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Convergence of Fourier Series in L^1-Metric using γ- General Monotone Coefficients

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Abstract:

In the present article , We have extended the result of S. Tikhonov[8] and used γ-general monotone coefficients to prove the L^1-convergence of the trigonometric series. Also,"$||g-S_n|| = o(1)$" considered as , the necessary and sufficient condition for the convergence in terms of coefficients.

Keywords: L^1- Convergence, γ- monotone coefficients, Trigonometric Series.

Introduction:

Let g be 2π -periodic measurable function i.e.

$$||g|| = ||g||_1 = \frac{1}{2\pi}\int_0^{2\pi} |g|\, dx < \infty$$

Let also the cosine and sine series be:

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \qquad (1.1)$$

and $\sum_{k=1}^{\infty} a_k \sin kx$ respectively. (1.2)

The partial sum of the above sine and cosine series be:

"$\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$ and $\sum_{k=1}^{n} a_k \sin kx$ respectively.

(1.1) (or(1.2)) converges in L^1 if

$||g(x)-S_n(g,x)|| = o(1)$ as $n\to\infty$" (1.3)

Mostly, in many cases , L^1-convergence class is defined as:

"$|a_n \log n| = o(1)$ as $n\to\infty$".

The above result is true for (1.1)(or(1.2)) with the coefficients that are "monotone decreasing coefficients"$(a_k \epsilon M)$[3] or "quasi-monotone coefficients" $(a_k \epsilon QM)$[3,6], or "O-regularly quasi-monotone coefficients" $(a_k \epsilon ORVQM)$[5,9]. We know,

$QM = [a_n \epsilon R_+ : \exists \zeta > 0$ such that $n^{-\zeta} a_n \downarrow 0]$ (1.4)

and the $ORVQM$ class is given by:

$ORVQM = [a_n \epsilon R_+ : \exists \lambda_n \uparrow . \lambda_{2n} \leq C\lambda_n$ such that $\frac{a_n}{\lambda_n} \downarrow 0]$

S.Tikhonov[8] has given the conditions on a_n, by which the accuracy of criterion can be proved:
"$||g(x) - S_n(g,x)|| = O(1)$ iff $|a_n| \log n = O(1)$" (1.5)

Many authors [2,4,10] discussed similar ideas like, . they have studied the following class of coefficients:

The general monotone class of coefficients is defined as:

"$GM = a_n \epsilon C : \sum_{v=n}^{2n-1} |a_v - a_{v+1}| \leq C|a_n|$

It is also known for series which are having having GM - coefficients , three criteria of convergence can be proved for trigonometric series in L_p- space: for $p = \infty, p = 1$ and $1 < p < \infty$ "

GBVS [4] and NBVS[10]-classes are defined as :

$GBVS = a_n \epsilon C : \sum_{v=n}^{2n-1} |a_v - a_{v+1}| \leq C \max_{n \leq v \leq n+N} |a_v|$ for some integer N

and

$NBVS = a_n \epsilon C : \sum_{v=n}^{2n-1} |a_v - a_{v+1}| \leq C(|a_n| + |a_{2n}|)$ respectively.

The following embeddings are true for the above mentioned classes:

$M \subset QM \subset ORVQM \subset GM \subset GBVS \cup NBVS$ (1.6)

For a more general class

"$a_n \epsilon C : \sum_{v=n}^{2n-1} |a_v - a_{v+1}| \leq C \sum_{v=[n/c]}^{[c_n]} \frac{|a_v|}{v}$ for some integer $c > 1$" (1.7)

"In this paper , we extend the result of S.Tikhonov[8] in which the L^1-convergence for the series with general behaviour of a_n is obtained. Firstly, if $\gamma_n \log_n = o(1)$ then it follows that sufficient conditions in problem of L^1-convergence, that is (1.4) implies (1.3). If the behaviour of coefficient is rather, that is if ,, e.g

$\sum_{k=0}^{n-1} \frac{\gamma_{k+n}}{k+1} + \sum_{k=0}^{n-1} \frac{\gamma_{-k+n}}{-k+1} \leq C \sum_{k=0}^{n-1} \frac{a_{k+n}}{k+1} + \sum_{k=0}^{n-1} \frac{a_{-k+n}}{-k+1}$ (1.8)

or if

$$\sum_{v=n}^{2n-1} |a_v - a_{v+1}| \leq C\gamma_n \equiv (\log^{-1} n \left(\max_{m \geq [n/c]} \frac{\log m}{m} \sum_{v=m}^{2m} |a_v| \right)$$

Then the criterion (1.5) is true."

General monotone coefficients:

Definition: "Let $\gamma = (\gamma_n)_{n=1}^{\infty}$ be a non-negative sequence. With the bound γ, or $a \epsilon GM(\gamma)$, the sequence of complex numbers $a = (a_n)_{n=1}^{\infty}$ is said to be γ-general monotone ,if the relation

$$\sum_{v=n}^{2n-1} |a_v - a_{v+1}| \leq C\gamma_n$$

holds for all integers n, where , C is independent of n."

Following are examples of the sequence γ_n:

"(1) ${}_1\gamma_n = |a_n|$,

(2) ${}_2\gamma_n = \sum_{k=n}^{n+N} |a_k|$, for some integer n

(3) ${}_3\gamma_n = \sum_{v=0}^{N} |a_c v_n|$ for some integers N , $c > 1$,

(4) ${}_4\gamma_n = |a_n| + \sum_{v=n+1}^{[c_n]} \frac{|a_v|}{v}$, for some $c > 1$,

(5) ${}_5\gamma_n = \sum_{v=[n/c]}^{[c_n]}$, for some $c > 1$,

(6) ${}_6\gamma_n = \log^{-1} n \left(\max_{m \geq [n/c]} \frac{\log m}{m} \sum_{v=m}^{2m} |a_v| \right)$, for some $c > 1$".

As from S.Tikhonov[7],

$GM({}_1\gamma + {}_2\gamma + {}_3\gamma + {}_4\gamma + {}_5\gamma) \equiv GM({}_5\gamma) \subset GM({}_6\gamma)$

Also, it is noted "$GM \equiv GM({}_1\gamma)$,$GBVS \equiv GM({}_2\gamma)$ and $NBVS \subset GM({}_3\gamma)$.

Now, we will explain the properties of $GM(\gamma)$-sequences , which are as follows:

Lemma 2.1.1[5] "If $a = (a_n)_{n=1}^{\infty} \epsilon GM(\gamma)$, then one has for any integer n

$|a_v| \leq C\gamma_n + |a_m|$, for any $v,m = n,\ldots..2n$.

$|a_v| \leq C\gamma_n + \frac{1}{n}\sum_{j=n+1}^{2n} |a_j|$, for any $v = n,\ldots..2n$.

$|a_n| \leq \frac{C}{n}\left(\sum_{v=[n/2]}^{n-1} \gamma_v + \sum_{j=n}^{2n-1} |a_j|\right)$,

$|a_n|\left(\sum_{v=1}^{[n/2]} d_v\right) \leq C(\sum_{v=1}^{[n/2]} d_v \gamma_{v+[n/2]} + \sum_{v=1}^{[n/2]} d_v a_{2(v+[n/2])}$ for any $d_v \geq 0$."

L^1-convergence of trigonometric series:

Before giving the proof of L^1-convergence(boundedness) classes of trigonometric series:

$||g(x)-S_n(x)||_1 = O(1)$, (3.1)

and also if

$\sum_{k=2} c_k e^{ikx}$ is the fourier series of $f \epsilon L^1$

then (3.1) is euivalent to

$||V_n(x)-S_n(x)||_1 = O(1)$ (3.2),

where , $V_n(x)$; (C-1)-means of $S_n(x) = S_n(g,x)$,i.e.

$$V_n(x) = \frac{1}{n+1}\sum_{v=0}^{n} S_v(x) = \frac{1}{n+1}\sum_{v=0}^{n}\left(\sum_{|k|\leq v} c_k e^{ikx}\right)$$

So, "we only study the condition(3.2) for the sequences $(c_k)_{k\epsilon Z}$ and the condition (3.2) is true if and only if the same holds for $(c_k)_{k>0} \equiv \ldots.,0,c_1,c_2,\ldots..$ and $(c_{-k})_{k>0} \equiv \ldots.,0,c_{-1},c_{-2},\ldots..$".

For the cosine series $\left(c_k = c_{-k} = \frac{a_k}{2}\right)$

$\frac{a_0}{2} + \sum_{k\geq 1} a_k \cos kx$.

the accuracy of condition (3.2) for $(c_k)_{k>0}$ is equivalent to
$||V_n(h,x)-S_n(h,x)||_1 = O(1)$
where , "$S_n(h,x) = \frac{a_0}{2} + \sum_{v=1}^{n} a_v \cos vx$ and $V_n(h,x) = \frac{1}{n+1}\sum_{v=0}^{n} S_v(h,x)$. Similar conditions are applied to the sine series.$(c_k = -c_{-k} = -ia_k/2)$.
Therefore, we study accuracy of condition (3.2) for the sequences $(c_n)_{n>0}$",
Secondly, we note that [1]
$||V_n(x)-S_n(x)||_1 \le C\left(\frac{1}{n+1}\sum_{j=1}^{n} ||S_j(x)-S_{[j/2]}(x)||_1 + \max_{k=[n/2]....n} ||S_k(x)-S_{[n/2]}(x)||_1\right)$
which follows from:
"$S_n(x)-V_n(x) = \frac{1}{n+1}\sum_{j=1}^{n} j\,C_j, C_j = c_j e^{ijx} + c_{-j}e^{-ijx}$
and
$\sum_{j=1}^{n} j\,C_j = \sum_{j=1}^{n} \tilde{C}_j + \sum_{j=[n/2]+1}^{n} (2j-n-1)\,C_j, \tilde{C}_j = \sum_{l=[j/2]+1}^{j} C_l.$

Now we will present sufficient condition for relation (3.2) which holds in terms of γ_n"
Theorem (3.1): "Let $c = (c_n)_{n=1}^{\infty} \epsilon GM(\gamma)$, where a non-negative sequence $\gamma = (\gamma_n)_{n=1}^{\infty}$, satisfies

$$\sum_{j=n}^{2n-1} \frac{\gamma_j + |c_j|}{j-n+1} + \sum_{j=n}^{2n-1} \frac{\gamma_{-j} + |c_{-j}|}{-j-n+1} = O(1) \qquad (3.3)$$

Then the sequence $(c_n)_{n=1}^{\infty}$ satisfies (3.2)."
Proof. By using [1], (3.2) is implied by

$$\max_{n \le m \le 2n} ||S_m(x)-S_{n-1}(x)||_1 = O(1) \qquad (3.4)$$

So, now we will show that the condition (3.3) guarantees the accuracy of condition (3.4). Indeed,
"$||S_m(x)-S_{n-1}(x)||_1 = ||\sum_{j=n}^{m} c_j e^{ijx} + \sum_{j=n}^{m} c_{-j} e^{-ijx}||$

$$= ||\sum_{j=n}^{m-1} \Delta c_j \sum_{k=n}^{j} e^{ikx} + c_m \sum_{k=n}^{m} e^{ikx} + \sum_{j=n}^{m-1} \Delta c_{-j} \sum_{k=n}^{j} e^{-ikx} + c_{-m}\sum_{k=n}^{m} e^{-ikx}$$

$$\le C\Big(\sum_{j=n}^{m-1} |\Delta c_j| \log[(j-n)+2] + |c_m|\log[(m-n)+2] + \sum_{j=n}^{m-1} |\Delta c_{-j}| \log[(-j-n)+2] + |c_{-m}|\log[(-m-n)+2]$$

$$\le C\sum_{k=n}^{2n-1} \frac{\gamma_k + |c_k|}{k-n+1} + C\sum_{k=n}^{2n-1} \frac{\gamma_{-k} + |c_{-k}|}{-k-n+1}"$$

Thus (3.3) implies (3.4). Hence (3.2) is proved.

Corollary 3.1.1: Let $c = (c_n)_{n=1}^{\infty} \epsilon GM(\gamma)$ such that
$(\gamma_n + |c_n|)\log n + (\gamma_{-n} + |c_{-n}|)\log n = O(1)$
then condition (3.2) holds true.

Corollary 3.1.2: Let $c = (c_n)_{n=1}^{\infty} \epsilon GM({}_6\gamma)$. Then
$|c_n|\log n = O(1)$ and $|c_{-n}\log n| = O(1)$
This proves (3.2).

Corollary 3.1.3: Let $c = (c_n)_{n=1}^{\infty} \epsilon GM(\gamma)$,where

$$\sum_{k=0}^{n-1} \frac{\gamma_{k+n}}{k+1} + \sum_{k=0}^{n-1} \frac{\gamma_{-k+n}}{-k+1} \leq C\left(\sum_{k=0}^{[\log n]-1} \frac{|c_k + [\log n]|}{k+1} + \sum_{k=0}^{[\log n]-1} \frac{|c_{-k} + [\log n]|}{-k+1}\right) = C\log n(c,l)$$

for some $l > 0$, Then condition

$$\sum_{k=n}^{2n} \frac{|c_k|}{k-n+1} + \sum_{k=n}^{2n} \frac{|c_{-k}|}{k-n+1} = O(1)$$

which implies (3.2).

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