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Computational Analysis of Inventory Model for Non- Instantaneous Deterioration following Weibull Distribution with Quadratic Demand using Preservation Technology

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Abstract:

In this study, two different inventory models for products with quadratic demand that deteriorate slowly according to a Weibull distribution are presented and computationally analyzed. The first model investigates the conventional method without using preservation technologies while second model uses preservation technologies to lessen the consequences of deterioration. Comparing the overall inventory expenses of each model is the goal, with an emphasis on how much money can be saved by incorporating preservation technologies into inventory management procedures. The study shows, using analytical techniques and numerical simulations, that the preservation technology model effectively lowers overall costs by increasing the shelf-life of inventory items and reducing the adverse financial impact of product deterioration. The investigation verifies that spending money on preservation technologies improves inventory sustainability by cutting waste and lowers overall inventory expenditures. The present study offers significant perspectives for enterprises seeking to enhance their inventory management tactics. It highlights the concrete advantages of utilizing preservation technology to handle declining products in a demand-sensitive setting.

Keywords: Inventory; Non- Instantaneous Deterioration; Weibull Distribution; Quadratic Demand Rate; Preservation Technology.

Introduction

In general, Demand that first rises linearly but subsequently falls exponentially when a replacement product becomes available is known as quadratic demand. A company which can produce an item or service at a lower cost may be able to gain a larger market share by offering it at a lower price. If the firm's cost advantage is inadequate company may be unable to compete with lower-cost producers. Furthermore, the shape of the demand curve may influence the firm's pricing strategy and profit capacity. It is crucial to note that demand for most goods and services is not perfectly quadratic and can be influenced by a variety of factors such as consumer affluence, replacement availability, product degradation, and overall economic conditions.

Due to the highly competitive global marketplace, inventory management has grown increasingly complex. As a result, numerous inventory researchers have created a number of efficient inventory models based on more realistic assumptions that perfectly mimic the reality of organizations. Many perishable products (such as vegetables, fruits, milk, and meat) degrade during storage, either owing to production or for other reasons. When a large number of other items (such as alcohol, radioactive compounds, and scents) are stored, practitioners may perceive disintegration. The rate of deterioration has a significant impact on inventories, sales, and order quantities, particularly for commodities that depreciate quickly, resulting in a higher total applied cost and a lower demand rate. The impact of degradation must be considered when managing these items' inventory because the practitioners' profitability may suffer as a result of product deterioration. Because of the items' initial quality (such as vegetables, fruits, milk, and meat, among others), degradation may occur later than when the practitioner receives them. The majority of degrading inventory models assume that deterioration occurs as soon as the inventory arrives at the retailer. However, certain commodities do not deteriorate with time. Non-instantaneous degradation describes the slow erosion of a product's quality or functioning over time. This can be caused by a number of circumstances, including typical wear and tear, exposure to external variables, or the natural ageing of the product's materials. There are several ways to for predicting and forecasting a product's non-instantaneous deterioration, including the use of statistical models such as the Weibull distribution and engineering models that account for the specific mechanisms by which the product is expected to deteriorate. To avoid or mitigate the effects of non-instantaneous deterioration, the product must be properly designed and maintained, as well as evaluated in the environment in which it will be used. This could entail making greater use of durable materials, implementing maintenance schedules, or applying protective coatings or other treatments to extend the product's life.

The Weibull distribution is frequently used in reliability engineering and quality control to model a product's failure rate over time. It is valuable because, depending on the shape and scale parameters, it can represent both early-life and wear-out failures. This study proposes an inventory model with a quadratic demand function of time and a partly backlogged shortage to improve inventory management operational performance for non-instantaneously decaying objects. Partial backlogging in inventory management refers to the practice of carrying over a portion of unsatisfied demand to the following time rather than losing it completely. This can be a useful strategy when it is neither possible or desirable to immediately meet all of the demand for a product, such as when the product has a short shelf life or production capacity is limited. Inventory models can make advantage of partial backlogging in a number of ways. A preset backlogging rate, for instance, could be specified by the model, meaning that a portion of unmet demand is carried over to the following time. Alternatively, the model might treat the backlog rate as a decision variable that can be optimized to lower the overall cost of inventory management. In any case, partial backlogging can influence the model's optimal inventory levels and ordering decisions. For example, If the backlog rate is low, it may be more cost-effective to maintain low inventory levels while incurring higher backlog expenses to reduce holding and ordering costs.

The remainder of the research is carried out in other parts. The review of literature is found in Section 2. Section 3 describes the notations and assumptions of the inventory model designed for things that don't deteriorate instantly with a quadratic demand function, where shortages are permitted and partially backlogged. Section 4 explains the solution technique and provides a numerical example. Section 5 includes a sensitivity analysis as well as graphs of the various parameters utilized in the model. The study's conclusion was explained in Section 6.

1. Literature Review

By discussing existing research that is relevant to this study effort and then comparing the studies in a tabular format, this part clarifies the research gap and prior research contributions:

In inventory management, the demand rate is the rate at which things are purchased or consumed by customers. In some circumstances, the demand rate is believed to be constant, implying that it does not fluctuate with time. This assumption is frequently made for the sake of simplicity since it facilitates inventory level analysis and optimization. However, in many circumstances, the demand rate is not constant and is determined by a number of variables including price, marketing efforts, seasonality, and competition. Considering the demand rate as constant (Kar, Roy, and Maiti 2008) and (Chung et al. 2019) developed the inventory models. But in general, the demand depends on the price of the product. Customers buy more products when the price is low as compared to the higher price. (Yang, Ouyang, and Wu 2009) and (Annadurai 2013) established the inventory model where the demand for the items depends on the price of the product. When a product is introduced to the market, demand rises linearly for a while, but once a new substitute becomes available, demand falls exponentially. This type of phenomenon of demand is known as quadratic demand. (Yadav 2014) and (Lakshmidevi and Maragatham 2015) have developed the models for quadratic demand function with different parameters. (Ahmed, Al-Khamis, and Benkherouf 2013) proposed a technique to identify the EOQ policy in a model of inventory that has ramp type demand and general deterioration rate. It has been discovered that the previous demand patterns do not accurately portray the desire for specific things, such as freshly launched fashion items, clothing, cosmetics, vehicles, and so on, for which the demand grows when they are eventually introduced into the market and then stays steady after a while. The idea of ramp-type demand is presented to consider such demand. A ramp-type demand grows until a particular point, after which it stabilizes and remains constant. (M Valliathal 2013), (Saha et al. 2018), (Mohanty, Kumar, and Goswami 2018), and (Khatri and Gothi 2020) developed the inventory models for the ramp-type demand function. Generally, in the market, a huge number of customers are enticed to superstores by the presentation of vast volumes of inventory with a wide variety, resulting in increased market demand. This type of demand is known as stock-dependent demand. By assuming the demand as stock-dependent (Singh, Malik, and Kumar n.d.), (Kumar and Kumar 2016), and (Sharma and Kumar Bansal 2017) have developed inventory models. While (Tat, Esmaeili, and Taleizadeh 2014) have developed an inventory model for demand rates that are deterministic.

Another issue that businesses face is the deterioration of commodities due to spoilage, dryness, and vaporization. Numerous investigations have been conducted to solve the issue of inventory system degradation. The product’s inventory depends on the deterioration of the product. (Bhunia and Maiti n.d.) provided an inventory model for degrading goods over a limited time horizon in which the demand rises linearly with time. The approach is created with the premise that the lengths of the subsequent replenishment cycles are constant. By considering the deterioration of the product as constant (Yang 2012) developed an inventory model. But sometimes the deterioration of the products changes over time, so (Xu et al. 2020) created a model for inventory for the variable deterioration rate. Various O.R. scientists and researchers were able to create an ideal replacement plan where the cost of replenishment remains constant throughout a given time frame for every cycle duration. By considering the stochastic deterioration rate (Mohanty et al. 2018) have developed an inventory model for the items with preservation technology. (Saha and Sen 2017) attempted to create a model of inventories for degrading products with negative exponential demand and time-dependent degradation. (Tat et al. 2014), and (Chung et al. 2019) have developed an Inventory model for goods that degrade instantly. (Mukherjee n.d.) has been developed an inventory model for items that degrade instantaneously by assuming that preservation technique can regulate the deterioration. In actuality, the items (vegetables, fruits, milk, and meat, among others) may not deteriorate right away when the practitioner receives them because of their original quality; alternatively, it may occur after some time has passed since the practitioner has received the products. Non-instantaneous degradation is the term used to describe this type of event. Many researchers like (Yang et al. 2009), (Vaish and Garg 2011), (Kumar and Kumar 2016), (Bishi and Kumar Sahu 2188), (Patel and Gor 2019) and (Khatri and Gothi 2020) developed the models of inventory for the non-instantaneous deteriorating items.

In inventory models, preservation technology refers to the incorporation of methods and plans intended to prolong the shelf life of perishable commodities or preserve their quality along a supply chain. This idea is essential in sectors where product deterioration can result in large losses, such the food, pharmaceutical, and chemical industries. Businesses hope to accomplish more effective inventory management, lower waste, raise customer happiness, and increase product quality by implementing preservation technologies. (Dye and Hsieh 2012) evolved an ideal replacement strategy for deteriorating goods that makes efficient use of preservation technologies. (Mishra 2013) created a model of deteriorating inventory with shortages and salvage value utilizing preservation technology. (Yang, Dye, and Ding 2015) created a preservation technology allocation and optimal dynamic trade credit model for a declining inventory. (Mishra 2016) created a revenue-sharing inventory model for products that are degrading under price-sensitive stock dependent demand for preservation technology investment. (Chandra Das et al. 2020) evolved A partial backlog and price-dependent demand inventory control system using preservation technologies. Priyamvada (2020) evolved an inventory model with need for controllable deterioration rate with shortages and preservation technology investment that is depending on price and stock. Arash Sepehri (2021) developed a production-inventory strategy that is sustainable with preservation technology and quality improvement investment, albeit with imperfect quality. (Priyamvada et al. 2021) created price-sensitive investment strategies for items with preservation technology using optimal inventory strategies

When shortages occur due to market uncertainty, practitioners are typically presented with two separate situations: backorders and sales opportunities. In reality, when there are shortages, buyers may wait for new items to come or switch to other accessible suppliers that can suit their needs. Complete back ordering occurs when every customer awaits the arrival of the new product they want.

Furthermore, partial back ordering occurs when some consumers wait for new items to arrive. Numerous scholars have examined partial back-ordering situations under the assumption that a fixed backlog rate that is, a specific proportion of customers wait for a backordered item occurs. (Yang et al. 2009) consider a non-instantaneous degrading inventory model for establishing optimal pricing and ordering strategies with price-dependent demand to reflect realistic situations. In the model, shortages are permitted and partially backlogged, with the backlogging rate being changeable and based on the time it takes for the next replenishment. (Kumar 2016) and (Sharma and Kumar Bansal 2017) have also developed an inventory model where shortages are permitted and partially backlogged for non-instantaneously deteriorating items with stock-dependent demand. (Khatri and Gothi 2020) have developed the inventory model by assuming the shortages as partially backlogged. (Sahoo and Paul 2021) invented a model for Weibull function of inventory level demand assuming the deterioration as the cubic function of time and shortages are not allowed. (Lesmono et al., 2022) developed an inventory model for price as well as inventory dependent demand and shortages are not allowed. (Khyati and Saxena 2022) has been developed an inventory model for deteriorating items following the Weibull distribution and the demand is assumed as time dependent while shortages are partially backlogged.

The literature review above demonstrates that few academics have studied on the constant deterioration rate, although in real-life circumstances, some goods decay over time. Some researchers have included a non-instantaneous deterioration rate in their inventory models. Various inventory models have been established sorts of demand rates (deterministic, stock dependent, quadratic, price sensitive, ramp, and constant). In this study, we estimated an inventory model for objects that deteriorate gradually. The demand is modelled as a quadratic function of time. The shortages are permitted and somewhat backlogged. In addition, the ideal solution is found via the development of an algorithm. The theoretical results are illustrated with a numerical example, and a sensitivity analysis of the optimal solution to pertinent factors is also performed.

2. Assumptions, Notations, Description and Formulation of the Inventory Model

An inventory model for non-instantaneously decaying items with quadratic demand and partially backlogged shortages is defined in this research effort.

2.1. Assumptions

There are some assumptions we need to develop the model:

i. Quadratic demand rate.

ii. The distribution of deterioration of products is two-parameter Weibull distribution.

iii. There is no replenishment or repair during the cycle length.

iv. A single warehouse and a single item are taken into consideration.

v. Lead time is zero.

vi. In a particular cycle, there is no replacement or repair of deteriorating parts.

vii. We consider the planning horizon to be finite since we know the product's life is finite.

viii. Shortages of goods are partially backlogged. The backlogging rate $e^{-\delta(L-t)}$, where δ is the backlogging parameter.

2.2. Notations:

The mathematical model is developed using the notations shown below:

i. $\theta(t) = \alpha\beta t^{\beta-1}$: The Weibull distribution deterioration rate where $0 < \alpha < 1$the scale parameter and $\beta > 0$ is the shape parameter.
ii. $D(t) = a + bt + ct^2$: Quadratic demand rate where, $a > b$ and $a > c$.
iii. ξ = the preservation indicator.
iv. L= Cycle Length.
v. Q_k=Ordered Quantity per Cycle.
vi. t_0= Time period before Deterioration.
vii. t_{k1}= Time period of Deterioration.
viii. Q_c= Ordering Cost per order.
ix. θ_c= Deterioration Cost per unit per unit time.
x. H_c= Holding Cost per unit per unit time.
xi. B_c= Backorder Cost per unit short item per unit time.
xii. δ= backlogging parameter.
xiii. $C_t(t, L)$ = Total Cost.

2.3. Description and Formulation of the Inventory Model (Without preservation technology):

The mathematical model for the above assumptions and notation can be described as follows:
$I_1(t)$ Describes the inventory level assuming there's no deterioration within the period $(0, t_0)$. This inventory level depends only on the demand rate. In this paper, the demand function, $D(t) = a + bt + ct^2$, is a quadratic function of time, t. In this function, D represents the demand for a product or service at a given time, t, and *a, b,* and *c* are constants. The term a represents the constant demand for the product or services, independent of time. The term bt represents the linear component of the demand function, which describes how the demand changes over time in a linear fashion. The term ct^2 represents the quadratic component of the demand function, which describe how the demand changes over time in a non-linear fashion. This demand function is useful for modelling scenarios in which demand for a product or service varies over time in a complex manner. $I_2(t)$ is the inventory level during the time period (t_0, t_{k1}) where the product has deteriorated. And $I_3(t)$ denotes the inventory level during the period (t_{k1}, L) when shortages have appeared.

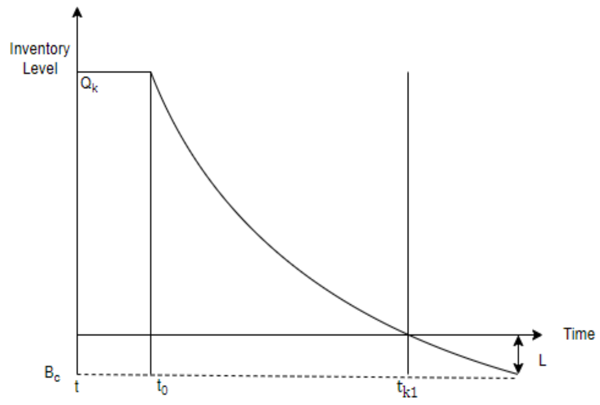

Figure 1. Inventory system for non-instantaneous deterioration with partial backlogging.

Let Q_k be the inventory level at time t. The inventory level is assumed to be constant within the period $(0, t_0)$ because there is no spoiling within this time period. This means that the items will not deteriorate until after the given time interval. After some time, the deterioration starts during the interval (t_0, t_{k1}). At t_{k1}, the inventory level drops to zero, and in the period (t_{k1}, L) there is a shortage. The model can be further developed by including additional assumptions and notation, such as the rate of deterioration, the cost of acquiring additional inventory, and the cost of lost sales due to the shortages. These additional factors can be used to optimize the inventory management strategy and minimize the overall cost of the system. The inventory levels of the model are governed by the following differential equations:

$$\frac{dI_1(t)}{dt} = -D(t), \quad 0 \le t \le t_0 \qquad (1)$$

Equation (1) is representing the inventory level when there is no deterioration of items and inventory depends only on demand.

$$\frac{dI_2(t)}{dt} + \theta(t)I_2(t) = -D(t), \quad t_0 \le t \le t_{k1} \qquad (2)$$

Equation (2) is representing the inventory level when there is deterioration of items and inventory depends on both demand and deterioration.

$$\frac{dI_3(t)}{dt} = -e^{-\mu(L-t)}D(t), \quad t_{k1} \le t \le L \qquad (3)$$

Equation (3) is representing the inventory level when there is shortages of items.

From equation (1), we get

$$\frac{dI_1(t)}{dt} = -(a + bt + ct^2)$$

$$I_1(t) = -\left(at + \frac{bt^2}{2} + \frac{ct^3}{3}\right) + c_1,$$

With Boundary conditions, $I_1(t) = Q_k$ at $t = 0$

Then,

$Q_k = c_1$

$$I_1(t) = Q_k - \left(at + \frac{bt^2}{2} + \frac{ct^3}{3}\right) \quad (4)$$

On solving equation (2), we have;

$$\frac{dI_2(t)}{dt} + \alpha\beta t^{\beta-1} I_2(t) = -(\mathrm{a} + \mathrm{bt} + \mathrm{c}t^2)$$

$$\mathrm{I}_2(\mathrm{t}) = -e^{-\alpha t^\beta}\left[at + \frac{bt^2}{2} + \frac{ct^3}{3} + \alpha\left(\frac{t^{\beta+1}}{\beta+1} + \frac{t^{\beta+2}}{\beta+2} + \frac{t^{\beta+3}}{\beta+3}\right)\right] + c_2 e^{-\alpha t^\beta} \quad (5)$$

If $\mathrm{I}_2(\mathrm{t}) = 0$ at $t = t_{k1}$ then

$$0 = -e^{-\alpha t_{k1}^\beta}\left[\mathrm{a}t_{k1} + \frac{bt_{k1}^2}{2} + \frac{bt_{k1}^3}{3} + \alpha\left(\frac{t_{k1}^{\beta+1}}{\beta+1} + \frac{t_{k1}^{\beta+2}}{\beta+2} + \frac{t_{k1}^{\beta+3}}{\beta+3}\right)\right] + c_2 e^{-\alpha t_{k1}^\beta}$$

$$c_2 = \left[\mathrm{a}t_{k1} + \frac{bt_{k1}^2}{2} + \frac{bt_{k1}^3}{3} + \alpha\left(\frac{t_{k1}^{\beta+1}}{\beta+1} + \frac{t_{k1}^{\beta+2}}{\beta+2} + \frac{t_{k1}^{\beta+3}}{\beta+3}\right)\right]$$

Putting the value of c_2 in equation (5)

$$\mathrm{I}_2(\mathrm{t}) = -e^{-\alpha t^\beta}\left[at + \frac{bt^2}{2} + \frac{ct^3}{3} + \alpha\left(\frac{t^{\beta+1}}{\beta+1} + \frac{t^{\beta+2}}{\beta+2} + \frac{t^{\beta+3}}{\beta+3}\right)\right] + \left[\mathrm{a}t_{k1} + \frac{bt_{k1}^2}{2} + \frac{bt_{k1}^3}{3} + \alpha\left(\frac{t_{k1}^{\beta+1}}{\beta+1} + \frac{t_{k1}^{\beta+2}}{\beta+2} + \frac{t_{k1}^{\beta+3}}{\beta+3}\right)\right] e^{-\alpha t^\beta}$$

$$\mathrm{I}_2(\mathrm{t}) = e^{-\alpha t^\beta}\left[a(t_{k1} - t) + \frac{b(t_{k1}^2 - t^2)}{2} + \frac{c(t_{k1}^3 - t^3)}{3} + \alpha\left(\frac{t_{k1}^{\beta+1} - t^{\beta+1}}{\beta+1} + \frac{t_{k1}^{\beta+2} - t^{\beta+2}}{\beta+2} + \frac{t_{k1}^{\beta+3} - t^{\beta+3}}{\beta+3}\right)\right] \quad (6)$$

Now, considering continuity of $I(t)$ at $t = t_0$ it follows from the equation (4) and (6)

$I_1(t_0) = I_2(t_0)$

We get,

$$Q_k - \left(at_0 + \frac{bt_0^2}{2} + \frac{ct_0^3}{3}\right) = e^{-\alpha t_0^\beta}\left[a(t_{k1} - t_0) + \frac{b(t_{k1}^2 - t_0^2)}{2} + \frac{c(t_{k1}^3 - t_0^3)}{3} + \alpha\left(\frac{t_{k1}^{\beta+1} - t_0^{\beta+1}}{\beta+1} + \frac{t_{k1}^{\beta+2} - t_0^{\beta+2}}{\beta+2} + \frac{t_{k1}^{\beta+3} - t_0^{\beta+3}}{\beta+3}\right)\right]$$

$$Q_k = e^{-\alpha t_0^\beta}\left[a(t_{k1} - t_0) + \frac{b(t_{k1}^2 - t_0^2)}{2} + \frac{c(t_{k1}^3 - t_0^3)}{3} + \alpha\left\{\frac{t_{k1}^{\beta+1} - t_0^{\beta+1}}{\beta+1} + \frac{t_{k1}^{\beta+2} - t_0^{\beta+2}}{\beta+2} + \frac{t_{k1}^{\beta+3} - t_0^{\beta+3}}{\beta+3}\right\}\right] + at_0 + \frac{bt_0^2}{2} + \frac{ct_0^3}{3} \quad (7)$$

Using equation (7) in equation (4)

$$\mathrm{I}_1(\mathrm{t}) = e^{-\alpha t_0^\beta}\left[a(t_{k1} - t_0) + \frac{b(t_{k1}^2 - t_0^2)}{2} + \frac{c(t_{k1}^3 - t_0^3)}{3} + \alpha\left\{\frac{t_{k1}^{\beta+1} - t_0^{\beta+1}}{\beta+1} + \frac{t_{k1}^{\beta+2} - t_0^{\beta+2}}{\beta+2} + \frac{t_{k1}^{\beta+3} - t_0^{\beta+3}}{\beta+3}\right\}\right] + a(t_0 - t) + \frac{b(t_0^2 - t^2)}{2} + \frac{c(t_0^3 - t^3)}{3} \quad (8)$$

From equation (3), we get

$$I_3(t) = -\left[(1-\mu L)\left(at + \frac{bt^2}{2} + \frac{ct^2}{3}\right) + \mu\left(\frac{at^2}{2} + \frac{bt^3}{3} + \frac{ct^4}{4}\right)\right] \quad (9)$$

Considering the model's assumptions, suppose the following different costs:
The ordering cost per cycle is $C_o = \frac{O_c}{L}$

3.3.1. The deterioration cost per cycle is $C_d = \frac{\theta_c}{L}\int_{t_0}^{t_{k1}} I_2(t)dt$

3.3.2. The inventory holding cost per cycle is

$$C_h = \frac{H_c}{L}\left[\int_0^{t_0} I_1(t)dt + \int_{t_0}^{t_{k1}} I_2(t)dt\right]$$

3.3.3. Backorder cost per cycle is: $C_b = \frac{B_c}{L}\int_{t_{k1}}^{L} -I_3(t)dt$

Consequently, the ordering cost, degrading cost, holding cost, and backorder cost are added up to determine the total inventory cost.

The total cost $C_t(t_{k1,}L) = C_o + C_d + C_h + C_b$ (10)

3.4 Description and Formulation of the Inventory Model (with preservation technology):

Let ξ is the preservation rate of products. We use the preservation technology during the time period in which the items start deteriorate i.e. (t_0, t_{k1}). The inventory levels of the model using preservation technology are governed by the following differential equations:

$$\frac{dI_{1p}(t)}{dt} = -D(t), \qquad 0 \le t \le t_0 \quad (11)$$

Equation (11) is representing the inventory level when there is no deterioration of items and inventory depends only on demand.

$$\frac{dI_{2p}(t)}{dt} + \theta(1-\xi)(t)I_{2p}(\boldsymbol{t}) = -D(t), \qquad t_0 \le t \le t_{k1} \quad (12)$$

Equation (12) is representing the inventory level when there is deterioration of items and inventory depends on both demand and deterioration.

$$\frac{dI_{3p}(t)}{dt} = -e^{-\mu(L-t)}D(t), \qquad t_{k1} \le t \le L \quad (13)$$

Equation (13) is representing the inventory level when there is shortages of items.

On solving equation (11) we get

$$I_{1p}(t) = I_1(t)$$

On solving equation (12) we get

$$I_{2p}(t) = e^{(1-\xi)}I_2(t)$$

On solving equation (13) we get

$$I_{3p}(t) = I_3(t)$$

$I_1(t)$, $I_2(t)$ and $I_3(t)$ are the inventory level of the without preservation technology inventory model.

Considering the model's assumptions, suppose the following different costs:

3.4. 1. The ordering cost per cycle is $\mathrm{C_o}^* = \mathrm{C_o} = \frac{O_c}{L}$

3.4. 2. The deterioration cost per cycle is $C_d{}^* = \frac{\theta_c}{L}\int_{t_0}^{t_{k1}} I_{2p}(t)dt$

$$C_d{}^* = \frac{\theta_c}{L} e^{(1-\xi)} B$$

3.4. 3. The inventory holding cost per cycle is

$$\mathrm{C_h}^* = \frac{H_c}{L}\left[\int_0^{t_0} I_{1p}(t)dt + \int_{t_0}^{t_{k1}} I_{2p}(t)dt\right]$$

$$C_h{}^* = \frac{H_c}{L}\left[A + e^{(1-\xi)}B\right]$$

3.4. 4. Backorder cost per cycle is: $\mathrm{C_b}^* = \mathrm{C_b} = \frac{B_c}{L}\int_{t_{k1}}^{L} -I_{3p}(t)dt$

Thus, the overall inventory cost is calculated as the sum of the ordering cost, degradation cost, holding cost, and backorder cost.

The total cost $C_t(t_{k1,}\mathrm{L}) = \mathrm{C_o}^* + C_d{}^* + \mathrm{C_h}^* + \mathrm{C_b}^*$

$$C_{tp}(t_{k1,}\mathrm{L}) = \frac{O_c}{L} + \frac{\theta_c}{L} e^{(1-\xi)} B + \frac{H_c}{L}\left[A + e^{(1-\xi)}B\right] + \frac{B_c}{L} C$$

$$C_{tp}(t_{k1,}\mathrm{L}) = \frac{O_c}{L} + \left(\frac{\theta_c}{L} + \frac{H_c}{L}\right) e^{(1-\xi)} B + A + \frac{B_c}{L} C \qquad (14)$$

Where,

$$A = a(t_{k1}t_0 - t_0{}^2) + \frac{b}{2}(t_{k1}{}^2 t_0 - t_0{}^3) + \frac{c}{3}(t_{k1}{}^3 t_0 - t_0{}^4) + \alpha\left(\frac{t_{k1}{}^{\beta+1}t_0 - t_0{}^{\beta+2}}{\beta+1} + \frac{t_{k1}{}^{\beta+2}t_0 - t_0{}^{\beta+3}}{\beta+2} + \frac{t_{k1}{}^{\beta+3}t_0 - t_0{}^{\beta+4}}{\beta+3}\right) - \alpha\left\{a\left(t_{k1}t_0{}^{\beta+1} - t_0{}^{\beta+2}\right) + \frac{b}{2}\left(t_{k1}{}^2 t_0{}^{\beta+1} - t_0{}^{\beta+3}\right) + \frac{c}{3}\left(t_{k1}{}^3 t_0{}^{\beta+1} - t_0{}^{\beta+4}\right)\right\}$$

And

$$B = \left\{\frac{at_{k1}{}^2}{2} + \frac{bt_{k1}{}^3}{3} + \frac{ct_{k1}{}^4}{4} - a\left(t_{k1}t_0 - \frac{t_0{}^2}{2}\right) - \frac{b}{2}\left(t_{k1}{}^2 t_0 - \frac{t_0{}^3}{3}\right) - \frac{c}{3}\left(t_{k1}{}^3 t_0 - \frac{t_0{}^4}{4}\right) + \alpha\left\{\left(\frac{t_{k1}{}^{\beta+2}}{\beta+2} + \frac{t_{k1}{}^{\beta+3}}{\beta+3} + \frac{t_{k1}{}^{\beta+4}}{\beta+4}\right) - \frac{1}{\beta+1}\left(\frac{at_{k1}{}^{\beta+2}}{\beta+2} + \frac{bt_{k1}{}^{\beta+3}}{\beta+3} + \frac{ct_{k1}{}^{\beta+4}}{\beta+4}\right) - t_0\left\{\frac{1}{\beta+1}\left(t_{k1}{}^{\beta+1} - \frac{t_0{}^{\beta+1}}{\beta+2}\right) + \frac{1}{\beta+2}\left(t_{k1}{}^{\beta+2} - \frac{t_0{}^{\beta+2}}{\beta+3}\right) + \frac{1}{\beta+3}\left(t_{k1}{}^{\beta+3} - \frac{t_0{}^{\beta+3}}{\beta+4}\right)\right\} + a\left(\frac{t_0{}^{\beta+1}t_{k1}}{\beta+1} - \frac{t_0{}^{\beta+2}}{(\beta+2)}\right) + \frac{b}{2}\left(\frac{t_0{}^{\beta+1}t_{k1}{}^2}{\beta+1} - \frac{t_0{}^{\beta+3}}{(\beta+3)}\right) + \frac{c}{3}\left(\frac{t_0{}^{\beta+1}t_{k1}{}^3}{\beta+1} - \frac{t_0{}^{\beta+4}}{(\beta+4)}\right)\right\}\right\}$$

$$C = \left[(1-\mu L)\left\{\frac{a}{2}(L^2 - t_{k1}{}^2) + \frac{b}{6}(L^3 - t_{k1}{}^3) + \frac{c}{12}(L^4 - t_{k1}{}^4)\right\} + \mu\left\{\frac{a}{6}(L^3 - t_{k1}{}^3) + \frac{b}{12}(L^4 - t_{k1}{}^4) + \frac{c}{20}(L^5 - t_{k1}{}^5)\right\}\right]$$

3. Solution Procedure

4.1. Solution Procedure (without preservation technology)

Our objective is to minimize total cost function $C_t(t_{k1}, L)$ subject to the decision variables t_{k1} and L. The values $t_{k1}{}^* and\ L^*$, for which the $C_t(t_{k1}, \mathrm{L})$ is minimum, are the solutions of equations. This optimization problem can be solved by the following solution procedure.

The necessary conditions for optimizing the $C_t(t_{k1}, \mathrm{L})$ are:

$$\frac{\partial C_t(t_{k1}, \mathrm{L})}{\partial t_{k1}} = 0 \tag{11}$$

$$\frac{\partial C_t(t_{k1}, \mathrm{L})}{\partial L} = 0 \tag{12}$$

And $\frac{\partial^2 C_t(t_{k1}, \mathrm{L})}{\partial^2 t_{k1}} > 0 , \frac{\partial^2 C_t(t_{k1}, \mathrm{L})}{\partial^2 L} > 0$ (13)

The convexity of the total cost function is obtained by the well – known Hessian matrix. Here, Hessian matrix of the total cost function is:

$$H(t_{k1}, \mathrm{L}) = \begin{vmatrix} \frac{\partial^2 C_t(t_{k1}, \mathrm{L})}{\partial^2 t_{k1}} & \frac{\partial^2 C_t(t_{k1}, \mathrm{L})}{\partial t_{k1} \partial L} \\ \frac{\partial^2 C_t(t_{k1}, \mathrm{L})}{\partial L \partial t_{k1}} & \frac{\partial^2 C_t(t_{k1}, \mathrm{L})}{\partial^2 L} \end{vmatrix} \tag{14}$$

Where the principal minor of $H(t_{k1}, \mathrm{L})$ are $H_1 > 0$ and $H_2 > 0$ which all are positive. Therefor the total inventory cost is convex.

or

$\left\{\left(\frac{\partial^2 C_t(t_{k1}, \mathrm{L})}{\partial^2 t_{k1}}\right) \times \left(\frac{\partial^2 C_t(t_{k1}, \mathrm{L})}{\partial^2 L}\right) - \left(\frac{\partial^2 C_t(t_{k1}, \mathrm{L})}{\partial L \partial t_{k1}}\right)^2\right\}$ >0 which is also called $H_2 > 0$.

Now using the equation (14), we get

This implies that the stationary point is the minimum. Using the software Mathematica-5.2, from these equations we can determine the optimum values of t_{k1} and L and total cost.

See the Appendix B for full solution procedure.

4.2. Solution Procedure (with preservation technology)

Our objective is to minimize total cost function $C_t(t_{k1}, L)$ subject to the decision variables t_{k1} and L. The values $t_{k1}{}^*\ and\ L^*$, for which the $C_t(t_{k1}, \mathrm{L})$ is minimum, are the solutions of equations. This optimization problem can be solved by the following solution procedure.

The necessary conditions for optimizing the $C_t(t_{k1}, \mathrm{L})$ are:

$$\frac{\partial C_{tp}(t_{k1}, \mathrm{L})}{\partial t_{k1}} = 0 \tag{15}$$

$$\frac{\partial C_{tp}(t_{k1}, \mathrm{L})}{\partial L} = 0 \tag{16}$$

And $\frac{\partial^2 C_{tp}(t_{k1}, \mathrm{L})}{\partial^2 t_{k1}} > 0 , \frac{\partial^2 C_{tp}(t_{k1}, \mathrm{L})}{\partial^2 L} > 0$ (17)

The convexity of the total cost function is obtained by the well – known Hessian matrix. Here, Hessian matrix of the total cost function is:

$$H(t_{k1},\mathrm{L}) = \begin{vmatrix} \frac{\partial^2 C_{tp}(t_{k1},\mathrm{L})}{\partial^2 t_{k1}} & \frac{\partial^2 C_{tp}(t_{k1},\mathrm{L})}{\partial t_{k1}\partial L} \\ \frac{\partial^2 C_{tp}(t_{k1},\mathrm{L})}{\partial L \partial t_{k1}} & \frac{\partial^2 C_{tp}(t_{k1},\mathrm{L})}{\partial^2 L} \end{vmatrix} \tag{18}$$

Where the principal minor of $H(t_{k1},\mathrm{L})$ are $H_1 > 0$ and $H_2 > 0$ which all are positive. Therefore, the total inventory cost is convex.

or

$\left\{\left(\frac{\partial^2 C_{tp}(t_{k1},\mathrm{L})}{\partial^2 t_{k1}}\right) \times \left(\frac{\partial^2 C_{tp}(t_{k1},\mathrm{L})}{\partial^2 L}\right) - \left(\frac{\partial^2 C_{tp}(t_{k1},\mathrm{L})}{\partial L \partial t_{k1}}\right)^2\right\}$ >0 which is also called $H_2 > 0$.

Now using the equation (14), we get

This implies that the stationary point is the minimum. Using the software Mathematica-5.2, from these equations we can determine the optimum values of ξ and L and total cost.

See the Appendix C for full solution procedure.

4. Algorithm

Step 1: Develop the differential equations of inventory levels for different time periods that is No Deterioration time period, Deterioration time period and shortages time period.

Step 2: Find inventory levels for without and with preservation technology i.e. $I_1(t)$, $I_2(t)$, $I_3(t)$ and $I_{1p}(t)$, $I_{2p}(t)$, $I_{3p}(t)$ respectively.

Step 3: Calculate various type of costs like ordering cost C_O and $C_O{}^*$, Deterioration cost C_d and $C_d{}^*$, Holding cost C_h and $C_h{}^*$, Backorder Cost C_b and $C_b{}^*$.

Step 4: Calculate total cost $TC = C_O + C_d + C_h + C_b$ and $TC = C_O{}^* + C_d{}^* + C_h{}^* + C_b{}^*$.

Step 5: Find derivatives $\frac{\partial \mathrm{C}_t(t_{k1},L)}{\partial L}$, $\frac{\partial \mathrm{C}_t(t_{k1},L)}{\partial t_{k1}}$, $\frac{\partial^2 \mathrm{C}_t(t_{k1},L)}{\partial t_{k1}{}^2}$, $\frac{\partial^2 \mathrm{C}_t(t_{k1},L)}{\partial L^2}$, $\frac{\partial^2 \mathrm{C}_t(t_{k1},L)}{\partial L^2 \partial t_{k1}{}^2}$ and $\frac{\partial \mathrm{C}_{tp}(t_{k1},L)}{\partial L}$, $\frac{\partial \mathrm{C}_{tp}(t_{k1},L)}{\partial t_{k1}}$, $\frac{\partial^2 \mathrm{C}_{tp}(t_{k1},L)}{\partial t_{k1}{}^2}$, $\frac{\partial^2 \mathrm{C}_{tp}(t_{k1},L)}{\partial L^2}$, $\frac{\partial^2 \mathrm{C}_{tp}(t_{k1},L)}{\partial L^2 \partial t_{k1}{}^2}$

Step 6: Check the optimality of objective function by the well – known Hessian matrix.

Step-7: Sensitivity analysis of total cost function by changing the various parameters. One parameter is changed at a time while the other parameters are left unchanged.

Step-8: Explain the observations obtained by sensitivity analysis.

Numerical Example and Sensitivity Analysis

We use a hypothetical example with values in this section. An inventory mechanism with the hypothetical values displayed below is examined to demonstrate the suggested idea. By taking $O_c = 200$, $\theta_c = 4, H_c = 5.5$, $B_c = 3, \alpha = 0.80, \beta = 1, a = 100, b = 50, c = 2, t_0 = 0.2, \mu = 0.08, \xi = 0.002$. The optimal values $t_{k1}{}^* = 0.088814\,units, L^* = 0.532434 units$ and the optimal average total cost $\mathrm{C_t} = 459.729$

Using the provided data in the right units, the aforementioned model serves as an example. The MATLAB R2015a programme is being used to solve the model. We looked at the ideal solution's sensitivity analysis.

Table 1 Sensitivity analysis of Parameters with t_{k1} Cycle Length and Total Optimal Cost without using preservation technology:

Parameter	% Change in Parameter	t_{k1}	L	Total Cost (C_t)
θ_c	4.1	0.088801	1.532434	459.8144
	4.2	0.088788	1.532428	459.8997
	4.3	0.088775	1.532422	459.9967
	4.4	0.088762	1.532416	460.1235
	4.5	0.088749	1.532410	460.1891
H_c	5.6	0.211491	1.658880	497.5417
	5.7	0.211500	1.658971	497.5671
	5.8	0.211508	1.659053	497.5742
	5.9	0.211515	1.659125	497.6067
	6	0.211521	1.659186	497.6447
α	0.81	0.195924	2.084810	456.2999
	0.82	0.195936	2.084940	456.4056
	0.83	0.195951	2.084956	456.4992
	0.84	0.195967	2.084974	456.5929
	0.85	0.195984	2.084992	456.6864
a	210	0.088735	1.532397	372.4004
	220	0.088753	1.532416	394.6049
	230	0.088772	1.532436	417.6147
	240	0.088792	1.532457	438.1748
	250	0.088813	1.532480	460.5146
μ	0.081	0.097779	1.834810	420.7069
	0.082	0.097700	1.835022	420.8415
	0.083	0.097822	1.835240	420.8614
	0.084	0.097845	1.835471	420.9461
	0.085	0.097869	1.835713	420.9734

Table 2 Sensitivity analysis of Parameters with t_{k1} Cycle Length and Total Optimal Cost with using preservation technology:

Parameter	% Change in Parameter	t_{k1}	L	Total Cost (C_t)
θ_c	4.1	0.097265	1.964337	403.0821
	4.2	0.097251	1.964330	403.1364
	4.3	0.097239	1.964323	403.1915
	4.4	0.097224	1.964316	403.2336
	4.5	0.097210	1.964307	403.2767
	5.6	0.219955	2.090783	442.6229

H_c	5.7	0.219946	2.090868	442.8682
	5.8	0.219935	2.090965	442.9936
	5.9	0.219922	2.091061	443.0325
	6	0.219912	2.091148	443.0518
α	0.81	0.204405	2.516713	428.9662
	0.82	0.204417	2.516729	429.2462
	0.83	0.204432	2.516743	429.5222
	0.84	0.204481	2.516756	429.7912
	0.85	0.202590	2.516768	430.0622
a	210	0.097221	1.963809	252.8652
	220	0.097240	1.963828	272.2076
	230	0.097260	1.963849	290.5571
	240	0.097269	1.938660	307.7256
	250	0.097284	1.938843	322.9746
μ	0.081	0.106265	2.241176	383.9746
	0.082	0.106345	2.243383	384.1092
	0.083	0.106427	2.245645	384.2335
	0.084	0.106508	2.248006	384.3677
	0.085	0.105872	2.210198	384.5003

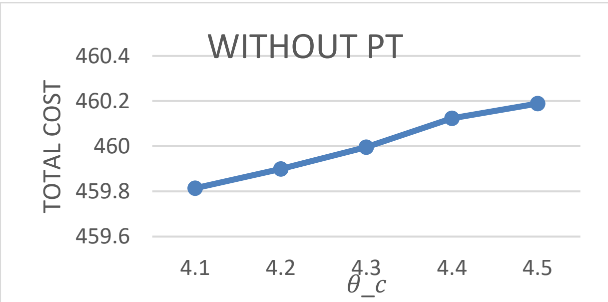

Figure 2. Sensitivity graph with respect to deterioration rate θ_c and total cost without using Preservation Technology

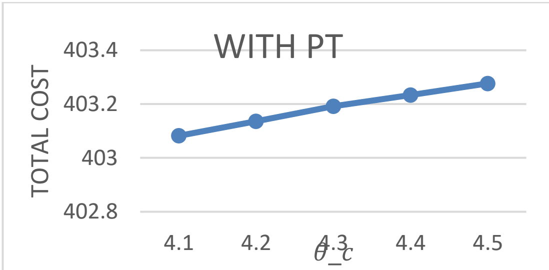

Figure 3. Sensitivity graph with respect to deterioration rate θ_c and total cost with using Preservation Technology

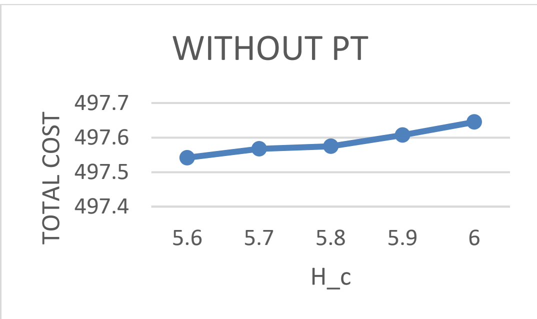

Figure-4: Sensitivity graph with respect to holding cost H_c and total cost without using Preservation Technology

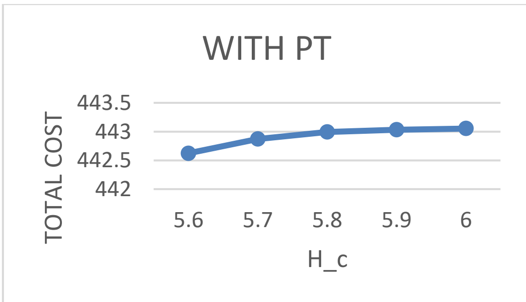

Figure-5: Sensitivity graph with respect to holding cost H_c and total cost with using Preservation Technology

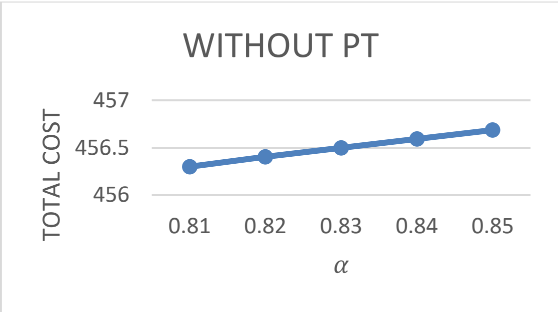

Figure-6: Sensitivity graph with respect to α and total cost without using Preservation Technology

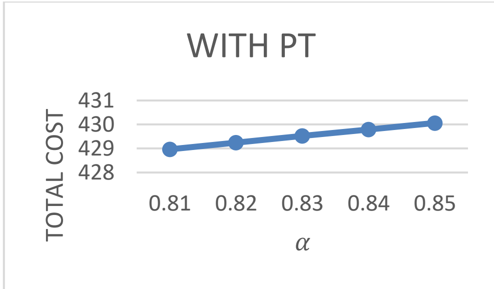

Figure-7: Sensitivity graph with respect to α and total cost with using Preservation Technology

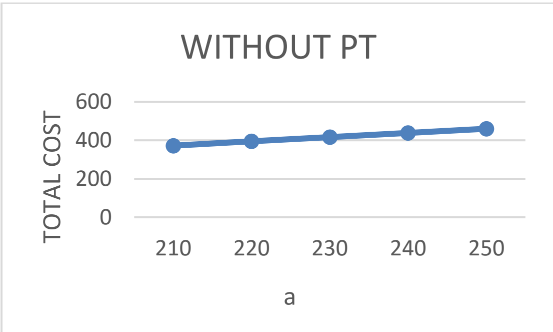

Figure-8: Sensitivity graph with respect to a (Demand) and total cost without using Preservation Technology

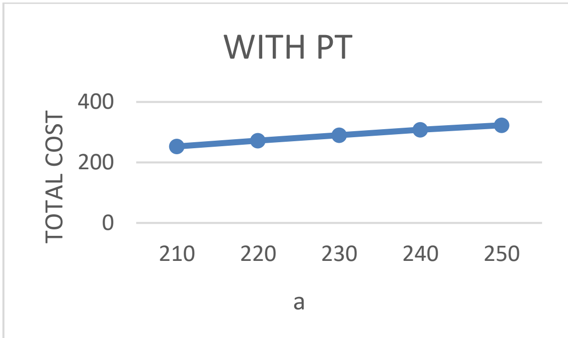

Figure-9: Sensitivity graph with respect to a (Demand) and total cost with using Preservation Technology

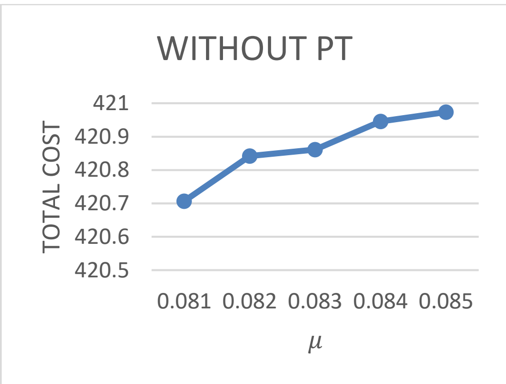

Figure-10: Sensitivity graph with respect to μ (Backorder parameter) and total cost without using Preservation Technology

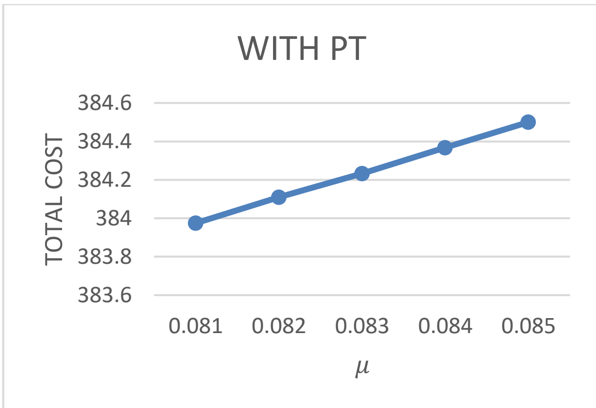

Figure-11: Sensitivity graph with respect to μ (Backorder parameter) and total cost with using Preservation Technology

This analysis is carried out both with and without the use of preservation technology, highlighting the effects of parameter variations under various conditions. The sensitivity analysis tables that are provided insightful information about how changes in different parameters impact important aspects of a system, specifically regarding (t_{k1}), cycle length (L), and total cost (C_t). θ_c, H_c, α, a, and μ are among the parameters that were investigated. A range of percentage adjustments is applied to each parameter, and the differences in t_{k1} and total cost that occur are noted. When θc is examined, for example, a percentage change from 4.1 to 4.5 results in a small change in t_{k1} but a significant effect on total cost, suggesting that this parameter is sensitive. Variations in H_c, α, a, and μ likewise show their individual effects on the cost and performance of the system.

However, the sensitivity analysis is also carried out with the presumption that preservation technology is being used. As a result, new conditions are introduced that may have an impact on the dynamics of the system. An enhanced comprehension of how preservation technique modifies the system's sensitivity to parameter variations can be gained by contrasting the outcomes of the two scenarios. As an example, the effect of θ_c on t_{k1} and total cost varies depending on whether preservation technology is used or not, demonstrating the technology's function in reducing some sensitivities.

Trends across parameters are also revealed by the analysis. Changes in α, for instance, have a greater impact on overall cost than do changes in t_{k1}, suggesting that α is important for cost optimisation. Furthermore, the parameter "a" shows a major impact on the overall cost, as demonstrated by the notable cost variations noted for various values of "a" in the absence of preservation technology. Furthermore, the system's sensitivity to changes in parameters emphasises how crucial it is to optimise these parameters throughout system design or operation. The statement emphasises the necessity of giving serious thought to the intended performance indicators, like cycle length, length, and cost efficiency, and making possible iterative improvements. The disparate outcomes with and without preservation technology further highlight the advantages and disadvantages of integrating such technologies into the system. To sum up, the sensitivity analysis offers a thorough understanding of how different factors affect system performance and cost in diverse scenarios, illuminating optimisation tactics and the function of preservation technologies in reducing sensitivities. This analysis is a useful tool for system design and decision-making when it comes to maximising workflow and resource efficiency.

5. Conclusion

The thorough analysis provided by the paper emphasizes how important preservation technologies are to inventory management, especially for goods that have quadratic demand and delayed deterioration according to a Weibull distribution. The study finds considerable cost savings by comparing scenarios with and without preservation technologies. These savings are related to longer shelf lives and less expensive effects of product deterioration. Sensitivity analysis of parameters such as θ, H, α, a, and μ highlights how important it is to optimise parameters for cost effectiveness, with "a" and α being highlighted as important variables. In order to increase workflow and resource efficiency, the study recommends strategic examination of performance indicators and iterative modifications. In the end, the research shows that preservation technologies are critical instruments for mitigating sensitivities, enhancing system efficiency, and cultivating long-term inventory management practices. These findings provide insightful information for companies looking to improve their inventory management strategies.

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Conflict of interest

The authors declare no conflict of interest.

Author Contribution

K. wrote and edited the original draft. K. analyzed the model. K. and A. K. S. did the software work. A. K. S. supervised the research.

Appendix A.

$$C_d=\frac{\theta_c}{L}\left[\frac{at_{k1}^2}{2}+\frac{bt_{k1}^3}{3}+\frac{ct_{k1}^4}{4}-a\left(t_{k1}t_0-\frac{t_0^2}{2}\right)-\frac{b}{2}\left(t_{k1}^2t_0-\frac{t_0^3}{3}\right)-\frac{c}{3}\left(t_{k1}^3t_0-\frac{t_0^4}{4}\right)+\alpha\left\{\left(\frac{t_{k1}^{\beta+2}}{\beta+2}+\frac{t_{k1}^{\beta+3}}{\beta+3}+\frac{t_{k1}^{\beta+4}}{\beta+4}\right)-\frac{1}{\beta+1}\left(\frac{at_{k1}^{\beta+2}}{\beta+2}+\frac{bt_{k1}^{\beta+3}}{\beta+3}+\frac{ct_{k1}^{\beta+4}}{\beta+4}\right)-t_0\left\{\frac{1}{\beta+1}\left(t_{k1}^{\beta+1}-\frac{t_0^{\beta+1}}{\beta+2}\right)+\frac{1}{\beta+2}\left(t_{k1}^{\beta+2}-\frac{t_0^{\beta+2}}{\beta+3}\right)+\frac{1}{\beta+3}\left(t_{k1}^{\beta+3}-\frac{t_0^{\beta+3}}{\beta+4}\right)\right\}+a\left(\frac{t_0^{\beta+1}t_{k1}}{\beta+1}-\frac{t_0^{\beta+2}}{(\beta+2)}\right)+\frac{b}{2}\left(\frac{t_0^{\beta+1}t_{k1}^2}{\beta+1}-\frac{t_0^{\beta+3}}{(\beta+3)}\right)+\frac{c}{3}\left(\frac{t_0^{\beta+1}t_{k1}^3}{\beta+1}-\frac{t_0^{\beta+4}}{(\beta+4)}\right)\right\}\right] \quad \text{(A1)}$$

$$C_h=\frac{H_c}{L}\left[a(t_{k1}t_0-t_0^2)+\frac{b}{2}(t_{k1}^2t_0-t_0^3)+\frac{c}{3}(t_{k1}^3t_0-t_0^4)+\alpha\left(\frac{t_{k1}^{\beta+1}t_0-t_0^{\beta+2}}{\beta+1}+\frac{t_{k1}^{\beta+2}t_0-t_0^{\beta+3}}{\beta+2}+\frac{t_{k1}^{\beta+3}t_0-t_0^{\beta+4}}{\beta+3}\right)-\alpha\left\{a\left(t_{k1}t_0^{\beta+1}-t_0^{\beta+2}\right)+\frac{b}{2}\left(t_{k1}^2t_0^{\beta+1}-t_0^{\beta+3}\right)+\frac{c}{3}\left(t_{k1}^3t_0^{\beta+1}-t_0^{\beta+4}\right)\right\}+\frac{at_{k1}^2}{2}+\frac{bt_{k1}^3}{3}+\frac{ct_{k1}^4}{4}-a\left(t_{k1}t_0-\frac{t_0^2}{2}\right)-\frac{b}{2}\left(t_{k1}^2t_0-\frac{t_0^3}{3}\right)-\frac{c}{3}\left(t_{k1}^3t_0-\frac{t_0^4}{4}\right)+\alpha\left\{\left(\frac{t_{k1}^{\beta+2}}{\beta+2}+\frac{t_{k1}^{\beta+3}}{\beta+3}+\frac{t_{k1}^{\beta+4}}{\beta+4}\right)-\frac{1}{\beta+1}\left(\frac{at_{k1}^{\beta+2}}{\beta+2}+\frac{bt_{k1}^{\beta+3}}{\beta+3}+\frac{ct_{k1}^{\beta+4}}{\beta+4}\right)-t_0\left\{\frac{1}{\beta+1}\left(t_{k1}^{\beta+1}-\frac{t_0^{\beta+1}}{\beta+2}\right)+\frac{1}{\beta+2}\left(t_{k1}^{\beta+2}-\frac{t_0^{\beta+2}}{\beta+3}\right)+\frac{1}{\beta+3}\left(t_{k1}^{\beta+3}-\frac{t_0^{\beta+3}}{\beta+4}\right)\right\}+a\left(\frac{t_0^{\beta+1}t_{k1}}{\beta+1}-\frac{t_0^{\beta+2}}{(\beta+2)}\right)+\frac{b}{2}\left(\frac{t_0^{\beta+1}t_{k1}^2}{\beta+1}-\frac{t_0^{\beta+3}}{(\beta+3)}\right)+\frac{c}{3}\left(\frac{t_0^{\beta+1}t_{k1}^3}{\beta+1}-\frac{t_0^{\beta+4}}{(\beta+4)}\right)\right\}\right]$$
(A2)

$$C_b=\frac{B_c}{L}\left[(1-\mu L)\left\{\frac{a}{2}(L^2-t_{k1}^2)+\frac{b}{6}(L^3-t_{k1}^3)+\frac{c}{12}(L^4-t_{k1}^4)\right\}+\mu\left\{\frac{a}{6}(L^3-t_{k1}^3)+\frac{b}{12}(L^4-t_{k1}^4)+\frac{c}{20}(L^5-t_{k1}^5)\right\}\right] \quad \text{(A3)}$$

Appendix B.

$$\frac{\partial C_t(t_{k1},\mathrm{L})}{\partial L} = -\frac{O_c}{L^2} - \left(\frac{H_c}{L^2}+\frac{\theta_c}{L^2}\right)\left[\frac{at_{k1}{}^2}{2}+\frac{bt_{k1}{}^3}{3}+\frac{ct_{k1}{}^4}{4} - a\left(t_{k1}t_0-\frac{t_0{}^2}{2}\right)-\frac{b}{2}\left(t_{k1}{}^2t_0-\frac{t_0{}^3}{3}\right)-\frac{c}{3}\left(t_{k1}{}^3t_0-\frac{t_0{}^4}{4}\right) + \alpha\left\{\left(\frac{t_{k1}{}^{\beta+2}}{\beta+2}+\frac{t_{k1}{}^{\beta+3}}{\beta+3}+\frac{t_{k1}{}^{\beta+4}}{\beta+4}\right)-\frac{1}{\beta+1}\left(\frac{at_{k1}{}^{\beta+2}}{\beta+2}+\frac{bt_{k1}{}^{\beta+3}}{\beta+3}+\frac{ct_{k1}{}^{\beta+4}}{\beta+4}\right)-t_0\left\{\frac{1}{\beta+1}\left(t_{k1}{}^{\beta+1}-\frac{t_0{}^{\beta+1}}{\beta+2}\right)+\frac{1}{\beta+2}\left(t_{k1}{}^{\beta+2}-\frac{t_0{}^{\beta+2}}{\beta+3}\right)+\frac{1}{\beta+3}\left(t_{k1}{}^{\beta+3}-\frac{t_0{}^{\beta+3}}{\beta+4}\right)\right\}+a\left(\frac{t_0{}^{\beta+1}t_{k1}}{\beta+1}-\frac{t_0{}^{\beta+2}}{(\beta+2)}\right)+\frac{b}{2}\left(\frac{t_0{}^{\beta+1}t_{k1}{}^2}{\beta+1}-\frac{t_0{}^{\beta+3}}{(\beta+3)}\right)+\frac{c}{3}\left(\frac{t_0{}^{\beta+1}t_{k1}{}^3}{\beta+1}-\frac{t_0{}^{\beta+4}}{(\beta+4)}\right)\right\}\right] - \frac{H_c}{L^2}\left[a(t_{k1}t_0-t_0{}^2)+\frac{b}{2}(t_{k1}{}^2t_0-t_0{}^3)+\frac{c}{3}(t_{k1}{}^3t_0-t_0{}^4)+\alpha\left(\frac{t_{k1}{}^{\beta+1}t_0-t_0{}^{\beta+2}}{\beta+1}+\frac{t_{k1}{}^{\beta+2}t_0-t_0{}^{\beta+3}}{\beta+2}+\frac{t_{k1}{}^{\beta+3}t_0-t_0{}^{\beta+4}}{\beta+3}\right)-\alpha\left\{a\left(t_{k1}t_0{}^{\beta+1}-t_0{}^{\beta+2}\right)+\frac{b}{2}\left(t_{k1}{}^2t_0{}^{\beta+1}-t_0{}^{\beta+3}\right)+\frac{c}{3}\left(t_{k1}{}^3t_0{}^{\beta+1}-t_0{}^{\beta+4}\right)\right\}\right] - \frac{B_c}{L^2}\left[(1-\mu L)\left\{\frac{a}{2}(L^2-t_{k1}{}^2)+\frac{b}{6}(L^3-t_{k1}{}^3)+\frac{c}{12}(L^4-t_{k1}{}^4)\right\}+\mu\left\{\frac{a}{6}(L^3-t_{k1}{}^3)+\frac{b}{12}(L^4-t_{k1}{}^4)+\frac{c}{20}(L^5-t_{k1}{}^5)\right\}\right]+\frac{B_c}{L}\left[(1-\mu L)\left\{aL+\frac{bL^2}{2}+\frac{cL^3}{3}\right\}-\mu\left\{\frac{a}{2}(L^2-t_{k1}{}^2)+\frac{b}{6}(L^3-t_{k1}{}^3)+\frac{c}{12}(L^4-t_{k1}{}^4)\right\}+\mu\left\{\frac{aL^2}{2}+\frac{bL^3}{3}+\frac{cL^4}{4}\right\}\right]$$

(B1)

and

$$\frac{\partial C_t(t_{k1},\mathrm{L})}{\partial t_{k1}} = \left(\frac{H_c}{L}+\frac{\theta_c}{L}\right)\left[at_{k1}+bt_{k1}{}^2+ct_{k1}{}^3-t_0(a+bt_{k1}+ct_{k1}{}^2) + \alpha\left\{\left(t_{k1}{}^{\beta+1}+t_{k1}{}^{\beta+2}+t_{k1}{}^{\beta+3}\right)-\frac{1}{\beta+1}\left(at_{k1}{}^{\beta+1}+bt_{k1}{}^{\beta+2}+ct_{k1}{}^{\beta+3}\right)-t_0\left\{t_{k1}{}^{\beta}+t_{k1}{}^{\beta+1}+t_{k1}{}^{\beta+2}\right\}+\frac{t_0{}^{\beta+1}}{\beta+1}(a+bt_{k1}+ct_{k1}{}^2)\right\}\right]+\frac{H_c}{L}\left[t_0(a+bt_{k1}+ct_{k1}{}^2)+\alpha t_0\left\{t_{k1}{}^{\beta}+t_{k1}{}^{\beta+1}+t_{k1}{}^{\beta+2}\right\}-\alpha t_0{}^{\beta+1}(a+bt_{k1}+ct_{k1}{}^2)\right]-\frac{B_c}{L}\left[(1-\mu L)\left\{at_{k1}+\frac{bt_{k1}{}^2}{2}+\frac{ct_{k1}{}^3}{3}\right\}+\mu\left\{\frac{at_{k1}{}^2}{2}+\frac{bt_{k1}{}^3}{3}+\frac{ct_{k1}{}^4}{4}\right\}\right]$$

(B2)

Also

$$\frac{\partial^2 C_t(t_{k1},\mathrm{L})}{\partial^2 L} = \frac{2O_c}{L^3}+2\left(\frac{H_c}{L^3}+\frac{\theta_c}{L^3}\right)\left[\frac{at_{k1}{}^2}{2}+\frac{bt_{k1}{}^3}{3}+\frac{ct_{k1}{}^4}{4} - a\left(t_{k1}t_0-\frac{t_0{}^2}{2}\right)-\frac{b}{2}\left(t_{k1}{}^2t_0-\frac{t_0{}^3}{3}\right)-\frac{c}{3}\left(t_{k1}{}^3t_0-\frac{t_0{}^4}{4}\right) + \alpha\left\{\left(\frac{t_{k1}{}^{\beta+2}}{\beta+2}+\frac{t_{k1}{}^{\beta+3}}{\beta+3}+\frac{t_{k1}{}^{\beta+4}}{\beta+4}\right)-\frac{1}{\beta+1}\left(\frac{at_{k1}{}^{\beta+2}}{\beta+2}+\frac{bt_{k1}{}^{\beta+3}}{\beta+3}+\frac{ct_{k1}{}^{\beta+4}}{\beta+4}\right)-t_0\left\{\frac{1}{\beta+1}\left(t_{k1}{}^{\beta+1}-\frac{t_0{}^{\beta+1}}{\beta+2}\right)+\frac{1}{\beta+2}\left(t_{k1}{}^{\beta+2}-\frac{t_0{}^{\beta+2}}{\beta+3}\right)+\frac{1}{\beta+3}\left(t_{k1}{}^{\beta+3}-\frac{t_0{}^{\beta+3}}{\beta+4}\right)\right\}+a\left(\frac{t_0{}^{\beta+1}t_{k1}}{\beta+1}-\frac{t_0{}^{\beta+2}}{(\beta+2)}\right)+\frac{b}{2}\left(\frac{t_0{}^{\beta+1}t_{k1}{}^2}{\beta+1}-\frac{t_0{}^{\beta+3}}{(\beta+3)}\right)+\frac{c}{3}\left(\frac{t_0{}^{\beta+1}t_{k1}{}^3}{\beta+1}-\frac{t_0{}^{\beta+4}}{(\beta+4)}\right)\right\}\right] + \frac{H_c}{L^3}\left[a(t_{k1}t_0-t_0{}^2)+\frac{b}{2}(t_{k1}{}^2t_0-t_0{}^3)+\frac{c}{3}(t_{k1}{}^3t_0-t_0{}^4)+\alpha\left(\frac{t_{k1}{}^{\beta+1}t_0-t_0{}^{\beta+2}}{\beta+1}+\frac{t_{k1}{}^{\beta+2}t_0-t_0{}^{\beta+3}}{\beta+2}+\frac{t_{k1}{}^{\beta+3}t_0-t_0{}^{\beta+4}}{\beta+3}\right)-\alpha\left\{a\left(t_{k1}t_0{}^{\beta+1}-t_0{}^{\beta+2}\right)+\right.$$

$\frac{b}{2}\left({t_{k1}}^2{t_0}^{\beta+1}-{t_0}^{\beta+3}\right)+\frac{c}{3}\left({t_{k1}}^3{t_0}^{\beta+1}-{t_0}^{\beta+4}\right)\}\Big]-\frac{B_c}{L^2}\Big[(1-\mu L)\left(aL+\frac{bL^2}{2}+\frac{cL^3}{3}\right)+(\mu-2L)\left\{\frac{a}{2}(L^2-{t_{k1}}^2)+\frac{b}{6}(L^3-{t_{k1}}^3)+\frac{c}{12}(L^4-{t_{k1}}^4)\right\}+\mu\left\{\frac{aL^2}{2}+\frac{bL^3}{3}+\frac{cL^4}{4}-\frac{2}{L}\left(\frac{a}{6}(L^3-{t_{k1}}^3)+\frac{b}{12}(L^4-{t_{k1}}^4)+\frac{c}{20}(L^5-{t_{k1}}^5)\right)\right\}\Big]+\frac{B_c}{L}\Big[(1-\mu L)\{a+bL+cL^2\}+\frac{\mu}{L}\left\{\frac{a}{2}(L^2-{t_{k1}}^2)+\frac{b}{6}(L^3-{t_{k1}}^3)+\frac{c}{12}(L^4-{t_{k1}}^4)-\left\{aL+\frac{bL^2}{2}+\frac{cL^3}{3}\right\}+L\left(\frac{a}{2}+\frac{2bL}{3}+\frac{3cL^2}{4}\right)\right\}\Big]$

(B3)

Equation (B3) is greater than zero as O_c, H_c, θ_c and B_c are the positive values and L is greater than t_{k1} which gives the positive value.

$\frac{\partial^2 C_t(t_{k1},\mathrm{L})}{\partial^2 t_{k1}}=\left(\frac{H_c}{L}+\frac{\theta_c}{L}\right)\Big[a+2bt_{k1}+3c{t_{k1}}^2-t_0(b+2ct_{k1})+\alpha\Big\{1+2t_{k1}+3{t_{k1}}^2-\frac{1}{\beta+1}\left(a(\beta+1){t_{k1}}^{\beta}+b(\beta+2){t_{k1}}^{\beta+1}+c(\beta+3){t_{k1}}^{\beta+2}\right)-t_0\{\beta{t_{k1}}^{\beta-1}+(\beta+1){t_{k1}}^{\beta}+(\beta+2){t_{k1}}^{\beta+1}\}+\frac{{t_0}^{\beta+1}}{\beta+1}(b+ct_{k1})\Big\}\Big]+\frac{H_c}{L}\big[t_0(b+ct_{k1}+)+\alpha t_0(\beta{t_{k1}}^{\beta-1}+(\beta+1){t_{k1}}^{\beta}+(\beta+2){t_{k1}}^{\beta+1})-\alpha{t_0}^{\beta+1}\{b+ct_{k1}\}\big]+\frac{B_c}{L}[(\mu L-1)\{a+bt_{k1}+c{t_{k1}}^2\}-\mu\{at_{k1}+b{t_{k1}}^2+c{t_{k1}}^3\}]$

(B4)

(B4) is greater than zero as all the terms are the positive and L is greater than t_{k1} which gives the positive value.

$\frac{\partial^2 C_t(t_{k1},L)}{\partial L\partial t_{k1}}=\frac{\partial^2 C_t(t_{k1},\mathrm{L})}{\partial t_{k1}\partial L}=-\left(\frac{H_c}{L^2}+\frac{\theta_c}{L^2}\right)\Big[at_{k1}+b{t_{k1}}^2+c{t_{k1}}^3-t_0(a+bt_{k1}+c{t_{k1}}^2)+\alpha\Big\{t_{k1}+{t_{k1}}^2+{t_{k1}}^3-\frac{1}{\beta+1}\left(a{t_{k1}}^{\beta+1}+b{t_{k1}}^{\beta+2}+c{t_{k1}}^{\beta+3}\right)-t_0\{{t_{k1}}^{\beta}+{t_{k1}}^{\beta+1}+{t_{k1}}^{\beta+2}\}+\frac{{t_0}^{\beta+1}}{\beta+1}(a+bt_{k1}+c{t_{k1}}^2)\Big\}\Big]-\frac{H_c}{L^2}\big[t_0(a+bt_{k1}+c{t_{k1}}^2+)+\alpha t_0({t_{k1}}^{\beta}+{t_{k1}}^{\beta+1}+{t_{k1}}^{\beta+2})-\alpha{t_0}^{\beta+1}\{a+bt_{k1}+c{t_{k1}}^2\}\big]-\frac{B_c}{L^2}\Big[(\mu L-1)\left\{at_{k1}+\frac{b{t_{k1}}^2}{2}+\frac{c{t_{k1}}^3}{3}\right\}-\mu\left\{\frac{a{t_{k1}}^2}{2}+\frac{b{t_{k1}}^3}{3}+\frac{c{t_{k1}}^4}{4}\right\}\Big]+\frac{B_c}{L}\Big[\mu\left\{at_{k1}+\frac{b{t_{k1}}^2}{2}+\frac{c{t_{k1}}^3}{3}\right\}\Big]$ (B5)

On squaring (B5) we will get the positive value, as the square of any negative value will be positive.

Appendix C.

$\frac{\partial C_t(t_{k1},\mathrm{L})}{\partial L}=-\frac{O_c}{L^2}-e^{(1-\xi)}\left(\frac{H_c}{L^2}+\frac{\theta_c}{L^2}\right)\Big[\frac{a{t_{k1}}^2}{2}+\frac{b{t_{k1}}^3}{3}+\frac{c{t_{k1}}^4}{4}-a\left(t_{k1}t_0-\frac{{t_0}^2}{2}\right)-\frac{b}{2}\left({t_{k1}}^2t_0-\frac{{t_0}^3}{3}\right)-\frac{c}{3}\left({t_{k1}}^3t_0-\frac{{t_0}^4}{4}\right)+\alpha\Big\{\left(\frac{{t_{k1}}^{\beta+2}}{\beta+2}+\frac{{t_{k1}}^{\beta+3}}{\beta+3}+\frac{{t_{k1}}^{\beta+4}}{\beta+4}\right)-\frac{1}{\beta+1}\left(\frac{a{t_{k1}}^{\beta+2}}{\beta+2}+\frac{b{t_{k1}}^{\beta+3}}{\beta+3}+\frac{c{t_{k1}}^{\beta+4}}{\beta+4}\right)-$

$t_0\left\{\frac{1}{\beta+1}\left(t_{k1}{}^{\beta+1}-\frac{t_0{}^{\beta+1}}{\beta+2}\right)+\frac{1}{\beta+2}\left(t_{k1}{}^{\beta+2}-\frac{t_0{}^{\beta+2}}{\beta+3}\right)+\frac{1}{\beta+3}\left(t_{k1}{}^{\beta+3}-\frac{t_0{}^{\beta+3}}{\beta+4}\right)\right\}+a\left(\frac{t_0{}^{\beta+1}t_{k1}}{\beta+1}-\frac{t_0{}^{\beta+2}}{(\beta+2)}\right)+$
$\frac{b}{2}\left(\frac{t_0{}^{\beta+1}t_{k1}{}^2}{\beta+1}-\frac{t_0{}^{\beta+3}}{(\beta+3)}\right)+\frac{c}{3}\left(\frac{t_0{}^{\beta+1}t_{k1}{}^3}{\beta+1}-\frac{t_0{}^{\beta+4}}{(\beta+4)}\right)\Big\}\Big]-\frac{H_c}{L^2}\Big[a(t_{k1}t_0-t_0{}^2)+\frac{b}{2}(t_{k1}{}^2t_0-t_0{}^3)+\frac{c}{3}(t_{k1}{}^3t_0-$
$t_0{}^4)+\alpha\left(\frac{t_{k1}{}^{\beta+1}t_0-t_0{}^{\beta+2}}{\beta+1}+\frac{t_{k1}{}^{\beta+2}t_0-t_0{}^{\beta+3}}{\beta+2}+\frac{t_{k1}{}^{\beta+3}t_0-t_0{}^{\beta+4}}{\beta+3}\right)-\alpha\Big\{a\left(t_{k1}t_0{}^{\beta+1}-t_0{}^{\beta+2}\right)+$
$\frac{b}{2}\left(t_{k1}{}^2t_0{}^{\beta+1}-t_0{}^{\beta+3}\right)+\frac{c}{3}\left(t_{k1}{}^3t_0{}^{\beta+1}-t_0{}^{\beta+4}\right)\Big\}\Big]-\frac{B_c}{L^2}\Big[(1-\mu L)\Big\{\frac{a}{2}(L^2-t_{k1}{}^2)+\frac{b}{6}(L^3-t_{k1}{}^3)+$
$\frac{c}{12}(L^4-t_{k1}{}^4)\Big\}+\mu\left\{\frac{a}{6}(L^3-t_{k1}{}^3)+\frac{b}{12}(L^4-t_{k1}{}^4)+\frac{c}{20}(L^5-t_{k1}{}^5)\right\}\Big]+\frac{B_c}{L}\Big[(1-\mu L)\Big\{aL+\frac{bL^2}{2}+$
$\frac{cL^3}{3}\Big\}-\mu\left\{\frac{a}{2}(L^2-t_{k1}{}^2)+\frac{b}{6}(L^3-t_{k1}{}^3)+\frac{c}{12}(L^4-t_{k1}{}^4)\right\}+\mu\left\{\frac{aL^2}{2}+\frac{bL^3}{3}+\frac{cL^4}{4}\right\}\Big]$
(C1)

and

$\frac{\partial C_t(t_{k1},\mathrm{L})}{\partial t_{k1}}=e^{(1-\xi)}\left(\frac{H_c}{L}+\frac{\theta_c}{L}\right)\Big[at_{k1}+bt_{k1}{}^2+ct_{k1}{}^3-t_0(a+bt_{k1}+ct_{k1}{}^2)+\alpha\Big\{\big(t_{k1}{}^{\beta+1}+$
$t_{k1}{}^{\beta+2}+t_{k1}{}^{\beta+3}\big)-\frac{1}{\beta+1}\left(at_{k1}{}^{\beta+1}+bt_{k1}{}^{\beta+2}+ct_{k1}{}^{\beta+3}\right)-t_0\left\{t_{k1}{}^{\beta}+t_{k1}{}^{\beta+1}+t_{k1}{}^{\beta+2}\right\}+$
$\frac{t_0{}^{\beta+1}}{\beta+1}(a+bt_{k1}+ct_{k1}{}^2)\Big\}\Big]+\frac{H_c}{L}\Big[t_0(a+bt_{k1}+ct_{k1}{}^2)+\alpha t_0\left\{t_{k1}{}^{\beta}+t_{k1}{}^{\beta+1}+t_{k1}{}^{\beta+2}\right\}-$
$\alpha t_0{}^{\beta+1}(a+bt_{k1}+ct_{k1}{}^2)\Big]-\frac{B_c}{L}\Big[(1-\mu L)\left\{at_{k1}+\frac{bt_{k1}{}^2}{2}+\frac{ct_{k1}{}^3}{3}\right\}+\mu\left\{\frac{at_{k1}{}^2}{2}+\frac{bt_{k1}{}^3}{3}+\frac{ct_{k1}{}^4}{4}\right\}\Big]$
(C2)

Also

$\frac{\partial^2 C_t(t_{k1},\mathrm{L})}{\partial^2 L}=\frac{2O_c}{L^3}+2e^{(1-\xi)}\left(\frac{H_c}{L^3}+\frac{\theta_c}{L^3}\right)\Big[\frac{at_{k1}{}^2}{2}+\frac{bt_{k1}{}^3}{3}+\frac{ct_{k1}{}^4}{4}-a\left(t_{k1}t_0-\frac{t_0{}^2}{2}\right)-\frac{b}{2}\Big(t_{k1}{}^2t_0-$
$\frac{t_0{}^3}{3}\Big)-\frac{c}{3}\left(t_{k1}{}^3t_0-\frac{t_0{}^4}{4}\right)+\alpha\Big\{\left(\frac{t_{k1}{}^{\beta+2}}{\beta+2}+\frac{t_{k1}{}^{\beta+3}}{\beta+3}+\frac{t_{k1}{}^{\beta+4}}{\beta+4}\right)-\frac{1}{\beta+1}\left(\frac{at_{k1}{}^{\beta+2}}{\beta+2}+\frac{bt_{k1}{}^{\beta+3}}{\beta+3}+\frac{ct_{k1}{}^{\beta+4}}{\beta+4}\right)-$
$t_0\left\{\frac{1}{\beta+1}\left(t_{k1}{}^{\beta+1}-\frac{t_0{}^{\beta+1}}{\beta+2}\right)+\frac{1}{\beta+2}\left(t_{k1}{}^{\beta+2}-\frac{t_0{}^{\beta+2}}{\beta+3}\right)+\frac{1}{\beta+3}\left(t_{k1}{}^{\beta+3}-\frac{t_0{}^{\beta+3}}{\beta+4}\right)\right\}+a\left(\frac{t_0{}^{\beta+1}t_{k1}}{\beta+1}-\frac{t_0{}^{\beta+2}}{(\beta+2)}\right)+$
$\frac{b}{2}\left(\frac{t_0{}^{\beta+1}t_{k1}{}^2}{\beta+1}-\frac{t_0{}^{\beta+3}}{(\beta+3)}\right)+\frac{c}{3}\left(\frac{t_0{}^{\beta+1}t_{k1}{}^3}{\beta+1}-\frac{t_0{}^{\beta+4}}{(\beta+4)}\right)\Big\}\Big]+\frac{H_c}{L^3}\Big[a(t_{k1}t_0-t_0{}^2)+\frac{b}{2}(t_{k1}{}^2t_0-t_0{}^3)+\frac{c}{3}(t_{k1}{}^3t_0-$
$t_0{}^4)+\alpha\left(\frac{t_{k1}{}^{\beta+1}t_0-t_0{}^{\beta+2}}{\beta+1}+\frac{t_{k1}{}^{\beta+2}t_0-t_0{}^{\beta+3}}{\beta+2}+\frac{t_{k1}{}^{\beta+3}t_0-t_0{}^{\beta+4}}{\beta+3}\right)-\alpha\Big\{a\left(t_{k1}t_0{}^{\beta+1}-t_0{}^{\beta+2}\right)+$
$\frac{b}{2}\left(t_{k1}{}^2t_0{}^{\beta+1}-t_0{}^{\beta+3}\right)+\frac{c}{3}\left(t_{k1}{}^3t_0{}^{\beta+1}-t_0{}^{\beta+4}\right)\Big\}\Big]-\frac{B_c}{L^2}\Big[(1-\mu L)\left(aL+\frac{bL^2}{2}+\frac{cL^3}{3}\right)+(\mu-$
$2L)\left\{\frac{a}{2}(L^2-t_{k1}{}^2)+\frac{b}{6}(L^3-t_{k1}{}^3)+\frac{c}{12}(L^4-t_{k1}{}^4)\right\}+\mu\Big\{\frac{aL^2}{2}+\frac{bL^3}{3}+\frac{cL^4}{4}-\frac{2}{L}\Big(\frac{a}{6}(L^3-t_{k1}{}^3)+$
$\frac{b}{12}(L^4-t_{k1}{}^4)+\frac{c}{20}(L^5-t_{k1}{}^5)\Big)\Big\}\Big]+\frac{B_c}{L}\Big[(1-\mu L)\{a+bL+cL^2\}+\frac{\mu}{L}\Big\{\frac{a}{2}(L^2-t_{k1}{}^2)+\frac{b}{6}(L^3-$

$t_{k1}{}^{3}) + \frac{c}{12}(L^4 - t_{k1}{}^{4}) - \left\{aL + \frac{bL^2}{2} + \frac{cL^3}{3}\right\} + L\left(\frac{a}{2} + \frac{2bL}{3} + \frac{3cL^2}{4}\right)\Big\}\Big]$

(C3)

Equation (C3) is greater than zero as O_c, H_c, θ_c and B_c are the positive values and L is greater than t_{k1} which gives the positive value.

$$\frac{\partial^2 C_t(t_{k1},\mathrm{L})}{\partial^2 t_{k1}} = e^{(1-\xi)}\left(\frac{H_c}{L} + \frac{\theta_c}{L}\right)\Big[a + 2bt_{k1} + 3ct_{k1}{}^{2} - t_0(b + 2ct_{k1}) + \alpha\Big\{1 + 2t_{k1} + 3t_{k1}{}^{2} - \frac{1}{\beta+1}\left(a(\beta+1)t_{k1}{}^{\beta} + b(\beta+2)t_{k1}{}^{\beta+1} + c(\beta+3)t_{k1}{}^{\beta+2}\right) - t_0\{\beta t_{k1}{}^{\beta-1} + (\beta+1)t_{k1}{}^{\beta} + (\beta+2)t_{k1}{}^{\beta+1}\} + \frac{t_0{}^{\beta+1}}{\beta+1}(b + ct_{k1})\Big\}\Big] + \frac{H_c}{L}\left[t_0(\, b + ct_{k1} +) + \alpha t_0(\beta t_{k1}{}^{\beta-1} + (\beta+1)t_{k1}{}^{\beta} + (\beta+2)t_{k1}{}^{\beta+1}) - \alpha t_0{}^{\beta+1}\{b + ct_{k1}\}\right] + \frac{B_c}{L}[(\mu L - 1)\{a + bt_{k1} + ct_{k1}{}^{2}\} - \mu\{at_{k1} + bt_{k1}{}^{2} + ct_{k1}{}^{3}\}]$$

(C4)

(C4) is greater than zero as all the terms are the positive and L is greater than t_{k1} which gives the positive value.

$$\frac{\partial^2 C_t(t_{k1},L)}{\partial L \partial t_{k1}} = \frac{\partial^2 C_t(t_{k1},\mathrm{L})}{\partial t_{k1}\partial L} = -e^{(1-\xi)}\left(\frac{H_c}{L^2} + \frac{\theta_c}{L^2}\right)\Big[at_{k1} + bt_{k1}{}^{2} + ct_{k1}{}^{3} - t_0(a + bt_{k1} + ct_{k1}{}^{2}) + \alpha\Big\{t_{k1} + t_{k1}{}^{2} + t_{k1}{}^{3} - \frac{1}{\beta+1}\left(at_{k1}{}^{\beta+1} + bt_{k1}{}^{\beta+2} + ct_{k1}{}^{\beta+3}\right) - t_0\{t_{k1}{}^{\beta} + t_{k1}{}^{\beta+1} + t_{k1}{}^{\beta+2}\} + \frac{t_0{}^{\beta+1}}{\beta+1}(a + bt_{k1} + ct_{k1}{}^{2})\Big\}\Big] - \frac{H_c}{L^2}\left[t_0(a + bt_{k1} + ct_{k1}{}^{2} +) + \alpha t_0(t_{k1}{}^{\beta} + t_{k1}{}^{\beta+1} + t_{k1}{}^{\beta+2}) - \alpha t_0{}^{\beta+1}\{a + bt_{k1} + ct_{k1}{}^{2}\}\right] - \frac{B_c}{L^2}\left[(\mu L - 1)\left\{at_{k1} + \frac{bt_{k1}{}^{2}}{2} + \frac{ct_{k1}{}^{3}}{3}\right\} - \mu\left\{\frac{at_{k1}{}^{2}}{2} + \frac{bt_{k1}{}^{3}}{3} + \frac{ct_{k1}{}^{4}}{4}\right\}\right] + \frac{B_c}{L}\left[\mu\left\{at_{k1} + \frac{bt_{k1}{}^{2}}{2} + \frac{ct_{k1}{}^{3}}{3}\right\}\right] \quad \text{(B5)}$$

On squaring (C5) we will get the positive value, as the square of any negative value will be positive.

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