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Domain Decomposition Method for a Class of Singularly Perturbed Differential-Difference Equations

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Abstract: In the current research, a domain decomposition approach is created to solve a class of differential difference equations with singular perturbations. By establishing a terminal point, the original problem's domain is split into two distinct parts, the inner region, and the outer region. The reduced problem is used to derive an implicit boundary condition at the terminal point, and a special finite difference approach is then used to solve the outer region problem. Applying a particular finite difference technique, the inner area problem is solved by using a stretching transformation and the known value at the terminal point from the outer solution. The method's convergence analysis is also presented. Three model examples are tested, and it is discovered that the numerical solution closely resembles the precise/available solution.

Keywords: Differential-difference equation, boundary layer, positive shift, negative shift, singular perturbation.

1. Introduction

The development of reliable numerical techniques for handling the most difficult issues, such as boundary layer problems, is a serious area of focus for mathematicians. Boundary Layer is the term used to refer to a region where the problem's solution changes abruptly. In actuality, the answer evolves quickly to suit the problem's stated requirements. The boundary layer phenomenon remains present when the highest order differential coefficient is multiplied by a small positive parameter in any ordinary differential equation, also it is called a singular perturbation problem. Furthermore, the boundary layer phenomenon can be seen in any differential equation that has at least one delay or advance parameter, commonly referred to as a delay or differential-difference equation. The boundary layer problems make it extremely challenging to solve these issues. These issues come up when modeling a variety of real-world phenomena in biology, engineering, and control theory, including variational issues with control theory, the human pupil-light reflex phenomenon, various models that describes diseases or physical processes, and modelling to estimate the time required to generate the action potentials in nerve cells by random synaptic inputs in dendrites in first exit time problems. Many perturbation techniques, including the WKB method and matched asymptotic expansions, are employed to address these issues. The experience, experiment and insight are required for such asymptotic expansions. Additionally, it can be difficult to match the coefficients in the solution expansions for the inner and outside regions according to the Matching Principle. Scientists began developing numerical techniques as a result. The existing numerical will yields sufficient output only when the step size is taken small as compared to the perturbation parameter due to the presence of the boundary layer, which is not only very costly but also time consuming. Therefore, the researchers are focusing to develop credible numerical techniques that can function with a respectable level of precision. In reality, the parameters ought to have no effect on these trustworthy numerical approaches. Such a numerical method's efficiency will be determined by its accuracy, simplicity of computation, and sensitivity of the parameters . The following books and papers provide a thorough theoretical and numerical treatment: [1-34]. To solve these types of problems, we have developed a domain decomposition method, which we discuss here. By establishing a terminal point, the primary problem's domain is split into two parts, inside and outside of the region.At the end point an implicit boundary condition is derived using reduced problem-solving, and a special finite difference strategy is then used to solve the outer area problem. Applying a particular finite difference technique, the inner area problem is solved by using a stretching transformation and the known value at the end point from the outer solution. The method's convergence analysis is also covered. Three model examples are tested, and it is discovered that the numerical solution closely resembles the precise/available solution.

2.1 Description of the Method

2.2 Type-1: Differential Equation with a negative shift with boundary layer

Differential equation with delay term of the form,

$$\varepsilon v''(ţ) + a(ţ)v'(ţ - \delta) + b(ţ)v(ţ) = f(ţ), \quad 0 \le ţ \le 1, \qquad (1)$$

with boundary conditions

$$v(ţ) = \varphi(ţ), \quad -\delta \le ţ \le 0, \qquad (2)$$

and $$v(1) = \beta, \qquad (3)$$

where, ε is the perturbation parameter such that $0 < \varepsilon \ll 1$, and the small delay parameter $0 < \delta = O(\varepsilon)$, $a(ţ)$, $b(ţ)$ and $f(ţ)$ are the functions which are C^{∞}in $(0,1)$. $\varphi(ţ)$ is continuous and bounded on $[0,1]$ and β is a constant with a finite value.

Now Taylor's series expansion yields

$$v'(ţ - \delta) \approx v'(ţ) - \delta v''(ţ) \qquad (4)$$

After equation (4) is substituted into equation (1), we obtain

$$\varepsilon' v''(ţ) + p(ţ)v'(ţ) + q(ţ)v(ţ) = f(ţ), \quad 0 \le ţ \le 1 \qquad (5)$$

Where the given boundary conditions

$$v(0) = \acute{\alpha} \qquad ţ \qquad (6)$$

$$v(1) = \beta \qquad (7)$$

where $\varepsilon' = \varepsilon - a(ţ)\delta$, $p(ţ) = a(ţ)$, $q(ţ) = b(ţ)$ and $\acute{\alpha}$ is a constant with a finite value. Also, when $a(ţ) \ge M > 0$ in [0, 1], boundary layer will be at $ţ = 0$, where M is a positive number.

2.2 Type-2: Differential Equation with mixed shift with boundary layer

The differential equation with mixed shift of the form given below is considered,

$$\varepsilon v''(ţ) + a(ţ)v'(ţ) + b(ţ)v(ţ - \delta) + c(ţ)v(ţ) + d(ţ)v(ţ + \eta) = f(ţ),$$
(8)

$0 \leq ţ \leq 1$ with boundary conditions

$$v(ţ) = \varphi(ţ), \text{ on } -\delta \leq ţ \leq 0, \tag{9}$$

$$v(ţ) = \gamma(ţ), \text{ on } 1 \leq ţ \leq 1 + \eta, \tag{10}$$

Now Taylor's series expansion yields

$$v(ţ - \delta) \approx v(ţ) - \delta v'(ţ) + \frac{\delta^2}{2} v''(ţ) \tag{11}$$

$$v(ţ + \eta) \approx v(ţ) + \eta v'(ţ) + \frac{\eta^2}{2} v''(ţ) \tag{12}$$

Substitute equations (11) and (12) into equation (8), we get

$$\varepsilon' v''(ţ) + p(ţ)v'(ţ) + q(ţ)v(ţ) = f(ţ), \; 0 \leq ţ \leq 1 \tag{13}$$

with boundary conditions

$$v(0) = \acute{\alpha} \tag{14}$$

$$v(1) = \beta \tag{15}$$

where

$$= \varepsilon + b(ţ)\frac{\delta^2}{2} + d(ţ)\frac{\eta^2}{2} \tag{16}$$

$$p(ţ) = a(ţ) - \delta b(ţ) + \eta d(ţ) \tag{17}$$

$$q(ţ) = b(ţ) + c(ţ) + d(ţ) \tag{18}$$

Equation (1) to equation (5) or equation (8) to equation (13) is a valid transition, Since $0 < \delta \ll 1$ and $0 < \eta \ll 1$. The details of this transition can be validated from El'sgol'ts and Norkin [11]). Further, it can be established that, equation (8) has unique solution and a boundary layer at $ţ = 0$,if $q(ţ) \leq 0, \; p(ţ) \geq M > 0$ in [0,1] ,where M is a positive number.

Consider $ţ_p \left(0 < ţ_p \ll 1\right)$ is the terminal point or thickness of the boundary layer. Reduced equation (i.e. $\varepsilon = 0$ in equation (5)or equation (13)) is valid in the outer region.
Hence, by putting $\varepsilon = 0$ in equation (5) or equation (13), we get reduced equation

$$p(ţ)v'(ţ) + q(ţ)v(ţ) = f(ţ), ţ_p \leq ţ \leq 1 \tag{19}$$

Evaluate equation (19) at $ţ = ţ_p$ and denoting $c_1 = p\left(ţ_p\right) \neq 0$, $c_2 = q(ţ)$ and $c_3 = f\left(ţ_p\right)$, we get

$$c_1 v'\left(ţ_p\right) + c_2 v\left(ţ_p\right) = c_3 \tag{20}$$

Assume that equation (20) is the implicit boundary condition at $ţ = ţ_p$. Terminal point $ţ_p$ is common point for both the regions, then inner and outer regions problems are defined on $0 \leq ţ \leq ţ_p$ and $ţ_p \leq ţ \leq 1$ respectively.
Hence, outer region problem is defined as

$$\varepsilon' v''(ţ) + p(ţ)v'(ţ) + q(ţ)v(ţ) = f(ţ), ţ_p \leq ţ \leq 1 \tag{21}$$

with boundary conditions

$$c_1 v'(ţ_p) + c_2 v(ţ_p) = c_3 \text{ and } v(1) = \beta \qquad (22)$$

and the inner region problem is defined as

$$\varepsilon' v''(ţ) + p(ţ)v'(ţ) + q(ţ)v(ţ) = f(ţ), \quad 0 \le ţ \le ţ_p \qquad (23)$$

with boundary conditions

$$v(0) = \alpha \text{ and } v(ţ_p) = \theta \qquad (24)$$

where θ is to be determined as described below

We solve equation (21) to equation (22) and get $v(ţ)$ over $ţ_p \le ţ \le 1$ and hence we get the value of $v(ţ_p)$ which is the explicit boundary condition for inner region problem.

In order to get the solution of outer region problem, we divide $[ţ_p, 1]$ into n sub-intervals of equal length $\hbar$ such that $ţ_p = ţ_0 < ţ_1 < ţ_2 < \cdots < ţ_n = 1$ and we have $ţ_i = ţ_p + i\hbar$, $i = 0, 1, 2, \ldots, n$. For simplicity assume that $p(x_i) = p_i$, $q(ţ_i) = q_i$, $f(ţ_i) = f_i$, $v(ţ_i) = v_i$ and $v(ţ_p) = v_0$.

Apply the special finite difference scheme on equation (21), we have

$$\varepsilon'\left(\frac{v_{i-1} - 2v_i + v_{i+1}}{\hbar^2}\right) + p_i\left(\frac{v_{i+1} - v_i}{\hbar} - \frac{\hbar}{2}v_i''\right) + q_i\left(\frac{v_{i+1} + v_{i-1}}{2}\right) = f_i, \; i = 0, 1, 2, \ldots, n-1. \qquad (25)$$

$$\varepsilon'\left(\frac{v_{i-1} - 2v_i + v_{i+1}}{\hbar^2}\right) + p_i\left(\frac{v_{i+1} - v_i}{\hbar} - \frac{h}{2}\left(\frac{f_i - p_i v_i' - q_i v_i}{\varepsilon'}\right)\right) + q_i\left(\frac{v_{i+1} + v_{i-1}}{2}\right) = f_i \qquad (26)$$

Substitute $v_i' = \frac{v_{i+1} - v_i}{\hbar}$ in the above equation (26) and after simplifying, we get

$$\left(\frac{\varepsilon'}{\hbar^2} + \frac{q_i}{2}\right)v_{i-1} - \left(\frac{2\varepsilon'}{\hbar^2} + \frac{p_i}{\hbar} + \frac{p_i p_{i+\frac{1}{2}}}{2\varepsilon'} - \frac{\hbar}{2\varepsilon'}p_i q_{i+\frac{1}{2}}\right)v_i + \left(\frac{\varepsilon'}{\hbar^2} + \frac{p_i}{\hbar} + \frac{p_i p_{i+\frac{1}{2}}}{2\varepsilon'} + \frac{q_i}{2}\right)v_{i+1} = f_i + \frac{\hbar}{2\varepsilon'}p_i f_{i+\frac{1}{2}} \qquad (27)$$

Rearrange the terms of equation (27), we get system of n equations

$$E_i v_{i-1} - F_i v_i + G_i v_{i+1} = H_i; \quad i = 0, 1, 2, \ldots, n-1. \qquad (28)$$

where

$$E_i = \frac{\varepsilon'}{\hbar^2} + \frac{q_i}{2}$$

$$F_i = \frac{2\varepsilon'}{\hbar^2} + \frac{p_i}{\hbar} + \frac{p_i p_{i+\frac{1}{2}}}{2\varepsilon'} - \frac{\hbar}{2\varepsilon'}p_i q_{i+\frac{1}{2}}$$

$$G_i = \frac{\varepsilon'}{\hbar^2} + \frac{p_i}{\hbar} + \frac{p_i p_{i+\frac{1}{2}}}{2\varepsilon'} + \frac{q_i}{2}$$

$$H_i = f_i + \frac{\hbar}{2\varepsilon'}p_i f_{i+\frac{1}{2}}$$

This is a tri-diagonal system of n equations with $n+1$ unknowns $v_{-1}, v_0, \ldots, v_{n-1}$.

To eliminate the unknown v_{-1}, we use implicit boundary condition (i.e., equation (20)). Apply second order central difference approximation on equation (20), we get

$$v_{-1} = \frac{2\hbar c_2}{c_1} v_0 + v_1 - \frac{2\hbar c_3}{c_1} \tag{29}$$

For $i = 0$ in equation (28) and by using equation (29), we get

$$\left(-F_0 + \frac{2\hbar c_2}{c_1} E_0\right) v_0 + (E_0 + G_0) v_1 = H_0 + \frac{2\hbar c_3}{c_1} E_0 \tag{30}$$

Now equation (28) and equation (30) provide n equations with n unknowns $v_0, v_1, \ldots, v_{n-1}$. From Thomas Algorithm, we get values of above unknowns.

In order to get the solution of inner region problem, we choose the stretching transformation

$$t = \frac{ţ}{\varepsilon} \tag{31}$$

From equation (31), we get

$$v(ţ) = v(t\varepsilon) = Y(t) \tag{32}$$

Therefore, equation (23) and equation (24) become

$$\xi Y''(t) + A(t)Y'(t) + \varepsilon B(t)Y(t) = \varepsilon F(t),\ 0 \le t \le t_p \tag{33}$$

with boundary conditions

$$Y(0) = \acute{\alpha} \text{ and } Y(t_p) = \theta \tag{34}$$

where $\xi = \frac{\varepsilon'}{\varepsilon}$, $A(t) = p(ţ)$, $B(t) = q(ţ)$, $F(t) = f(ţ)$ and $t_p = \frac{ţ_p}{\varepsilon}$. We divide $[0, t_p]$ into n_1 sub-intervals of equal length $\hbar = \frac{t_p - 0}{n_1}$ such that $0 = t_0 < t_1 < t < \cdots < t_{n_1} = t_p$ and we have $t_i = t_0 + i\hbar$, $i = 0, 1, 2, \ldots, n_1$.

Apply the special finite difference scheme on equation (33) and proceeding in a similar fashion, we get

$$E_i Y_{i-1} - F_i Y_i + G_i Y_{i+1} = H_i; \quad i = 1, 2, \ldots, n_1 - 1. \tag{35}$$

In which,

$$E_i = \frac{\xi}{\hbar^2} + \frac{\varepsilon B_i}{2}$$

$$F_i = \frac{2\xi}{\hbar^2} + \frac{A_i}{\hbar} + \frac{A_i A_{i+\frac{1}{2}}}{2\xi} - \frac{\hbar}{2\xi} A_i \varepsilon B_{i+\frac{1}{2}}$$

$$G_i = \frac{\xi}{\hbar^2} + \frac{A_i}{\hbar} + \frac{A_i A_{i+\frac{1}{2}}}{2\xi} + \frac{\varepsilon B_i}{2}$$

$$H_i = \varepsilon F_i + \frac{\hbar}{2\xi} A_i \varepsilon F_{i+\frac{1}{2}}$$

This is a tri-diagonal system of $n_1 - 1$ equations with $n_1 - 1$ unknowns $Y_1, Y_2, \ldots, Y_{n_1-1}$. From Thomas Algorithm, we get values of above unknowns by using explicit boundary conditions defined in equation (34). Finally, we combine both solutions to get the solution of original problem. We repeat the process for various choice of t_p until the solution profiles do not differ materially from iteration to iteration. $\mathcal{P}$

3. Convergence Analysis

In matrix-vector form,the equation (28) can be written as,

$$\mathcal{P}Y = C \quad (36)$$

where $\mathcal{P} = [m_{ij}]$ is a "tridiagonal matrix" of order n, $0 \le i, j \le n-1$ with

$m_{ii+1} = \varepsilon' + p_i \hbar + \frac{p_i^2}{2\varepsilon'}\hbar^2 + \frac{p_i p_i'}{4\varepsilon'}\hbar^3 + \frac{q_i}{2}\hbar^2$

$m_{ii} = -2\varepsilon' - p_i \hbar - \frac{p_i^2}{2\varepsilon'}\hbar^2 - \frac{p_i p_i'}{4\varepsilon'}\hbar^3 + \frac{p_i q_i}{2\varepsilon'}\hbar^3 + \frac{p_i q_i'}{4\varepsilon'}\hbar^4$

$m_{ii-1} = \varepsilon' + \frac{q_i}{2}\hbar^2$

$Y = (\mathrm{v}_0, \mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_{n-1})^t$

and $C = (\mathrm{d}_i)$ is a column vector

$\mathrm{d}_i = \mathrm{f}_i \hbar^2 + \frac{p_i \mathrm{f}_i}{2\varepsilon'}\hbar^3 + \frac{p_i \mathrm{f}_i'}{4\varepsilon'}\hbar^4$

where, $0 \le i \le n-1$ such that the "local truncation error" is given by

$$T_i(\hbar) = \hbar^3 \left[\frac{p_i q_i}{2\varepsilon'}\mathrm{v}_i + \frac{p_i^2}{2\varepsilon'}\mathrm{v}_i' + \frac{p_i}{2}\mathrm{v}_i'' - \frac{p_i \mathrm{f}_i}{2\varepsilon'}\right] + O(\hbar^4) \quad (37)$$

We also have,

$$\mathcal{P}\bar{Y} - T(\hbar) = C \quad (38)$$

where $\bar{Y} = (\overline{\mathrm{v}_0}, \overline{\mathrm{v}_1}, \overline{\mathrm{v}_2}, \dots, \overline{\mathrm{v}_{n-1}})^t$ represents the exact solution,

$T(\hbar) = (T_0(\hbar), T_1(\hbar), T_2(\hbar), \dots, T_{n-1}(\hbar))^t$ represents column vector of truncation error.

Equation (36) and (38), yields

$$\mathcal{P}(\bar{Y} - Y) = T(\hbar) \quad (39)$$

Thus we obtained the error equation $\mathcal{P}E = T(\hbar)$ (40)

here, $E = \bar{Y} - Y = (e_0, e_1, e_2, \dots, e_{n-1})^t$.

If S_i denotes the sum of i^{th} row elements of matrix $\mathcal{P}$, we have

$S_i = -\varepsilon' \frac{2\hbar c_2}{c_1} + \left(q_i - \frac{2\hbar c_2}{c_1}\frac{q_i}{2}\right)\hbar^2 + O(\hbar^3)$ for $i = 0$

$S_i = q_i \hbar^2 + O(\hbar^3)$ for $i = 1, 2, 3, \dots, n-2$

$S_i = -\varepsilon' - p_i\hbar + \frac{q_i}{2}\hbar^2 - \frac{p_i^2}{2\varepsilon'}\hbar^2 + O(\hbar^3)$ for $i = n-1$

Since $1 \gg \varepsilon > 0$, $O(\varepsilon) = \delta > 0$ and $O(\varepsilon) = \eta > 0$,It can be proved that for sufficiently small step size, $\mathcal{P}$ is irreducible and monotone (Mohanty and Jha [25]). So $\mathcal{P}$is invertible and $\mathcal{P}^{-1}$ contains nonnegative elements.

Equation (40) becomes

$$E = \mathcal{P}^{-1}T(\hbar) \tag{41}$$

Taking norm on both sides of equation(41), we get

$$||E|| \leq ||\mathcal{P}^{-1}||.\,||T(\hbar)|| \tag{42}$$

Let us consider $\mathcal{P}^{-1} = [u_{ki}]$, $0 \leq k, i \leq n-1$. As $u_{ki} \geq 0$, so from matrix analysis we obtained

$$\sum_{i=0}^{n-1} u_{ki}S_i = 1, \quad k = 0, 1, 2, \ldots, n-1 \tag{43}$$

From equation (43), we get

$$\sum_{i=0}^{n-1} u_{ki} \leq \frac{1}{\underbrace{\min S_i}_{0\leq i\leq n-1}} = \frac{1}{\hbar^2|\Re|} \tag{44}$$

where $\Re$ is a constant independent of $\hbar$.

We define $||\mathcal{P}^{-1}|| = \underbrace{\max \sum_{i=0}^{n-1} |u_{ki}|}_{0\leq k\leq n-1}$ and $||T(\hbar)|| = \underbrace{\max|T_i(\hbar)|}_{0\leq i\leq n-1} = \hbar^3\Omega$

where Ω is a constant not dependent on $\hbar$.

The error at k^{th} tuple can be obtained from equation (41) as shown below

$$e_k = \sum_{i=0}^{n-1} u_{ki}\, T_i(\hbar), \; k = 0, 1, 2, \ldots, n-1 \tag{45}$$

Also from equation (42), we can obtained

$$||E|| \leq \underbrace{\max \sum_{i=0}^{n-1}|u_{ki}|}_{0\leq k\leq n-1} . \underbrace{\max|T_i(\hbar)|}_{0\leq i\leq n-1} \tag{46}$$

In equation (46) substituting the equation (44) , we have

$$||E|| = \underbrace{\max |e_i|}_{0\leq i\leq n-1} = |e_j| \leq \frac{1}{\hbar^2|\Re|}\hbar^3\Omega = \hbar\mathcal{K} \tag{47}$$

where $\mathcal{K}$ is a constant independent on $\hbar$ and

$|e_j| = \max(|e_0|, |e_1|, |e_2|, \ldots, |e_{n-1}|)^t$.

Hence, $||E|| = O(\hbar)$.

Hence, order of convergence is linear for uniform h. Also, similar analysis is applicable for equation (35).

4. Numerical Experiments

Three model examples with a left boundary layer are solved in this section, and the solutions are contrasted with the precise/available solutions. Equation (8)'s precise solution is provided by (with assumptions $f(ţ) = f, \varphi(ţ) = \varphi$ and $\gamma(ţ) = \gamma$ are constant)

$$v(ţ) = c_1 e^{m_1 ţ} + c_2 e^{m_2 ţ} + f/c' \tag{48}$$

where

$$c' = b + c + d$$

$$m_1 = \left[-(a - \delta b + \eta d) + \sqrt{(a - \delta b + \eta d)^2 - 4\varepsilon c'}\right]/2\,\varepsilon$$

$$m_2 = \left[-(a - \delta b + \eta d) - \sqrt{(a - \delta b + \eta d)^2 - 4\varepsilon c'}\right]/2\,\varepsilon$$

$$c_1 = [-f + \gamma c' + e^{m_2}(f - \varphi c')]/[(e^{m_1} - e^{m_2})c']$$

$$c_2 = [f - \gamma c' + e^{m_1}(-f + \varphi c')]/[(e^{m_1} - e^{m_2})c']$$

Example 1. Consider the delay differential equation

$$\varepsilon v''(ţ) + v'(ţ - \delta) - v(ţ) = 0,\ 0 \le ţ \le 1; \quad v(0) = 1 \text{ and } v(1) = 1.$$

The exact solution is given by

$$v = ((1 - e^{m_2})e^{m_1 ţ} + (e^{m_1} - 1)e^{m_2 ţ})/(e^{m_1} - e^{m_2})$$

where

$$m_1 = \frac{-1 - \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)} \text{ and } m_2 = \frac{-1 + \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)}$$

Tables and graphs for each numerical solution, correct solution, comparison solution, and boundary layer action are presented.

Example 2. Consider the differential-differential equation

$$\varepsilon v''(ţ) + v'(ţ) - 2\,v(ţ - \delta) - 5\,v(ţ) + v(ţ + \eta) = 0, \qquad 0 \le ţ \le 1;$$
$$\text{with } v(0) = 1 \text{ and } v(1) = 1.$$

Tables and graphs for each numerical solution, correct solution, comparison solution, and boundary layer action are presented.

Example 3. Consider the differential-differential equation

$$\varepsilon v''(ţ) + v'(ţ) - 3\,v(ţ) + 2\,v(ţ + \eta) = 0,\ 0 \le ţ \le 1; \text{with } v(0) = 1 \text{ and } v(1) = 1.$$

Tables and graphs for each numerical solution, correct solution, comparison solution, and boundary layer action are presented.

5. Conclusion

A domain decomposition method is developed to solve singularly perturbed differential equation with a delay term and singularly perturbed differential difference equation with a boundary layer on left. The method's convergence analysis is also explained. On the terminal points the method is iterative and easy to implement. From the graphs and tables, it can be observed numerical solution approximates available/exact solution very well.

Table-1: $\hbar = 0.01, \varepsilon = 0.01, \delta = 0.001$ and $ţ_p = 5$ (Example 1)

ţ	Numerical Solution v(ţ)	Exact Solution v_1(ţ)	Result by [14] v_c(ţ)
0.0	1.00000000	1.00000000	1.00000000
0.0002	0.98609488	0.98613127	0.98609448
0.0025	0.84682098	0.84722305	0.84681709
0.0125	0.52939705	0.53064865	0.52939083
0.0225	0.42845543	0.42999744	0.42845196
0.0325	0.39810211	0.39975084	0.39810068
0.0425	0.39078268	0.39247786	0.39078230
0.05	0.39060054	0.39231668	0.39060054
0.2	0.45313525	0.45251844	0.45252424
0.4	0.55229471	0.55173078	0.55173608
0.6	0.67315321	0.67269491	0.67269922
0.8	0.82045915	0.82017980	0.82018243
1.0	1.00000000	1.00000000	1.00000000

Fig.1

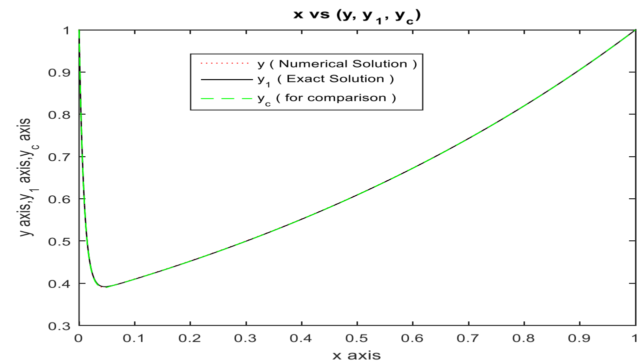


Table-2: $\hbar = 0.01, \varepsilon = 0.01, \delta = 0.001$ and $ţ_p = 10$(Example 1)

ţ	Numerical Solution v(ţ)	Exact Solution v_1(ţ)	Result by [14] v_c(ţ)
0.0	1.00000000	1.00000000	1.00000000
0.0002	0.98614338	0.98613127	0.98614297
0.0025	0.84735632	0.84722305	0.84735234
0.0125	0.53105838	0.53064865	0.53105189
0.0225	0.43049859	0.42999744	0.43049478
0.0325	0.40028469	0.39975084	0.40028289
0.0425	0.39302581	0.39247786	0.39302504
0.0725	0.39955252	0.39898403	0.39955247

0.1	0.41041063	0.40982631	0.41041063
0.4	0.55229471	0.55173078	0.55173608
0.6	0.67315321	0.67269491	0.67269922
0.8	0.82045915	0.82017980	0.82018243
1.0	1.00000000	1.00000000	1.00000000

Fig. 2.

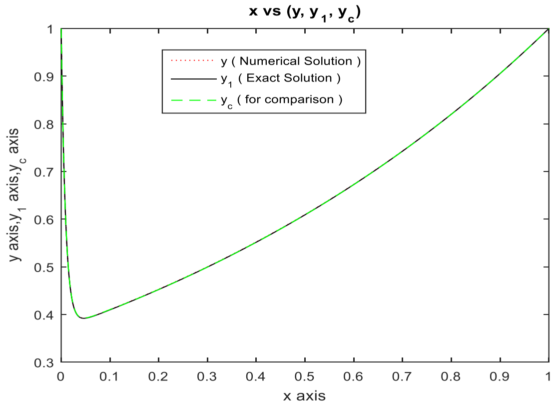


Table-3: $\hbar = 0.01, \varepsilon = 0.01, \delta = 0.001, \eta = 0.003$ and ţ$_p = 5$ (Example 2)

ţ	Numerical Solution v(ţ)	Exact Solution v_1(ţ)	Result by [14] v_c(ţ)
0.0	1.00000000	1.00000000	1.00000000
0.0002	0.97899848	0.97907102	0.97682844
0.0025	0.76696173	0.76778195	0.74603804
0.0125	0.26534558	0.26813298	0.23175433
0.0225	0.09175473	0.09543749	0.07286034
0.0325	0.03167862	0.03585315	0.02381896
0.0425	0.01088467	0.01540628	0.00873649
0.05	0.00484252	0.00959213	0.00484252
0.2	0.01126315	0.01086936	0.01064421
0.4	0.03457362	0.03366303	0.03313868
0.6	0.10612799	0.10425625	0.10317080
0.8	0.32577291	0.32288737	0.32120212
1.0	1.00000000	1.000000000	1.00000000

Fig. 3.

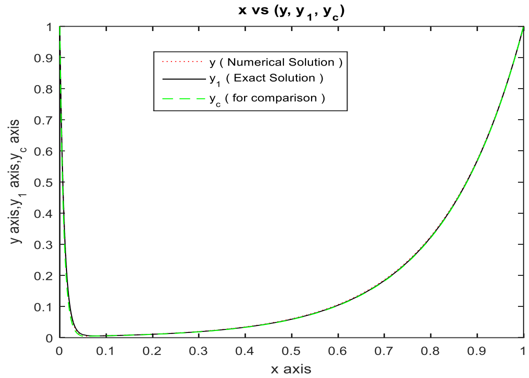


Table-4: $\hbar = 0.01, \varepsilon = 0.01, \delta = 0.001, \eta = 0.003$ and $ţ_p = 10$(Example 2)

ţ	Numerical Solution v(ţ)	Exact Solution v_1(ţ)	Result by [14] v_c(ţ)
0.0	1.00000000	1.00000000	1.00000000
0.0002	0.97908080	0.97907102	0.97687963
0.0025	0.76788132	0.76778195	0.74660295
0.0125	0.26835003	0.26813298	0.23352704
0.0225	0.09563686	0.09543749	0.07509981
0.0325	0.03602976	0.03585315	0.02630158
0.0425	0.01557301	0.01540628	0.01139892
0.0725	0.00592021	0.00574021	0.00567612
0.1	0.00640977	0.00620077	0.00640977
0.4	0.03457362	0.03366303	0.03313868
0.6	0.10612799	0.10425625	0.10317080
0.8	0.32577291	0.32288737	0.32120212
1.0	1.00000000	1.00000000	1.00000000

Fig. 4.

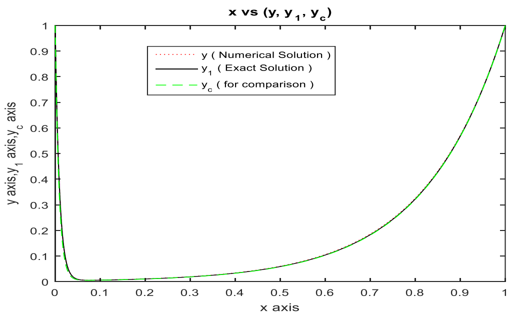


Table-5: $\hbar = 0.01, \varepsilon = 0.01, \delta = 0.001, \eta = 0.003$ and $ţ_p = 5$ (Example 3)

ţ	Numerical Solution v(ţ)	Exact Solution v_1(ţ)	Result by [14] v_c(ţ)
0.0	1.00000000	1.00000000	1.00000000
0.0002	0.98742258	0.98747674	0.98606052
0.0025	0.85982792	0.86044696	0.84656687
0.0125	0.55206208	0.55421829	0.52989337
0.0225	0.44289692	0.44573439	0.42996715
0.0325	0.40573420	0.40887695	0.40019551
0.0425	0.39469111	0.39797921	0.39314804
0.05	0.39306201	0.39641285	0.39306201
0.2	0.45554500	0.45497060	0.45462985
0.4	0.55449606	0.55397160	0.55366039
0.6	0.67494074	0.67451508	0.67426245
0.8	0.82154777	0.82128867	0.82113485
1.0	1.00000000	1.00000000	1.00000000

Fig. 5.

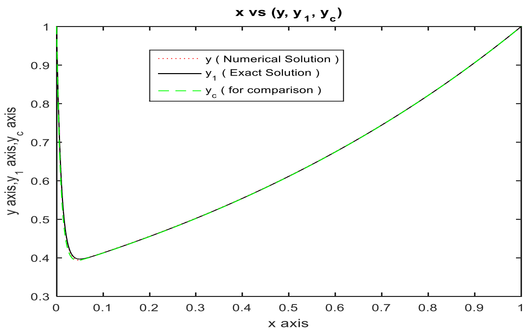
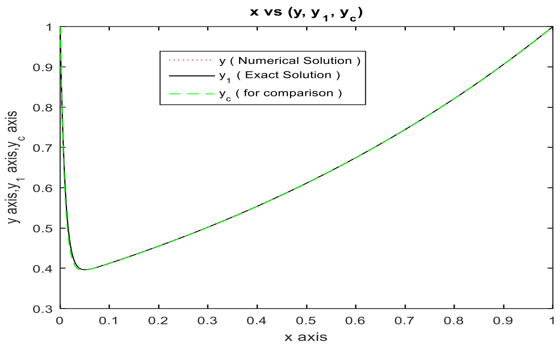


Table-6: $\hbar = 0.01, \varepsilon = 0.01, \delta = 0.001, \eta = 0.003$ and $ţ_p = 10$(Example 3)

ţ	Numerical Solution v(ţ)	Exact Solution v_1(ţ)	Result by [14] v_c(ţ)
0.0	1.00000000	1.00000000	1.00000000
0.0002	0.98749758	0.98747674	0.98610701
0.0025	0.86066538	0.86044696	0.84707959
0.0125	0.55476476	0.55421829	0.53148022
0.0225	0.44629934	0.44573439	0.43191560
0.0325	0.40941420	0.40887695	0.40227517
0.0425	0.39849591	0.39797921	0.39528450
0.0725	0.40220859	0.40170141	0.40199046
0.1	0.41286006	0.41234128	0.41286006
0.4	0.55449606	0.55397160	0.55366039

0.6	0.67494074	0.67451508	0.67426245
0.8	0.82154777	0.82128867	0.82113485
1.0	1.00000000	1.00000000	1.00000000

Fig. 6.



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