**An Aspect of Basic (q-) Integral for q-Basic Series**

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**Abstract**

In the present aspect, several transformations of q-basic series with bilateral q-basic series have been established by employing the notion of basic (q-) integral. Some of these lead to relationships between the two q-series products. Following from these findings are some highly intriguing modifications of the basic and bilateral basic q-series. This article presents a handful of the numerous outcomes that are illustrative of the many results attained.

**Keywords:** Basic (q-) integral, basic and bilateral basic q-series, transformation.

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1. **Introduction**

The idea of q-basic series was initially presented by Euler in 1748. Heine, however, generalised the $2F_1$ series of Gauss in order to turn this into a systematic theory. Cauchy's q-binomial theorem, Jacob's triple product identity, and Heine's transformation formula are likely the pillars that will allow the theory of q-basic hypergeometric series to be developed further through summation and transformation formulae. Since then, a number of mathematicians, including Aggrwal [1], Jackson [4-6], Andrews [2] and Shrivastav et. al. [13-14] have used techniques Bailey's transform symmetric & asymmetric Bailey's transform and contributed to the summations and transformations of q-basic and bilateral q-basic identities. Later Shrivastav et. al. [7-10] established a number of transformations on the same. A new approach of establishing the multi sum identities for q-series was also given by Shrivastav et. al. [12]. One might consult the book by Gasper and Rahman [3, 15] for more information.

Next, an expression in mathematics that generalizes a known expression and reduces to the known expression in the limit $q>1$ is referred to as a q-analog, also known as a q-extension or q-generalization. Factorial, binomial coefficient, derivative, integral, and Fibonacci numbers are only a few examples of concepts that have q-analogs. Even q-Fourier analysis has been accomplished by Koornwinder, Suslov, and Bustoz.

This paper's primary goals are to find novel summations and transformations identities q-basic and to provide a straightforward demonstration of bilateral q-basic identity using basic (q-) integral. Further, special cases of these identities have been discussed. The section 2, cover some standard definitions and identities. The key results' proofs are presented in section 3 as well.

2. **A Few Common Terminologies and Identities:**

General Basic Hypergeometric series

$\Phi_{n}[a_1, a_2, a_3, \ldots \ldots a_A; b_1, b_2, b_3 \ldots \ldots b_B; q, x]$
In which there are always A of the \( a \) parameters and B of the \( b \) parameters. In such a case, the product of products

\[
(a_1; q)_n (a_2; q)_n \ldots \ldots \ldots (a_A; q)_n
\]

It can be shortened still further to

\[
((a ; q))_n
\]

where it is understood that there are always A of the \( a \) parameters.

Some other forms of q-shifted factorials are

\[
(a_1; q)_n (a_2; q)_n \ldots \ldots \ldots (a_A; q)_n = (a_1, a_2, a_3, \ldots, a_A; q)_n
\]

\[(a; q)_n = (1 - a)(1 - aq) \ldots \ldots \ldots (1 - aq^{n-1})\]

In another form, the function defined as

\[
(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)
\]

For \(|q| < 1\). Since the above infinite product diverges when \( a \neq 0 \), and \(|q| \geq 1\), whenever \((a; q)_\infty\) appears in a formula, assuming that \(|q| < 1\).

The general Basic Bilateral series is written as

\[
\Lambda^\Psi\{a_1, a_2, \ldots, a_A; b_1, b_2, \ldots, b_B; q, r; z\} = \sum_{n=-\infty}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \ldots \ldots \ldots (a_A; q)_n}{(b_1; q)_n (b_2; q)_n \ldots \ldots \ldots (b_B; q)_n} z^n
\]

This series is convergent for \(|q| < 1\), for all values, real or complex, of the parameters \( a_1, a_2, \ldots, a_A \) and \( b_1, b_2, \ldots, b_B \) and for \(|z| \leq 1\).

Some important identity, given below which will be used

For any number \( a \) and \( q \) real or complex and \(|q| < 1\)

\[
(a; q)_n = (a)_n = \left\{ (1 - a)(1 - aq)(1 - aq^2) \ldots \ldots (1 - aq^{n-1}); n > 0 \right\}
\]

\[
(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)
\]

\[
(a; q)_2n = (a; q)_n (aq^n; q)_n
\]

\[
(a_1, a_2 \ldots \ldots a_n; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \ldots \ldots (a_n; q)_\infty
\]

\[
(a; q)_n = \frac{(-q/a)^n q^{n(n-1)/2}}{(q/a; q)_n}
\]

The basic (q-) integral is defined by

\[
\int_0^x f(t) d_q t = x(1 - q) \sum_{n=0}^{\infty} f(x q^n) q^n
\]

The following q-basic series of \( \Lambda^\Psi \) will be employed in given analysis.

\[
\Lambda^\Psi\{a, c; b, acz; q, r; z\} = \frac{(a, b/a, az, q/az, cz; q)_\infty}{(b, q/a, z, b/az, acz; q)_\infty}
\]

3. Main Findings:

\[
\sum_{r=0}^{\infty} \frac{(a, c; q)_r}{(b, acz; q)_r} (zx)^r \Phi_1\{aczx, q; cx, acxq^r; q ; q \} = (q, b/a, az, q/az, czx; q)_\infty \Phi_1\{aczx, q; czx, q ; q \}
\]

\[
\sum_{r=0}^{\infty} \frac{(a, c; q)_r}{(b, aczx; q)_r} (zx)^r \Phi_1\{aczx, q; acxq^r; q ; q^{1+r} \}
\]

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\[
\sum_{r=0}^{\infty} \frac{(a, cx; q)_r}{(b, acx; q)_r} (z)^r \left\{ \sum_{n=0}^{\infty} \frac{(ax^n; q)_n^n}{(bx^n; q)_n^n} \right\} = \frac{(q, b/a, cz, q/ax, axz; q)_\infty}{(bx, q/ax, z, b/az, acx; q)_\infty} \sum_{n=0}^{\infty} \frac{(ax^n; q)_n^{1-n/axz; q)}{(ax, q^{1-n}/ax; q)_n^{1-n}}
\]

4. Attestation

I. (i) On substituting \( c \to ct \), in (2.11)

\[
\mathcal{I}_2 \left[ \frac{a, ct}{b, acxt} ; q, z \right] = \frac{(q, b/a, az, q/az, czt; q)_\infty}{(b, q/a, z, b/az, acxt; q)_\infty}
\]

And taking \( f(t) = \frac{(q, b/a, az, q/az, czt; q)_\infty}{(b, q/a, z, b/az, acxt; q)_\infty} \) in (2.10)

\[
\int_0^x \frac{(q, b/a, az, q/az, czt; q)_\infty}{(b, q/a, z, b/az, acxt; q)_\infty} dt = x(1 - q) \sum_{n=0}^{\infty} \frac{(q, b/a, az, q/az, czxq^n; q)_\infty}{(b, q/a, z, b/az, acxq^n; q)_\infty} q^n
\]

On simplifying the (4.2)

\[
\int_0^x \frac{(q, b/a, az, q/az, czt; q)_\infty}{(b, q/a, z, b/az, acxt; q)_\infty} dt = x(1 - q) \frac{(a, cx, czxq^n; q)_\infty}{(b, acx, czx; q)_\infty} \Phi_2 \left[ a; cx, czxq^n; q, q \right]
\]

(ii) Taking \( f(t) = \sum_{r=0}^{\infty} \frac{(a, ct; q)_r}{(b, acxt; q)_r} z^r \) in (2.10),

\[
\int_0^x \frac{\left( \sum_{r=0}^{\infty} \frac{(a, ct; q)_r}{(b, acxt; q)_r} z^r \right)}{q^n} dt = x(1 - q) \sum_{n=0}^{\infty} \frac{\left( \sum_{r=0}^{\infty} \frac{(a, cxq^n; q)_r}{(b, acxq^n; q)_r} z^r \right)}{q^n}
\]

On simplifying the (4.4)

\[
\int_0^x \frac{\left( \sum_{r=0}^{\infty} \frac{(a, ct; q)_r}{(b, acxt; q)_r} z^r \right)}{q^n} dt = x(1 - q) \sum_{r=0}^{\infty} \frac{(a, cx, czxq^n; q)_\infty}{(b, acx, czx; q)_\infty} \Phi_2 \left[ a; cx, czxq^n; q, q \right]
\]

On equaling (4.3) & (4.5), our main finding (3.1) obtained.

II. (i) On substituting \( z \to zt \), in (2.11)

\[
\mathcal{I}_2 \left[ \frac{a, C}{b, aczt} ; q, zt \right] = \frac{(q, b/a, azt, q/azt, czt; q)_\infty}{(b, q/a, zt, b/azt, aczt; q)_\infty}
\]

And taking \( f(t) = \frac{(q, b/a, azt, q/azt, czt; q)_\infty}{(b, q/a, zt, b/azt, aczt; q)_\infty} \) in (2.10)

\[
\int_0^x \frac{(q, b/a, azt, q/azt, czt; q)_\infty}{(b, q/a, zt, b/azt, aczt; q)_\infty} dt = x(1 - q) \sum_{n=0}^{\infty} \frac{(q, b/a, aztq^n, q/aztq^n, cztq^n; q)_\infty}{(b, q/a, ztq^n, b/aztq^n, acztq^n; q)_\infty} q^n
\]

On simplifying the (4.7)

\[
\int_0^x \frac{(q, b/a, azt, q/azt, czt; q)_\infty}{(b, q/a, zt, b/azt, aczt; q)_\infty} dt = x(1 - q) \frac{(a, czx, czbtq^n; q)_\infty}{(b, acxz, czbtq^n; q)_\infty} \Phi_2 \left[ a; czx, czbtq^n; q, q \right]
\]

(ii) Taking \( f(t) = \sum_{r=0}^{\infty} \frac{(a, C; q)_r}{(b, aczt; q)_r} (zt)^r \) in (2.10),

\[
\int_0^x \frac{\left( \sum_{r=0}^{\infty} \frac{(a, C; q)_r}{(b, aczt; q)_r} (zt)^r \right)}{q^n} dt = x(1 - q) \sum_{n=0}^{\infty} \frac{\left( \sum_{r=0}^{\infty} \frac{(a, C; q)_r}{(b, acztq^n; q)_r} (ztq^n)^r \right)}{q^n}
\]

On simplifying the (4.9)

\[
\int_0^x \frac{\left( \sum_{r=0}^{\infty} \frac{(a, C; q)_r}{(b, aczt; q)_r} (zt)^r \right)}{q^n} dt = x(1 - q) \sum_{r=0}^{\infty} \frac{(a, C; q)_r}{(b, aczt; q)_r} (zt)^r \Phi_2 \left[ a; cxzq^n, q; q^{1+r} \right]
\]

On equaling (4.8) & (4.10), our main finding (3.2) obtained.

III. (i) On substituting \( a \to at \), \( b \to bt \) in (2.11)

\[
\mathcal{I}_2 \left[ \frac{a, ct}{b, aczt} ; q, z \right] = \frac{(q, b/a, azt, q/azt, czt; q)_\infty}{(bt, q/atz, z, b/atz, aczt; q)_\infty}
\]
And taking \( f(t) = \frac{q}{b/a, azx, q/azx, cz; q} \) in (2.10)

\[
\sum_{n=0}^{\infty} \frac{(q, b/a, azxq^n, q/azxq^n, cz; q)_n}{(bxq^n, q/azxq^n, z, b/az, aczxq^n; q)_n} q^n
\]

On oversimplifying the (4.12)

\[
\sum_{n=0}^{\infty} \frac{(a, c, q)_n}{(azxq^n, aczxq^n; q)_n} q^n
\]

(ii) Taking \( f(t) = \sum_{r=0}^{\infty} \frac{(at, c, q)_r}{(bt, aczt, q)_r} (z)^r \) in (2.10),

\[
\sum_{r=0}^{\infty} \frac{(a, cx; q)_r}{(bxq^n, aczxq^n; q)_r} (z)^r
\]

On oversimplifying the (4.14)

\[
\sum_{r=0}^{\infty} \frac{(a, c, q; q)_r}{(b, acx; q)_r} (z)^r
\]

On equaling (4.13) & (4.15), our main finding (3.3) obtained.

5. Special Cases

In this section, certain interesting finding has been made from the main finding.

(i). By employing the q-basic bilateral series (4.1) in (3.1), the one of interesting finding in form of product of q-basic and q-basic bilateral series have been obtained as

\[
\sum_{r=0}^{\infty} \frac{(a, c, q; q)_r}{(b, acxq^n; q)_r} (z)^r
\]

(ii). By employing the q-basic bilateral series (4.6) in (3.2), the one of interesting finding in form of product of q-basic and q-basic bilateral series have been obtained as

\[
\sum_{r=0}^{\infty} \frac{(a, c, q; q)_r}{(b, acxq^n; q)_r} (z)^r
\]

6. Conclusion

We conclude with the remark that the technique used here can be employed to yield a variety of interesting results involving q-basic and q-basic bilateral series which recursion formulas may find applications in numerous branches of mathematics, mathematical physics, engineering, and associated areas of study.

References