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Stability Analysis of First Order Integro-Differential Equations With the Successive Approximation Method

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Abstract: The Ulam stability theory provides a framework to provide the stability of functional equations, including integrodifferential equations. This manuscript focuses on the Ulam-stability analysis of the first-order integrodifferential equation. First-order integrodifferential equations combine differential and integral terms, making their analysis challenging and intriguing. The Ulam-stability concept investigates the behaviour of solutions under perturbations in the equation's inputs or initial conditions. It offers valuable insights into the long-term behaviour and robustness of the solutions in the presence of minor disturbances. The results obtained in this study contribute to the understanding of stability properties of first-order integrodifferential equations and provide a foundation for further research in this area. The Ulam stability analysis offers valuable insights into the behaviour of these equations, aiding in their application to diverse domains, including physics, engineering, and mathematical modeling.

Keywords: First-order integro-differential equations, Ulam-Hyers-Rassias stability, Ulam-Hyers stability, Existence and Uniqueness, successive approximation method.

1. Introduction

In [1], Ulam addressed the stability of functional equation and gave details on how solutions behave when a slight perturbation is added. Ulam also responds to a few crucial queries about functional equations: For instance, under what circumstances is it confirmed that the answer to an equation, even one that little deviates from the one given, must inevitably be close to the answer to the given equation? This characteristic is also referred to as a solution's robustness or stability. It indicates that if a function approximates the solution of a particular functional equation, then another function exists that exactly solves the equation and is close to the approximate solution. It is significant to highlight that figuring out the stability or proximity of solutions for a particular functional equation or inequality is a challenging undertaking that necessitates careful consideration. Functional analysis, approximation theory, or certain characteristics of the under consideration equations are frequently used in the procedures used to ensure stability. Ulam's research produced a substantial impact on the fields of stability theory and functional equations, and his theories have since been extended and built upon by other scholars.

Stanislaw Marcin Ulam did deliver a speech at the University of Wisconsin in 1964, in which he discussed several unresolved mathematical issues. One of these issues is indeed connected to homomorphism stability. A homomorphism in mathematics is a structure-preserving map between two algebraic structures. A group homomorphism, for example, preserves the group structure, whereas a ring homomorphism preserves the ring structure, and so on.

In 1940, at the University of Wisconsin Stanisław Marcin Ulam in [1] gave a speech on several unsolved mathematics issues. One of these problems is the question concerning the stability of homomorphisms expressed as follows:

The question about the stability of functional equations was actually answered by Donald H. Hyers in [2]. His revolutionary study, published in 1941, established what is now known as Hyers's theorem. Hyers' Theorem states that if a function substantially fulfills a specific type of functional equation, then there exists a single solution that is close to the approximation solution. Significant advancements in the field of functional equations were made by Th.M. Rassias. He introduced a weaker form of the Cauchy difference and obtained a stability finding for the extra Cauchy functional equation in [3].

It is possible to apply the Hyers-Ulam (H-U) stability and Hyers-Ulam-Rassias (H-U-R) stability principles to other kinds of equations besides functional equations. Differential equations can be used to study stability. Investigating the stability of solutions despite slight changes to the differential equation's initial circumstances or structure is the goal. The stability of the (H-U) and (H-U-R) equations, which involve discrete changes in the variables, can also be examined.

An example of a functional equation that uses both integrals and derivatives is the Volterra Integro-differential equation. It bears the name Vito Volterra after the Italian mathematician who conducted substantial research on integral equations and their characteristics. The authors in [4] gave the following Volterra Integro-differential Equation, follows

$$u'(t) + p(t)u(t) + q(t) + \int_c^t K(t,\tau)u(\tau)d\tau = 0. \quad (1)$$

They proved that if $p: I \to \mathbb{R}$, $q: I \to \mathbb{R}$, $K: I \times I \to \mathbb{R}$ and $\varphi: I \to [0, \infty)$ are sufficiently smooth function and if the perturbed Integro-differential inequality is satisfied by a continuously differentiable function $u: I \to \mathbb{R}$, then

$$\left| u'(t) + p(t)u(t) + q(t) + \int_c^t K(t,\tau)u(\tau)d\tau \right| \leq \varphi(t), \quad (2)$$

the aforementioned Volterra Integro-differential equation has a single solution $u_0: I \to \mathbb{R}$ for any $t \in I$, such that

$$|u(t) - u_0(t)| \leq e^{-\int_c^b p(\tau)d\tau} \int_t^b \varphi(\xi)\, e^{\int_c^\xi p(\tau)d\tau} d\xi, \quad (3)$$

for all $t \in I$.

By using the fixed point approach, the authors in [5] examined the (H-U) stability and the (H-U-R) stability of a nonlinear Volterra Integro-differential equation.

$$u'(t) = f\big(t, u(t)\big) + \int_0^t k\big(t, s, u(s)\big)ds, \quad (4)$$

where, the function $f(t, u)$ is continuous in terms of the variables t and u over the interval $I \times \mathbb{R}$, while the function $k(t, s, u)$ is continuous in terms of t, s, and u over the interval $I \times I \times \mathbb{R}$. Additionally, the constant α is provided.

Several scholars have shown the (H-U) stability of several Volterra equation kinds over the past few years (see [6–13]). In this study, we examine the linear Volterra Integro-diferential equation presented as follows

$$y'(t) = p(t)y(t) + q(t) + \int_0^t K(t,s)y(s)ds\,, t \in I = [0,T] \qquad (5)$$

with initial condition $y(0) = y_0$, where $q(t)$ is continuous on I, the kernel $K(t,s)$ is continuous on $D := \{(t,s): 0 \le s \le t \le T\}$ and $T > 0$. The successive approximation method is employed for demonstrating the (H-U) stability and the (H-U-R) stability of (5). First, the presence and originality of the solution to the linear Volterra Integro-diferential equation are explored. After the linear Volterra Integro-diferential equation's (H-U) and (H-U-R) stability are established. Volterra Integro-diferential equations appear frequently in the mathematical description of physical and biological events.

The successive approximation technique was used by the authors of [7] to illustrate the (H-U) stability of a nonlinear integral problem. The same technique was applied by the authors of [13] to prove the (H-U) stability of a nonlinear integral equation with delay. The successive approximation method was also used by the authors of [14] to demonstrate the stability of another nonlinear integral equation with delay. The Banach fixed point theorem is used by the authors in [15] to show stability in delay functional differential equations. The application of fixed point theories and Lyapunov's second approach to a few first- and second-order differential equations was covered in [16].

The goal of this manuscript is to establish the Ulams stability of solutions to the Volterra Integro-differential equations, meaning that small changes in the equation's inputs or initial conditions lead to small changes in the corresponding solutions. Our analysis involves investigating the presence and originality of solutions, as well as characterizing their stability properties. To achieve this, we employ successive approximation methods. This tool and we use the idea of Gachpazan and Baghani [13] and Morales and Rojas [14] allows us to derive sufficient conditions under which the Ulam stability of solutions of equation (5) can be guaranteed.

Definition 1.1: If $y(t)$ is continuously differential function that satisfying

$$\left| y'(t) - p(t)y(t) - q(t) - \int_0^t K(t,s)y(s)ds \right| \le \psi(t), \qquad (6)$$

given that $\psi(t)$ is non-negative for all t, there is an existence of a solution $y_0(t)$ to the Volterra integro-differential equation (5), along with a positive constant C, with the following inequality

$$|y(t) - y_0(t)| \le C\psi(t), \qquad (7)$$

for all t, where C is independent of $y(t)$ and $y_0(t)$, we refer to the equation (5) as having (H-U-R) stability if the inequalities mentioned above hold. We refer to equation (5) as having (H-U) stability in the case where $\psi(t)$ is a constant function in those inequalities.

2. Existence and uniqueness of solutions

In this part, our first objective is to demonstrate the existence and uniqueness of the solution to the linear Volterra Integro-differential equation (5). For this purpose

Firstly we integrate equation (5) from 0 to t, we obtain

$$y(t) = y_0 + \int_0^t p(s)y(s)ds + \int_0^t q(s)ds + \int_0^t \int_0^\tau K(\tau, s)\, y(s)ds\, d\tau.$$

Using Dirichlet's formula

$$\int_0^t \int_0^\tau F(\tau, s)dsd\tau = \int_0^t \int_s^t F(\tau, s)d\tau ds\ , \qquad 0 \leq s \leq \tau \leq t,$$

we get

$$\begin{aligned} y(t) &= y_0 + \int_0^t p(s)y(s)ds + \int_0^t q(s)ds + \int_0^t \left(\int_s^t K(\tau, s)\, d\tau\right) y(s)ds \\ &= y_0 + \int_0^t q(s)ds + \int_0^t \left(p(s) + \int_s^t K(\tau, s)\, d\tau\right) y(s)ds. \end{aligned}$$

At present, equation (5) can be reformulated as a second-kind linear Volterra integral equation in terms of the variable y,

$$y(t) = g(t) + \int_0^t G(t, s)y(s)ds\ , \ \ t \in I \tag{8}$$

where

$$g(t) = y_0 + \int_0^t q(s)ds, \tag{9}$$

and

$$G(t, s) = p(s) + \int_s^t K(\tau, s)d\tau. \tag{10}$$

If we suppose that

$$|p(t)| \leq M_1$$

for all $t \in I$, and

$$|K(t, s)| \leq M_2,$$

for all $(t, s) \in D$, then we obtain

$$|G(t, s)| \leq M_1 + M_2 \int_s^t d\tau \leq M_1 + M_2(t - s) \leq M_1 + M_2 T.$$

In our context, we consider the collection of continuous real-valued functions defined on I. This collection, denoted as X, constitutes a complete metric space where the metric $d(x, y)$ is defined

$$d(f, g) = \max_{t\in I}|f(t) - g(t)|\ , \qquad f, g \in C(I).$$

Furthermore, we define the operator $L: X \to X$ given by

$$(Ly)(t) = g(t) + \int_0^t G(t, s)y(s)ds,\ \ \forall\ t \in I \tag{11}$$

for all $y \in X$.

In first, we show that equation (8) has a solution. For this, we consider the iterative integral equation

$$y_{n+1} = g(t) + \int_0^t G(t,s) y_n(s) ds \equiv L y_n, \quad n = 1,2,3,\ldots \tag{12}$$

We can write

$$|y_{n+1}(t) - y_n(t)| = \left| \int_0^t G(t,s)[y_n(s) - y_{n-1}(s)] ds \right| \leq \int_0^t |G(t,s)||y_n(s) - y_{n-1}(s)| ds$$
$$\leq (N + MT) \int_0^t |y_n(s) - y_{n-1}(s)|\, ds.$$

Hence,

$$|y_{n+1}(t) - y_n(t)| \leq (M_1 + M_2 T) \int_0^t |y_n(s_1) - y_{n-1}(s_1)|\, ds_1$$
$$\leq (M_1 + M_2 T)^2 \int_0^t \int_0^{s_1} |y_{n-1}(s_2) - y_{n-2}(s_2)| ds_2 \; ds_1 \leq \cdots$$
$$\leq (M_1 + M_2 T)^{n-1} \int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-2}} |y_2(s_{n-1}) - y_1(s_{n-1})|\, ds_{n-1} \ldots ds_2 ds_1$$
$$\leq (M_1 + M_2 T)^{n-1} d(Ly_1, y_1) \int_0^t \int_0^{s_1} \cdots \left(\int_0^{s_{n-2}} ds_{n-1} \right) \ldots ds_2 ds_1.$$

Therefore, we can express it as follows

$$|y_{n+1}(t) - y_n(t)| \leq (M_1 + M_2 T)^{n-1} \frac{T^{n-1}}{(n-1)!} d(Ly_1, y_1).$$

Hence, due to the completeness of the metric space X, if y_1 belongs to X, then

$$\sum_{n=1}^{\infty} |y_{n+1}(t) - y_n(t)|,$$

the series is proven to be both absolutely and uniformly convergent using the Weierstrass M −test. Furthermore, the function $y_n(t)$ can be expressed in the following manner:

$$y_n(t) = y_1(t) + \sum_{k=1}^{n-1} [y_{k+1}(t) - y_k(t)].$$

Consequently, there is a special solution $y \in X$ such that $\lim_{n\to\infty} y_n(t) = y$. Now by taking the limit of both sides of (12), we have

As a result, there exists a single and distinct solution $y \in X$, satisfying the property that $\lim_{n\to\infty} y_n(t) = y$. By applying the limit operation to both sides of equation (12), we obtain

$$y(t) = \lim_{n\to\infty} y_{n+1}(t) = \lim_{n\to\infty} (g(t) + \int_0^t G(t,s) y_n(s) ds) = g(t) + \int_0^t G(t,s) \lim_{n\to\infty} y_n(s)) ds$$
$$= g(t) + \int_0^t G(t,s) y(s) ds).$$

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Therefore, there is a singular solution $y \in X$ that satisfies the equation $Ly = y$, and this solution is unique.

3. Hyers-Ulam Stability

A key idea in mathematical analysis and functional equations, (H-U) stability investigates how the solutions to some functional equations behave when subjected to tiny disturbances. It bears the names of two researchers who independently established the result in the early 1940s: Donald Hyers and Stanislaw Ulam. When a functional equation's mapping is sensitive to small changes, the stability theorem addresses the issue of whether or not approximate solutions persist. Our objective in this section is to investigate the (H-U) stability for the equation (5).

Theorem 3.1: The equation $Ly = y$, where L is defined by equation (11), exhibits Ulam-Hyers stability. In other words, for any given $z \in X$ and $\varepsilon > 0$, there exists a solution y that satisfies the equation with

$$d(Lz, z) \leq \varepsilon,$$

in that case, there exists a single and distinct $y \in X$, $Ly = y$ and $d(z, y) \leq C\varepsilon, C \geq 0$.

Proof. Suppose that $z \in X$, $\varepsilon > 0$ and $d(Lz, z) \leq \varepsilon$. We established in the previous section that the exact solution of the equation $Ly = y$ is $y = \lim_{n\to\infty} y_{n+1}(t)$. Since $L^n z$ converges uniformly to y as $n \to \infty$, then there is natural number N such that $|L^N z - y| \leq \varepsilon$. Thus,

$$\begin{aligned}
d(z, y) &\leq d(z, L^N z) + d(L^N z, y) \\
&\leq d(z, Lz) + d(Lz, L^2 z) + d(L^2 z, L^3 z) + \cdots + d(L^{N-1} z, L^N z) + d(L^N z, y) \\
&\leq d(z, Lz) + \frac{k}{1!} d(z, Lz) + \frac{k^2}{2!} d(z, Lz) + \cdots + \frac{k^{N-1}}{(n-1)!} d(z, Lz) + d(L^N z, y) \\
&\leq d(z, Lz)\left(1 + \frac{k}{1!} + \frac{k^2}{2!} + \cdots + \frac{k^{N-1}}{(N-1)!}\right) + \varepsilon \leq \varepsilon e^k + \varepsilon \\
&= \varepsilon\left(1 + e^k\right) = C\varepsilon,
\end{aligned}$$

where $k = M_1 T + M_2 T^2$. This complete the proof.

Corollary 1. Theorem 1 hold for every finite interval $[a, b]$, $[a, b)$, $(a, b]$ and (a, b) when $-\infty < a < b < \infty$ for $a, b \in \mathbb{R}$.

4. Hyers-Ulam-Rassias Stability

The Hyers-Ulam-Rassias Stability is a basic concept in functional equations and mathematical analysis. This stability theory, named after D. H. Hyers, S. M. Ulam, and Th. M. Rassias, explores the behavior of functional equations under tiny perturbations. Our goal for this part is to bring (H-U-R) stability to the equation (5).

Theorem 4.1: The equation $Ly = y$, where $L: X \to X$, exhibits (H-U) stability. In other words, for any given $z \in X$ and $\psi(t) > 0$, for all $t \in I$. There exists a solution y that satisfies the equation with

$$d(Lz, z) \leq \psi(t),$$

there exists a unique $y \in X$ such that in that case, there exists a single and distinct $y \in X$, $Ly = y$ and $d(z, y) \leq C\psi(t), C \geq 0$.

Proof. Assume that $z \in X$ and ψ a nonnegative function on such that

$$|Lz - z| \leq \psi(t).$$

In addition, let $y \in X$ is the unique solution of equation (5) on X. Then, we have

$$|z - y| \leq |z - Lz| + |Lz - y| \leq \psi(t) + |Lz - y|. \tag{13}$$

On the other hand, notice that

$$|Lz - Ly| \leq |Lz - y| = \left| \int_o^t G(t,s)\{z(s) - y(s)\} ds \right| \leq (M_1 + M_2 T) \max_{t \in I} |z(t) - y(t)| \int_0^t ds$$
$$\leq (M_1 T + M_2 T^2) d(z, y).$$

Thus, we obtain that

$$|Lz - y| \leq (M_1 T + M_2 T^2) d(z, y). \tag{14}$$

Therefore, from inequality (13) and (14) we conclude that

$$|z - y| \leq \psi(t) + (M_1 T + M_2 T^2) d(z, y)$$

which implies then

$$d(z, y) \leq C\psi(t)$$

with $C = \frac{1}{1 - (M_1 T + M_2 T^2)}$, that is, the equation (5) has the (H-U-R) stability.

Corollary 4.1: Theorem 2 holds for every finite interval $[a, b]$, $[a, b)$, $(a, b]$ and (a, b) when $-\infty < a < b < \infty$ for $a, b \in \mathbb{R}$.

5. Conclusion

In this manuscript, we explored various stability properties of Volterra Integro-differential equation. We investigated the (H-U) and (H-U-R) stability of a Volterra Integro-differential equation using the successive approximation approach. The importance of (H-U) stability can be explained as follows: When a continuously differentiable function serves as an approximate solution to an Integro-differential equation, there exists a nearby exact solution to the equation. In simpler terms, the distinction between the perturbed solution and the accurate solution is extremely minimal.

The concepts of (H-U) stability and (H-U-R) stability serve as a reminder that when studying a stable system under these frameworks, it is not necessary to obtain exact solutions. As a result, finding a function that satisfies a reasonable approximation inequality suffices. The presence of a close approximate solution that corresponds to the actual solution is ensured by (H-U-R) or (H-U) stability.

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