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Evaluation of Infinite Integrals Involving the H-Function of Two Variables

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Abstract:

In this research, we delve into the intricate realm of infinite integrals featuring the H-function of two variables, a powerful mathematical construct that encompasses various special functions. By exploring the H-function's versatility, we uncover novel solutions to integrals that emerge in diverse fields such as applied mathematics, physics, and engineering. Through rigorous analysis, we derive concise closed-form expressions for a range of infinite integral classes, shedding new light on their properties and applications.

Keywords : H-function, infinite integrals, special functions.

1.Introduction:-

A natural generalization of ${}_2F_1$ is the generalized hypergeometric function, the so-called ${}_pF_q$. This broader class of functions encompasses p parameters analogous to a and b, and q parameters akin to c, yielding a versatile series

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p; \\ \\ b_1, \ldots, b_q; \end{matrix} z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots\ldots (a_p)_n \; z^n}{(b_1)_n \ldots\ldots (b_q)_n \; n!} $$

$$ = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_n \, z^n}{\prod_{j=1}^{q} (b_j)_n \; n!}, \qquad (1) $$

is known as the generalized hypergeometric series and the function ${}_pF_q$ is called generalized hypergeometric function of variable z. ${}_pF_q$ is not defined if any denominator parameter b_q is a negative integer or zero. If any numerator parameter a_p is zero or a negative integer, the series terminates. If ${}_pF_q$ does not terminate, it converges

(i) for all finite z if $p \le q$;
(ii) for $|z| < 1$ if $p = q + 1$,

(iii) for $|z| = 1$ if $p = q + 1$ and $R\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) > 0$

and diverges for all $z \neq 0$ if $p > q + 1$.

Mittal and Gupta [2, p. 117] has given the following notation of the H-function of two variables as:

$$H^{0,\, n_1:m_2,\, n_2;m_3,\, n_3}_{p_1,\, q_1:p_2,\, q_2;p_3,\, q_3}\left[\begin{matrix} x \\ y \end{matrix}\,\middle|\,\begin{matrix} (a_j;\alpha_j, A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \\ (b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3} \end{matrix}\right]$$

$$= \frac{-1}{4\pi^2} \int_{L_1}\int_{L_2} \phi_1(\xi,\eta)\,\theta_2(\xi)\,\theta_3(\eta)x^{\xi} y^{\eta}\, d\xi\, d\eta, \qquad (2)$$

where

$$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{n_1} \Gamma(1 - a_j + \alpha_j\xi + A_j\eta)}{\prod^{p_1} \Gamma(a_j - \alpha_j\xi - A_j\eta) \prod^{q_1} \Gamma(1 - b_j + \beta_j\xi + B_j\eta)},$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_2} \Gamma(d_j - \delta_j\xi) \prod_{j=1}^{n_2} \Gamma(1 - c_j + \gamma_j\xi)}{\prod^{q_2} \Gamma(1 - d_j + \delta_j\xi) \prod^{p_2} \Gamma(c_j - \gamma_j\xi)},$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_3} \Gamma(f_j - F_j\eta) \prod_{j=1}^{n_3} \Gamma(1 - e_j + E_j\eta)}{\prod^{q_3} \Gamma(1 - f_j + F_j\eta) \prod^{p_3} \Gamma(e_j - E_j\eta)},$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, q_i, n_i and m_j are non negative integers such that $p_i \geq n_i \geq 0$, $q_i \geq 0$, $q_j \geq m_j \geq 0$, (i = 1, 2, 3; j = 2, 3). Also, all the A's, α's, B's, β's, γ's, δ's, E's, and F's are assumed to the positive quantities for standardization purpose.

The contour L_1 is in the ξ-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j - \delta_j\xi)$ $(j = 1, ..., m_2)$ lie to the right, and the poles of $\Gamma(1 - c_j + \gamma_j\xi)$

$(j = 1, ..., n_2)$, $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ $(j = 1, ..., n_1)$ to the left of the contour.

The contour L_2 is in the η-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j - F_j\eta)$ $(j = 1, ..., m_3)$ lie to the right, and the poles of $\Gamma(1 - e_j + E_j\eta)$ $(j = 1, ..., n_3)$, $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ $(j = 1, ..., n_1)$ to the left of the contour.

The contour L_2 is in the η-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j - F_j\eta)$ $(j = 1, ..., m_3)$ lie to the right, and the poles of $\Gamma(1 - e_j + E_j\eta)$ $(j = 1, ..., n_3)$, $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ $(j = 1, ..., n_1)$ to the left of the contour.

The function, defined by (2), is analytic function of x and y if

$$R = \sum^{p_1} \alpha_j + \sum^{p_2} \gamma_j - \sum^{q_1} \beta_j - \sum^{q_2} \delta_j < 0,$$

$$S = \sum^{p_1} A_j + \sum^{p_3} F_j - \sum^{q_1} B_j - \sum^{q_3} F_j < 0,$$

The H-function of two variables given by (2) is convergent if

$$U = -\sum^{p_1} \alpha_j - \sum^{q_1} \beta_j - \sum^{m_2} \delta_j - \sum^{q_2} \delta_j + \sum^{n_2} \gamma_j - \sum^{p_2} \gamma_j > 0, \qquad (3)$$

$$V = -\sum^{p_1} A_j - \sum^{q_1} B_j - \sum^{m_3} F_j - \sum^{q_3} F_j + \sum^{n_3} E_j - \sum^{p_3} E_j > 0, \qquad (4)$$

and $|\arg x| < \frac{1}{2} U\pi$, $|\arg y| < \frac{1}{2} V\pi$.

2.SOME FINITE AND INFINITE INTEGRALS :-

Integrals are useful in connection with the study of certain boundary value problems. It is also helpful for obtaining the Fourier series and expansion formula. It also used in the study of statistical distribution, probability and integral equation.

The evaluation of finite and infinite integrals involving H-functions and other generalized hypergeometric functions has been extensively explored by numerous researchers, including Chaurasia [3], Chandel [4], Garg [5], Jopshi [6], Shrivastava [7], Ahmad [8], Mishra [9], Nigam

[10], Pandey [11], Patel [12], Shrivastava [13], Singh [14], Shrivastava [15], Tripathi [16], and others.

Looking importance and usefulness of integral in various fields we have established some new integrals of various types, which will be helpful in the study of boundary value problems, expansion formula, Fourier series, statistical distribution, probability and integral equation.

Extending the research endeavors of Chaurasia [3], Chandel [4], Garg [5], Jopshi [6], Shrivastava [7], Ahmad [8], Mishra [9], Nigam [10], Pandey [11], Patel [12], Shrivastava [13], Singh [14], Shrivastava [15], Tripathi [16], and other notable contributors (Section 2.2), this study evaluates infinite integrals of the H-function of two variables. Furthermore, Section 2.3 presents the evaluation of finite integrals involving this function.

2.1 FORMULA USED:

In our investigation we shall need the following results:

From Erdelyi [17, p.284, (2)]:

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+\sigma+1}\Gamma(\sigma+1)\Gamma(\alpha+n+1)\Gamma(\sigma-\beta+1)}{n!\Gamma(\sigma-\beta-n+1)\Gamma(\alpha+\sigma+n+2)}, \quad (5)$$

where $\mathrm{Re}\,\alpha > -1, \mathrm{Re}\,\sigma > -1$.

From Dixon [18]:

$$\int_{-\infty}^{\infty}\frac{\sin(cx)}{\Gamma(\alpha+x)\Gamma(\beta-x)}dx = \frac{[2\cos(\frac{c}{2})]^{\alpha+\beta-2}\sin[\frac{1}{2}c(\beta-\alpha)]}{\Gamma(\alpha+\beta+1)}, \quad (6)$$

provided that $\mathrm{Re}\,(\alpha+\beta) < 1$, $0 < c < \pi$.

2.2 INFINITE INTEGRALS INVOLVING H-FUNCTION OF TWO VARIABLE:

In this section, we shall establish following integrals:

$$\int_{-\infty}^{\infty}\sin(cx)\, H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2+2;p_3,q_3}\left[{\zeta \atop \eta}\middle|{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(1-\alpha-x,u),(1-\beta+x,u):(f_j,F_j)_{1,q_3}}\right]dx$$

$$= [2\cos(\frac{c}{2})]^{\alpha+\beta-2}\sin[\frac{1}{2}c(\beta-\alpha)]\, H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2+1;p_3,q_3}\left[{(2\cos\frac{c}{2})^{2u}\zeta \atop \eta}\middle|{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(-\alpha-\beta,2u):(f_j,F_j)_{1,q_3}}\right], \quad (7)$$

provided that $\mathrm{Re}\,(\alpha+\beta) < 1$, $0 < c < \pi$, $|\arg\zeta| < \frac{1}{2}U\pi$, $|\arg\eta| < \frac{1}{2}V\pi$, where U and V are given in (3) and (4) respectively.

$$\int_{-\infty}^{\infty}\sin(cx)\,H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2+1,q_2+1;p_3,q_3}\left[{\zeta \atop \eta}\middle|{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(\beta-x,-u):(e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(1-\alpha-x,u):(f_j,F_j)_{1,q_3}}\right]dx$$

$$= [2\cos(\tfrac{c}{2})]^{\alpha+\beta-2}\sin[\tfrac{1}{2}c(\beta-\alpha)]\,H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2+1;p_3,q_3}\left[{(2\cos\frac{c}{2})^{2u}\zeta \atop \eta}\middle|{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(-\alpha-\beta,2u):(f_j,F_j)_{1,q_3}}\right], \quad (8)$$

provided that Re (α + β) < 1, 0 < c < π, |arg ζ| < ½U, |arg η| < ½V, where U and V are given in (3) and (4) respectively.

$$\int_{-\infty}^{\infty}\sin(cx)\,H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2+1,q_2+1;p_3,q_3}\left[{\zeta \atop \eta}\middle|{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(\alpha+x,-u):(e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(1-\beta+x,u):(f_j,F_j)_{1,q_3}}\right]dx$$

$$= [2\cos(\tfrac{c}{2})]^{\alpha+\beta-2}\sin[\tfrac{1}{2}c(\beta-\alpha)]\,H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2+1;p_3,q_3}\left[{(2\cos\frac{c}{2})^{2u}\zeta \atop \eta}\middle|{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(-\alpha-\beta,2u):(f_j,F_j)_{1,q_3}}\right], \quad (9)$$

provided that Re (α + β) < 1, 0 < c < π, |arg ζ| < ½U, |arg η| < ½V, where U and V are given in (3) and (4) respectively.

$$\int_{-\infty}^{\infty}\sin(cx)\,H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2+2,q_2;p_3,q_3}\left[{\zeta \atop \eta}\middle|{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(\alpha+x,u),(\beta-x,u):(e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}\right]dx$$

$$= [2\cos(\tfrac{c}{2})]^{\alpha+\beta-2}\sin[\tfrac{1}{2}c(\beta-\alpha)]\,H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2+1,q_2;p_3,q_3}\left[{(2\cos\frac{c}{2})^{-2u}\zeta \atop \eta}\middle|{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(1+\alpha+\beta,2u):(e_j,E_j)_{1,p_3} \atop (b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}\right], \quad (10)$$

provided that Re (α + β) < 1, 0 < c < π, |arg ζ| < ½U, |arg η| < ½V, where U and V are given in (3) and (4) respectively.

$$\int_{-\infty}^{\infty}\sin(cx)\,H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2+1,q_2+1;p_3,q_3}$$

$$\left[{}^{\zeta}_{\eta}\Big|{}^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(\alpha+x,u):(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(1-\beta+x,-u):(f_j,F_j)_{1,q_3}}\right]dx$$

$$= [2\cos(\tfrac{c}{2})]^{\alpha+\beta-2}\sin[\tfrac{1}{2}c(\beta-\alpha)]\, H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2+1,q_2;p_3,q_3}$$

$$\left[{}^{(2\cos\frac{c}{2})^{-2u}\zeta}_{\eta}\Big|{}^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(1+\alpha+\beta,2u):(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}\right], \qquad (11)$$

provided that Re (α + β) < 1, 0 < c < π, |arg ζ| < ½U□, |arg η| < ½V□, where U and V are given in (3) and (4) respectively.

$$\int_{-\infty}^{\infty}\sin(cx)\, H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2+1,q_2+1;p_3,q_3}$$

$$\left[{}^{\zeta}_{\eta}\Big|{}^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(\beta-x,u):(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(1-\alpha-x,-u):(f_j,F_j)_{1,q_3}}\right]dx \qquad \text{(expression continue)}$$

$$= [2\cos(\tfrac{c}{2})]^{\alpha+\beta-2}\sin[\tfrac{1}{2}c(\beta-\alpha)]\, H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2+1,q_2;p_3,q_3}$$

$$\left[{}^{(2\cos\frac{c}{2})^{-2u}\zeta}_{\eta}\Big|{}^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(1+\alpha+\beta,2u):(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}\right], \qquad (12)$$

provided that Re (α + β) < 1, 0 < c < π, |arg ζ| < ½U□, |arg η| < ½V□, where U and V are given in (3) and (4) respectively.

Proof (5):

The result (5) can be established by replacing the H-function of two variable on the left hand side as contour integral (2), we get

$$\int_{-\infty}^{\infty}\sin(cx)\,[\tfrac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\rho,\sigma)\theta_2(\rho)\theta_3(\sigma)$$

$$\frac{1}{\Gamma(\alpha+x+u\rho)\Gamma(\beta-x+u\rho)}\zeta^{\rho}\eta^{\sigma}d\rho d\sigma]dx$$

interchanging the order of integral involved in the process, evaluating the inner integral with the help of (8) and applying (2) the definition of H-function, the value of the integral is obtained. On using the same procedure as above, the integrals (8) to (12) are established.

PARTICULAR CASES:

On choosing c = π/2 in (7) and (8), we get following results, which are useful in space science and used in explanation of quantum gravitational:

$$\int_{-\infty}^{\infty}\sin(\frac{\pi}{2}x)\, H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2+2;p_3,q_3}$$

$$\left[{}^{\zeta}_{\eta}\Big|{}^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(1-\alpha-x,u),(1-\beta+x,u):(f_j,F_j)_{1,q_3}}\right]dx$$

$$= [\sqrt{2}]^{\alpha+\beta-2}\sin[\tfrac{\pi}{4}(\beta-\alpha)]\, H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2,q_2+1;p_3,q_3}$$

$$[{}^{(\sqrt{2})^{2u}\zeta}_{\eta}|^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(-\alpha-\beta,2u):(f_j,F_j)_{1,q_3}}] \quad (13)$$

provided that Re (α + β) < 1, |arg ζ| < ½ U□, |arg η| < ½ V□, where U and V are given in (3) and (4) respectively.

$$\int_{-\infty}^{\infty} \sin\left(\frac{\pi}{2}x\right) H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2+2,q_2;p_3,q_3}$$

$$[{}^{\zeta}_{\eta}|^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(\alpha+x,u),(\beta-x,u):(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}]dx$$

$$= [\sqrt{2}]^{\alpha+\beta-2}\sin\left[\frac{\pi}{4}(\beta-\alpha)\right] H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1;p_2+1,q_2;p_3,q_3}$$

$$[{}^{(\sqrt{2})^{-2u}\zeta}_{\eta}|^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(1+\alpha+\beta,2u):(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}], \quad (14)$$

provided that Re (α + β) < 1, arg ζ|<½Uπ, |arg η| <½Vπ, where U and V are given in (3) and (4) respectively.

2.3 FINITE INTEGRALS INVOLVING H-FUNCTION OF TWO VARIABLES

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\sigma}P_m^{(\alpha,\beta)}(x)H[{}^{\zeta}_{\eta(1-x)^{\lambda}}]dx$$

$$= \frac{2^{\alpha+\sigma+1}\Gamma(\sigma+1)\Gamma(\sigma-\beta+1)}{m!\Gamma(\sigma-\beta-m+1)} H^{0,n_1;m_2,n_2;m_3,n_3+1}_{p_1,q_1;p_2,q_2;p_3+1,q_3+1}$$

$$[{}^{\zeta}_{2^{\lambda}\eta}|^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(-\alpha-m,\lambda),(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3},(-1-\alpha-\sigma-m,\lambda)}], \quad (15)$$

$$\mathrm{Re}\,\alpha + \lambda \min_{1\le j\le m_3}\left[\mathrm{Re}\frac{f_j}{F_j}\right] > -1, \mathrm{Re}\,\sigma > -1;$$

$$\int_{-1}^{1}(1-y)^{h}(1+y)^{\rho}P_n^{(h,k)}(y)H[{}^{\zeta(1-y)^{\mu}}_{\eta}]dy$$

$$= \frac{2^{h+\rho+1}\Gamma(\rho+1)\Gamma(\rho-k+1)}{n!\Gamma(\rho-k-n+1)} H^{0,n_1;m_2,n_2+1;m_3,n_3}_{p_1,q_1;p_2+1,q_2+1;p_3,q_3}$$

$$[{}^{2^{\mu}\zeta}_{\eta}|^{(a_j,\alpha_j;A_j)_{1,p_1}:(-h-n,\mu),(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(-1-h-\rho-n,\mu):(f_j,F_j)_{1,q_3}}], \quad (16)$$

$$\mathrm{Re}\,h + \mu \min_{1\le j\le m_2}\left[\mathrm{Re}\frac{d_j}{\delta_j}\right] > -1, \mathrm{Re}\,\rho > -1;$$

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\sigma}P_m^{(\alpha,\beta)}(x)H[{}^{\zeta}_{\eta(1-x)^{-\lambda}}]dx$$

$$= \frac{2^{\alpha+\sigma+1}\Gamma(\sigma+1)\Gamma(\sigma-\beta+1)}{m!\Gamma(\sigma-\beta-m+1)} H^{0,n_1;m_2,n_2;m_3+1,n_3}_{p_1,q_1;p_2,q_2;p_3+1,q_3+1}$$

$$[_{2^{-\lambda}\eta}^{\zeta}|_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(m+\alpha+1,\lambda),(f_j,F_j)_{1,q_3}}^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3},(\alpha+\sigma+m+2,\lambda)}], \quad (17)$$

$$\text{Re}\,\alpha-\lambda\max_{1\le j\le n_3}[\text{Re}\frac{e_j-1}{E_j}]>-1, \text{Re}\,\sigma>-1;$$

$$\int_{-1}^{1}(1-y)^h\,(1+y)^\rho P_n^{(h,k)}(y)H[_\eta^{\zeta(1-y)^{-\mu}}]dy$$

$$=\frac{2^{h+\rho+1}\Gamma(\rho+1)\Gamma(\rho-k+1)}{n!\Gamma(\rho-k-n+1)}H_{p_1,q_1;p_2+1,q_2+1;p_3,q_3}^{0,n_1;m_2+1,n_2;m_3,n_3}$$

$$[_{\ \ \eta}^{2^{-\mu}\zeta}|_{(b_j,\beta_j;B_j)_{1,q_1}:(1+n+h,\mu),(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(2+n+\rho+h,\mu):(e_j,E_j)_{1,p_3}}], \quad (18)$$

$$\text{Re}\,h-\mu\max_{1\le j\le n_2}[\text{Re}\frac{c_j-1}{\gamma_j}]>-1, \text{Re}\,\rho>-1;$$

$$\int_{-1}^{1}(1-x)^\alpha\,(1+x)^\sigma P_m^{(\alpha,\beta)}(x)H[_{\eta(1+x)^\lambda}^{\zeta}]dx$$

$$=\frac{2^{\alpha+\sigma+1}\Gamma(\alpha+m+1)}{m!}H_{p_1,q_1;p_2,q_2;p_3+2,q_3+2}^{0,n_1;m_2,n_2;m_3,n_3+2}$$

$$[_{2^\lambda\eta}^{\zeta}|_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3},(-\sigma+\beta+m,\lambda),(-1-m-\alpha-\sigma,\lambda)}^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(-\sigma,\lambda),(-\sigma+\beta,\lambda),(e_j,E_j)_{1,p_3}}], \quad (19)$$

$$\text{Re}\,\sigma+\lambda\min_{1\le j\le m_3}[\text{Re}\frac{f_j}{F_j}]>-1, \text{Re}\,\beta>-1;$$

$$\int_{-1}^{1}(1-y)^h\,(1+y)^\rho P_n^{(h,k)}(y)H[_\eta^{\zeta(1+y)^{\mu}}]dy$$

$$=\frac{2^{h+\rho+1}\Gamma(h+n+1)}{n!}H_{p_1,q_1;p_2+2,q_2+2;p_3,q_3}^{0,n_1;m_2,n_2+2;m_3,n_3}$$

$$[_{\ \ \eta}^{2^{\mu}\zeta}|_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(-\rho+k+n,\mu),(-1-h-\rho-n,\mu):(f_j,F_j)_{1,q_3}}^{(a_j,\alpha_j;A_j)_{1,p_1}:(-\rho,\mu),(-\rho+k,\mu),(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}}], \quad (20)$$

$$\text{Re}\,\rho+\mu\min_{1\le j\le m_2}[\text{Re}\frac{d_j}{\delta_j}]>-1, \text{Re}\,k>-1;$$

$$\int_{-1}^{1}(1-x)^\alpha\,(1+x)^\sigma P_m^{(\alpha,\beta)}(x)H[_{\eta(1+x)^{-\lambda}}^{\zeta}]dx$$

$$=\frac{2^{\alpha+\sigma+1}\Gamma(\alpha+m+1)}{m!}H_{p_1,q_1;p_2,q_2;p_3+2,q_3+2}^{0,n_1;m_2,n_2;m_3+2,n_3}$$

$$[_{2^{-\lambda}\eta}^{\zeta}|_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(1+\sigma,\lambda),(1+\sigma-\beta,\lambda),(f_j,F_j)_{1,q_3}}^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3},(1+\sigma-\beta-m,\lambda),(2+m+\alpha+\sigma,\lambda)}], \quad (21)$$

$$\operatorname{Re}\sigma-\lambda\max_{1\le j\le n_3}[\operatorname{Re}\frac{e_j-1}{E_j}]>-1,\operatorname{Re}\beta>-1;$$

$$\int_{-1}^{1}(1-y)^h\,(1+y)^\rho P_n^{(h,k)}(y)H[{}^{\zeta(1+y)^{-\mu}}_{\eta}]dy$$

$$=\frac{2^{h+\rho+1}\Gamma(h+n+1)}{n!}\,H^{0,n_1;m_2+2,n_2;m_3,n_3}_{p_1,q_1;p_2+2,q_2+2;p_3,q_3}$$

$$[{}^{2^{-\mu}\zeta}_{\eta}|{}^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(1+\rho-k-n,\mu),(2+h+\rho+n,\mu):(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(1+\rho,\mu),(1+\rho-k,\mu),(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}}],\qquad(22)$$

$$\operatorname{Re}\rho-\mu\max_{1\le j\le n_2}[\operatorname{Re}\frac{c_j-1}{\gamma_j}]>-1,\operatorname{Re}k>-1;$$

and $\lambda>0,\mu>0,U>0,V>0,|\arg\zeta|<\frac{1}{2}U\pi$, $|\arg\eta|<\frac{1}{2}V\pi$, where U and V are given in (3) and (4).

Proof of (15):

To establish (15), expressing the H-function in the integrand as (3), changing the order of the x-integral and ξ, η-integral, evaluating the inner-integral with the help of (5), the value of the integral (15) is obtained. On using the same procedure as above, the integrals (16) to (22) are established.

Conclusion:

This research has successfully explored the evaluation of finite integrals involving H-functions of two variables, providing new analytical expressions and insights into their properties. The findings contribute to the advancement of mathematical and statistical theory, with potential applications in physics, engineering, and data science.

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