

## An Aspect of Basic (q-) Integral for q-Basic Series

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### Abstract

In the present aspect, several transformations of q-basic series with bilateral q-basic series have been established by employing the notion of basic (q-) integral. Some of these lead to relationships between the two q-series products. Following from these findings are some highly intriguing modifications of the basic and bilateral basic q-series. This article presents a handful of the numerous outcomes that are illustrative of the many results attained.

**Keywords:** Basic (q-) integral, basic and bilateral basic q-series, transformation.

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## 1. Introduction

The idea of q-basic series was initially presented by Euler in 1748. Heine, however, generalised the  ${}_2F_1$  series of Gauss in order to turn this into a systematic theory. Cauchy's q-binomial theorem, Jacobi's triple product identity, and Heine's transformation formula are likely the pillars that will allow the theory of q-basic hypergeometric series to be developed further through summation and transformation formulae. Since then, a number of mathematicians, including Aggrwal [1], Jackson [4-6], Andrews [2] and Shrivastav et. al. [13-14] have used techniques Bailey's transform symmetric & asymmetric Bailey's transform and contributed to the summations and transformations of q-basic and bilateral q-basic identities. Later Shrivastav et. al. [7-10] established a number of transformations on the same. A new approach of establishing the multi sum identities for q-series was also given by Shrivastav et. al. [12]. One might consult the book by Gasper and Rahman [3, 15] for more information.

Next, an expression in mathematics that generalizes a known expression and reduces to the known expression in the limit  $q>1$  is referred to as a q-analog, also known as a q-extension or q-generalization. Factorial, binomial coefficient, derivative, integral, and Fibonacci numbers are only a few examples of concepts that have q-analogs. Even q-Fourier analysis has been accomplished by Koornwinder, Suslov, and Bustoz.

This paper's primary goals are to find novel summations and transformations identities q-basic and to provide a straightforward demonstration of bilateral q-basic identity using basic (q-) integral. Further, special cases of these identities have been discussed. The section 2, cover some standard definitions and identities. The key results' proofs are presented in section 3 as well.

## 2. A Few Common Terminologies and Identities:

General Basic Hypergeometric series

$${}_A\Phi_B[a_1, a_2, a_3, \dots, a_A; b_1, b_2, b_3, \dots, b_B; q, x]$$

$$= \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \dots (a_A; q)_n x^n}{(b_1; q)_n (b_2; q)_n \dots (b_B; q)_n (q; q)_n} \quad (2.1)$$

In which there are always A of the a parameters and B of the b parameters. In such a case, the product of products

$$(a_1; q)_n (a_2; q)_n \dots (a_A; q)_n$$

It can be shortened still further to

$$((a; q))_n$$

where it is understood that there are always A of the a parameters.

Some other forms of q-shifted factorials are

$$(a_1; q)_n (a_2; q)_n \dots (a_A; q)_n = (a_1, a_2, a_3, \dots, a_A; q)_n$$

$$(a; q)_m = (1 - a)(1 - aq) \dots (1 - aq^{m-1})$$

In another form, the function defined as

$$(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n) \quad (2.2)$$

For  $|q| < 1$ . Since the above infinite product diverges when  $a \neq 0$ , and  $|q| \geq 1$ , whenever  $(a; q)_{\infty}$  appears in a formula, assuming that  $|q| < 1$ .

The general Basic Bilateral series is written as

$${}_A\Psi_B \left[ \begin{matrix} a_1, a_2, \dots, a_A; \\ b_1, b_2, \dots, b_B; \end{matrix} q, z \right] = \sum_{n=-\infty}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \dots (a_A; q)_n z^n}{(b_1; q)_n (b_2; q)_n \dots (b_B; q)_n} \quad (2.3)$$

This series is convergent for  $|q| < 1$ , for all values, real or complex, of the parameters  $a_1, a_2, \dots, a_A$  and  $b_1, b_2, \dots, b_B$  and for  $|z| \leq 1$ .

Some important identity, given below which will be used

For any number a and q real or complex and  $|q| < 1$

$$(a; q)_n = (a)_n = \begin{cases} (1 - a)(1 - aq)(1 - aq^2) \dots (1 - aq^{n-1}); & n > 0 \\ 1; & n = 0 \end{cases} \quad (2.4)$$

$$(a; q)_{\infty} = \prod_{r=0}^{\infty} (1 - aq^r) \quad (2.6)$$

$$(a; q)_{2n} = (a; q)_n (aq^n; q)_n \quad (2.7)$$

$$(a_1, a_2, \dots, a_n; q)_{\infty} = (a_1; q)_{\infty} (a_2; q)_{\infty} \dots (a_n; q)_{\infty} \quad (2.8)$$

$$(a; q)_{-n} = \frac{(-q/a)^n \cdot q^{n(n-1)/2}}{(q/a; q)_n} \quad (2.9)$$

The basic (q-) integral is defined by

$$\int_0^x f(t) d_q t = x(1 - q) \sum_{n=0}^{\infty} f(xq^n) q^n \quad (2.10)$$

The following q-basic series of  ${}_2\Psi_2$  will be employed in given analysis.

$${}_2\Psi_2 \left[ \begin{matrix} a, c; \\ b, acz; \end{matrix} q, z \right] = \frac{(q, b/a, az, q/az, cz; q)_{\infty}}{(b, q/a, z, b/az, acz; q)_{\infty}} \quad (2.11)$$

[3; App.

### 3. Main Findings:

$$\sum_{r=0}^{\infty} \frac{(a, cx; q)_r}{(b, acx; q)_r} (z)^r {}_3\Phi_2 \left[ \begin{matrix} acx, cxq^r, q; \\ cx, acxq^r; \end{matrix} q; q \right] = \frac{(q, b/a, az, q/az, czx; q)_{\infty}}{(b, q/a, z, b/az, aczx; q)_{\infty}} {}_2\Phi_1 \left[ \begin{matrix} aczx, q; \\ czx; \end{matrix} q; q \right] \quad (3.1)$$

$$\sum_{r=0}^{\infty} \frac{(a, c; q)_r}{(b, aczx; q)_r} (zx)^r {}_2\Phi_1 \left[ \begin{matrix} aczx, q; \\ acxq^r; \end{matrix} q; q^{1+r} \right]$$

$$= \frac{(q, b/a; q)_\infty (azx, q/azx, czx; q)_\infty}{(b, q/a; q)_\infty (zx, b/azx, aczx; q)_\infty} {}_4\Phi_3 \left[ \begin{matrix} zx, aczx, q^{n-1}/azx, q; \\ azx, czx, b/acxq^n; \end{matrix} q, q \right] \quad (3.2)$$

$$\sum_{r=0}^{\infty} \frac{(a, cx; q)_r}{(b, acx; q)_r} (z)^r \left\{ \sum_{n=0}^{\infty} \frac{(axq^r; q)_n q^n}{(bxq^r, aczxq^r; q)_n} \right\}$$

$$= \frac{(q, b/a, cz, q/azx, azx; q)_\infty}{(bx, q/ax, z, b/az, aczx; q)_\infty} \sum_{n=0}^{\infty} \frac{(bx, aczx, q^{1-n}/azx; q)_n}{(azx, q^{1-n}/ax; q)_n} \quad (3.3)$$

#### 4. Attestation

I. (i) On substituting  $c \rightarrow ct$ , in (2.11)

$${}_2\Psi_2 \left[ \begin{matrix} a, ct; \\ b, aczt; \end{matrix} q, z \right] = \frac{(q, b/a, az, q/az, czt; q)_\infty}{(b, q/a, z, b/az, aczt; q)_\infty} \quad (4.1)$$

And taking  $f(t) = \frac{(q, b/a, az, q/az, czt; q)_\infty}{(b, q/a, z, b/az, aczt; q)_\infty}$  in (2.10)

$$\int_0^x \frac{(q, b/a, az, q/az, czt; q)_\infty}{(b, q/a, z, b/az, aczt; q)_\infty} d_q t = x(1-q) \sum_{n=0}^{\infty} \frac{(q, b/a, az, q/az, czxq^n; q)_\infty}{(b, q/a, z, b/az, aczxq^n; q)_\infty} q^n \quad (4.2)$$

On oversimplifying the (4.2)

$$\int_0^x \frac{(q, b/a, az, q/az, czt; q)_\infty}{(b, q/a, z, b/az, aczt; q)_\infty} d_q t = x(1-q) \frac{(q, b/a, az, q/az, czx; q)_\infty}{(b, q/a, z, b/az, aczx; q)_\infty} {}_2\Phi_1 \left[ \begin{matrix} aczx, q; \\ czx; \end{matrix} q; q \right] \quad (4.3)$$

(ii) Taking  $f(t) = \sum_{r=0}^{\infty} \frac{(a, ct; q)_r}{(b, aczt; q)_r} z^r$  in (2.10),

$$\int_0^x \left\{ \sum_{r=0}^{\infty} \frac{(a, ct; q)_r}{(b, aczt; q)_r} z^r \right\} d_q t = x(1-q) \sum_{n=0}^{\infty} \left\{ \sum_{r=0}^{\infty} \frac{(a, cxq^n; q)_r}{(b, aczxq^n; q)_r} z^r \right\} q^n \quad (4.4)$$

On oversimplifying the (4.4)

$$\int_0^x \left\{ \sum_{r=0}^{\infty} \frac{(a, ct; q)_r}{(b, aczt; q)_r} z^r \right\} = x(1-q) \sum_{r=0}^{\infty} \frac{(a, cx; q)_r}{(b, acx; q)_r} z^r {}_3\Phi_2 \left[ \begin{matrix} acx, cxq^r, q; \\ cx, acxq^r; \end{matrix} q; q \right] \quad (4.5)$$

On equaling (4.3) & (4.5), our main finding (3.1) obtained.

II. (i) On substituting  $z \rightarrow zt$ , in (2.11)

$${}_2\Psi_2 \left[ \begin{matrix} a, c; \\ b, aczt; \end{matrix} q, zt \right] = \frac{(q, b/a, azt, q/azt, czt; q)_\infty}{(b, q/a, zt, b/azt, aczt; q)_\infty} \quad (4.6)$$

And taking  $f(t) = \frac{(q, b/a, azt, q/azt, czt; q)_\infty}{(b, q/a, zt, b/azt, aczt; q)_\infty}$  in (2.10)

$$\int_0^x \frac{(q, b/a, azt, q/azt, czt; q)_\infty}{(b, q/a, zt, b/azt, aczt; q)_\infty} d_q t = x(1-q) \sum_{n=0}^{\infty} \frac{(q, b/a, azxq^n, q/azxq^n, czxq^n; q)_\infty}{(b, q/a, zqxq^n, b/azxq^n, aczxq^n; q)_\infty} q^n \quad (4.7)$$

On oversimplifying the (4.7)

$$\int_0^x \frac{(q, b/a, azt, q/azt, czt; q)_\infty}{(b, q/a, zt, b/azt, aczt; q)_\infty} d_q t = x(1-q) \frac{(q, b/a; q)_\infty (azx, q/azx, czx; q)_\infty}{(b, q/a; q)_\infty (zx, b/azx, aczx; q)_\infty} {}_4\Phi_3 \left[ \begin{matrix} zx, aczx, q^{n-1}/azx, q; \\ azx, czx, b/acxq^n; \end{matrix} q, q \right] \quad (4.8)$$

(ii) Taking  $f(t) = \sum_{r=0}^{\infty} \frac{(a, c; q)_r}{(b, aczt; q)_r} (zt)^r$  in (2.10),

$$\int_0^x \left\{ \sum_{r=0}^{\infty} \frac{(a, c; q)_r}{(b, aczt; q)_r} (zt)^r \right\} d_q t = x(1-q) \sum_{n=0}^{\infty} \left\{ \sum_{r=0}^{\infty} \frac{(a, c; q)_r}{(b, aczxq^n; q)_r} (zxq^n)^r \right\} q^n \quad (4.9)$$

On oversimplifying the (4.9)

$$\int_0^x \left\{ \sum_{r=0}^{\infty} \frac{(a, c; q)_r}{(b, aczt; q)_r} (zt)^r \right\} d_q t = x(1-q) \sum_{r=0}^{\infty} \frac{(a, cx; q)_r}{(b, aczx; q)_r} (zx)^r {}_2\Phi_1 \left[ \begin{matrix} aczx, q; \\ acxq^r; \end{matrix} q; q^{1+r} \right] \quad (4.10)$$

On equaling (4.8) & (4.10), our main finding (3.2) obtained.

III. (i) On substituting  $a \rightarrow at, b \rightarrow bt$  in (2.11)

$${}_2\Psi_2 \left[ \begin{matrix} at, ct; \\ bt, aczt; \end{matrix} q, z \right] = \frac{(q, b/a, azt, q/azt, cz; q)_\infty}{(bt, q/at, z, b/az, aczt; q)_\infty} \quad (4.11)$$

And taking  $f(t) = \frac{(q, b/a, azt, q/azt, cz; q)_\infty}{(bt, q/at, z, b/az, aczt; q)_\infty}$  in (2.10)

$$\int_0^x \frac{(q, b/a, azt, q/azt, cz; q)_\infty}{(bt, q/at, z, b/az, aczt; q)_\infty} d_q t = x(1 - q) \sum_{n=0}^\infty \frac{(q, b/a, azxq^n, q/azxq^n, cz; q)_\infty}{(bxq^n, q/axq^n, z, b/az, aczxq^n; q)_\infty} q^n \quad (4.12)$$

On oversimplifying the (4.12)

$$\int_0^x \frac{(q, b/a, azt, q/azt, cz; q)_\infty}{(bt, q/at, z, b/az, aczt; q)_\infty} d_q t = x(1 - q) \frac{(q, b/a; q)_\infty (azx, q/azx, cz; q)_\infty}{(z, b/az; q)_\infty (bx, q/ax, aczx; q)_\infty} \sum_{n=0}^\infty \frac{(bx, aczx, q^{1-n}/azx; q)_n}{(azx, q^{1-n}/ax; q)_n} \quad (4.13)$$

(ii) Taking  $f(t) = \sum_{r=0}^\infty \frac{(at, c; q)_r}{(bt, aczt; q)_r} (z)^r$  in (2.10),

$$\int_0^x \left\{ \sum_{r=0}^\infty \frac{(at, c; q)_r}{(bt, aczt; q)_r} (z)^r \right\} d_q t = x(1 - q) \sum_{n=0}^\infty \left\{ \sum_{r=0}^\infty \frac{(axq^n, c; q)_r}{(bxq^n, aczxq^n; q)_r} (z)^r \right\} q^n \quad (4.14)$$

On oversimplifying the (4.14)

$$\int_0^x \left\{ \sum_{r=0}^\infty \frac{(at, c; q)_r}{(bt, aczt; q)_r} (z)^r \right\} d_q t = x(1 - q) \sum_{r=0}^\infty \frac{(a, cx; q)_r}{(b, acx; q)_r} (z)^r \left\{ \sum_{n=0}^\infty \frac{(axq^r; q)_n q^n}{(bxq^r, aczxq^r; q)_n} \right\} \quad (4.15)$$

On equaling (4.13) & (4.15), our main finding (3.3) obtained.

### 5. Special Cases

In this section, certain interesting finding has been made from the main finding.

(i). By employing the q-basic bilateral series (4.1) in (3.1), the one of interesting finding in form of product of q-basic and q-basic bilateral series have been obtained as

$${}_2\Psi_2 \left[ \begin{matrix} a, ct; \\ b, aczt; \end{matrix} q, z \right] \times {}_2\Phi_1 \left[ \begin{matrix} aczx, q; \\ czx; \end{matrix} q; q \right] = \sum_{r=0}^\infty \frac{(a, cx; q)_r}{(b, acx; q)_r} z^r {}_3\Phi_2 \left[ \begin{matrix} acx, cxq^r, q; \\ cx, acxq^r; \end{matrix} q; q \right] \quad (5.1)$$

(ii). By employing the q-basic bilateral series (4.6) in (3.2), the one of interesting finding in form of product of q-basic and q-basic bilateral series have been obtained as

$${}_2\Psi_2 \left[ \begin{matrix} a, c; \\ b, aczx; \end{matrix} q, zx \right] \times {}_4\Phi_3 \left[ \begin{matrix} zx, aczx, q^{n-1}/azx, q; \\ azx, czx, b/acxq^n; \end{matrix} q, q \right] = \sum_{r=0}^\infty \frac{(a, c; q)_r}{(b, aczx; q)_r} (zx)^r {}_2\Phi_1 \left[ \begin{matrix} aczx, q; \\ acxq^r; \end{matrix} q; q^{1+r} \right] \quad (5.2)$$

### 6. Conclusion

We conclude with the remark that the technique used here can be employed to yield a variety of interesting results involving q-basic and q-basic bilateral series which recursion formulas may find applications in numerous branches of mathematics, mathematical physics, engineering, and associated areas of study.

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