}{5}\right\rceil$ respectively. We observe that $2\left\lceil\frac{n}{5}\right\rceil$ number of subdivision vertices are introduced and the FD - set of H_1 is $J_1 = J$. This implies $|J_1| < |J'|$ and $\gamma_{fd}(H_1) < \gamma_{fd}(H)$. Hence $Sd^{-}_{\gamma_{fd}}(H) = 2\left\lceil\frac{n}{5}\right\rceil$. For $n = 7, 12,$ we observe that $\gamma(H) = \gamma_{fd}(H)$ and by the theorem (4.5), we have $Sd^{-}_{\gamma_{fd}}(H) = 0.$

$Case\ (iv)\ n \equiv 3\ (mod\ 5)$

In this case, we clearly see that $\gamma(H) < \gamma_{fd}(H)$,$\forall\, n \geq 5$ and by theorem (4.6), $Sd^{-}_{\gamma_{fd}}(H) > 0,$ From the above argument in $Sd^{+}_{\gamma_{fd}}(H),$ we observe that $Sd^{+}_{\gamma_{fd}}(H) = 1$ and $Sd^{-}_{\gamma_{fd}}(H) > 0.$ The γ_{fd} - set is a 1FD - set of H. Consider the γ - set of H,

$$J = \left\{u_{5k-3}: 1 \leq k \leq \left\lceil\frac{n}{5}\right\rceil\right\} \cup \left\{v_{5k-4}: 1 \leq k \leq \left\lceil\frac{n}{5}\right\rceil\right\} \cup \left\{v_{5k-2}: 1 \leq k \leq \left\lfloor\frac{n}{5}\right\rfloor\right\}$$

Here we observe that $|N(v_{5k-3}) \cap J| = 3,\ \forall\, 1 \leq k \leq \left\lfloor\frac{n}{5}\right\rfloor$ and $|N(v_n) \cap J| = 2,$ $|N(v_{n-1}) \cap J| = 2.$ Here all the vertices of $v \in V \setminus J$ are dominated by exactly one vertex of J except $u_1, v_{n-1}, v_{5k-3},\ \forall\, 1 \leq k \leq \left\lfloor\frac{n}{5}\right\rfloor$. Let H_1 be a graph obtained by subdividing any two incident edges with $v_{5k-3},\ \forall\, 1 \leq k \leq \left\lfloor\frac{n}{5}\right\rfloor$ by a subdivision vertices x_{k1} and $x_{k2},\ \forall\, 1 \leq k \leq \left\lfloor\frac{n}{5}\right\rfloor$ respectively. Note that the totally $2\left\lfloor\frac{n}{5}\right\rfloor$ number of edges are subdivided along with the edges $v_{n-2}v_{n-1}$ and $u_{n-1}u_1$ by the subdivision vertices w_1 and w_2 respectively. Now we observe that all the vertices of $v \in V \setminus J$ are dominated by one vertex of J, this clearly shows that J is a 1FD -set of H_1 and FD - set of H_1 is $J_1 = J$. This implies $|J_1| < |J'|$ and $\gamma_{fd}(H_1) < \gamma_{fd}(H)$. Here $2\left\lfloor\frac{n}{5}\right\rfloor + 2$ number of subdivision vertices are introduced. Hence $Sd^{-}_{\gamma_{fd}}(H) = 2\left\lceil\frac{n}{5}\right\rceil$.

$Case\ (v)\ n \equiv 4\ (mod\ 5)$

In this case, we clearly see that $\gamma(H) < \gamma_{fd}(H)$,$\forall\, n \geq 5$ except $n = 9$ and by theorem (4.6), $Sd^{-}_{\gamma_{fd}}(H) > 0,$ From the above argument in $Sd^{+}_{\gamma_{fd}}(H),$ we observe that $Sd^{+}_{\gamma_{fd}}(H) = 1$ and $Sd^{-}_{\gamma_{fd}}(H) > 0.$ The γ_{fd} - set is a 1FD - set of H. Consider J is the γ - set of H and defined as,

$$J = \left\{u_{5k-3}: 1 \leq k \leq \left\lceil\frac{n}{5}\right\rceil\right\} \cup \left\{v_{5k-4} \cup v_{5k-2}: 1 \leq k \leq \left\lceil\frac{n}{5}\right\rceil\right\}$$

Here we observe that $|N(v_{5k-3}) \cap J| = 3, \forall 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil$ and $|N(v_n) \cap J| = 2$, $|N(v_n) \cap J| = 2$. all the vertices of $v \in V \setminus J$ are dominated by exactly one vertex of J except $v_{n-1}, u_n, v_{5k-3},$ $\forall 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil$. Let H_1 be a graph obtained by subdividing any two incident edges with $v_{5k-3}, \forall 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil$ by a subdivision vertices x_{k1} and $x_{k2}, \forall 1 \leq k \leq \left\lceil \frac{n}{5} \right\rceil$ respectively. Note that $2\left\lceil \frac{n}{5} \right\rceil$ number of edges are subdivided along with the edges $v_n v_{n-1}$ and $u_n u_{n-2}$ by the subdivision vertices w_1 and w_2 respectively. Now we observe that all the vertices of $v \in V \setminus J$ are dominated by one vertex of J, this clearly shows that J is a 1FD -set of H_1 and FD - set of H_1 is $J_1 = J$. This implies $|J_1| < |J'|$ and $\gamma_{fd}(H_1) < \gamma_{fd}(H)$. Here totally $2\left\lceil \frac{n}{5} \right\rceil + 2$ number of subdivision vertices are introduced. Hence $Sd^{-}_{\gamma_{fd}}(H) = 2\left\lceil \frac{n}{5} \right\rceil + 2 = 2(\left\lceil \frac{n}{5} \right\rceil + 1)$. For $n = 9,$ we observe that $\gamma(H) = \gamma_{fd}(H)$ by the theorem (4.5), we have $Sd^{-}_{\gamma_{fd}}(H) = 0$.

5. Conclusion

The fair domination subdivision number, a key concept in graph theory, has important implications for optimizing network design. By identifying the minimum number of edge subdivisions needed for a fair domination set, this parameter provides insights into the resilience and efficiency of communication networks, social systems, and other graph-modelled structures. Understanding this metric enables the formulation of strategies to bolster network robustness, decrease susceptibility to failures, and enhance resource distribution, ultimately supporting the development of more resilient and equitable network architectures.

6. Future Work

We propose several open problems for further research in fair domination. First, it would be valuable to determine the fair domination subdivision number for additional classes of graphs. Another area of interest is to characterize the classes of graphs for which the relation $Sd^{-}_{\gamma_{fd}}(T) = 1$ holds. Additionally, a generalization of the fair domination subdivision number for Petersen graphs, specifically $P(n, m)$, presents an intriguing direction for exploration.

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