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An Embedding of Space of Continuous Functions of a Hausdorff, Locally Compact Space

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Abstract

Assume X is a Hausdorff, locally compact space. This paper demonstrates the space of continuous functions on $K(X)$, which is, the open sets in the topology of Vietoris on the collection of all compact subspaces of X embedded into $K(C(K(X))$ with Fell topology. Combining with another embedding theorem, it is confirmed that C(X) is placed within K(C(K(X)))

Keywords: Hyperspace, locally compact, regular, compact-open, Vietoris, Fell, topology.

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1. INTRODUCTION

Let X Serve as a topological space let $K(X)$ constitute the relations of nonempty compact subsets of X. $K(X)$ Provided concerning Vietoris set topology. For a subset $E \subseteq X$, we introduce two operations: These include $E^-$and E^+ which are subsets of $K(X)$.

For $E \subset X$, we compose $E^-$and E^+ for the following subsets of $K(X)$. $E^- = \{K \in K(X): K \cap E \neq \emptyset\}$ and $E^+ = \{K \in K(X): K \subset E\}$. In E^- the sets are the intersection on E but in E^+, the sets don't contain things in the complement of E. Vietoris topology is constructed using a sub base that comprises of sets like

V^- where V any accessible set in X also other sets like W^+ where W is also an open set in X. Detailed discussion on Vietoris topology is given in [1],[2],[9].

Let $C(X)$ and $C(K(X))$ act the topology of smooth functions X and $K(X)$ respectively. We endow $C(X)$ and $C(K(X))$ Supplied with the compact-open topology. Compact-open structure on C(X) has subbases that include specific elements in their formation.

$$\langle K: U\rangle = \{f \in C(X): f(K) \subset U\}$$

where $K \subset X$ acts as compact and $U \subset X$ is accessible. On $C(K(X))$, compact-open topology has a sub-base as $\langle \bar{K}: \bar{U}\rangle = \{f \in C\big(K(X)\big): \bar{f}(\bar{K}) \subset \bar{U}\}$ where $\bar{K} \subset K(X)$ acts as compact and $\bar{U} \subset K(X)$ seems accessible. See [5],[6],[7].

$f \in C(X)$ induces a *Flow function* $\bar{f}$ over $C(K(X))$ given by means of $\bar{f}(K) = f(K), \forall K \in K(X)$.

Define a partial order $\leq$ on $C(K(X))$ by $\bar{f} \leq \bar{g}$ in case that $f(K) \subseteq g(K), \forall K \in K(X)$.

We verify that $(C(K(X)), \leq)$ is a topological poset.

Let $(\bar{f})^* = \{\bar{g} \in C(K(X)) : \bar{g} \leq \bar{f}\}$ be the principal ideal and $(\bar{f})_* = \{\bar{g} \in C(K(X)) : \bar{f} \leq \bar{g}\}$ be the principal filter.

$(\bar{f})^*$ and $(\bar{f})_*$ are compact sets in $C(K(X))$. As of $C(K(X))$ continuous function set on $K(X)$, we take Fell topology on $K(C(K(X)))$.

Fell topology [10], the topology involves subsets in the configurations of V^- where V is open and W^+, where W has the compact complement. See [1],[3],[9],[17].

The set $K(X)$ with Vietoris topology is represented as $(K(X))_V$ and space $K(C(K(X)))$ using Fell topology is represented as $K(C(K(X)))_F$.

In this work with some assumptions on the set X, we clarify that $C(X)$ is fixed in $C(K(X))$ The role

$\phi : C(X) \to C(K(X))_V$ specified as $\phi(f) = \bar{f}$ is continuous as well as embeds $C(X)$ into $C(K(X))_V$. Then we prove that $\psi : C(K(X))_V \to K(C(K(X))_V)_F$ specified as $\psi(\bar{f}) = (\bar{f})^*$ is continuous and embeds

$C(K(X))_V$ into $C(K(X))_V)_F$. Hence the main theorem of the paper is,

Theorem. $\mathbf{C(X) \xrightarrow{\Phi} C(K(X))_V \xrightarrow{\Psi} K(C(K(X))_V)_F}$ **is an embedding.**

Many authors have studied induced mappings in different contexts [11],[12],[14]. Most of the embedding theorems on the hyperspace available in the literature are for real-valued functions. Embedding of the given space to its hyperspace is studied in [4],[8],[13],[14]. An advantage of Fell topology on the hyperspace of continuous functions over Vietoris topology, from the dynamics point of view is given in [12]. But as far as the knowledge of the authors is concerned, embedding into the hyperspace the concept of continuous functions with the Fell topology is a relatively recent addition to the list of topologies. In section 2, we verify that $C(K(X))_V$ represents a topological poset. We will prove some results on the continuity of ϕ and ψ in section 3 and finally, the proof of the main theorem is given in section 4.

2. $(C(K(X))_V, \leq)$ Acts as a Topological Poset

Proposition 1. *Assuming X acts as a Hausdorff set then* $C(K(X))_V, \leq)$ *represent a topological poset.*

Proof. Consider a net $\{(\bar{f}_\lambda, \bar{g}_\lambda)\}$ in $C(K(X))_V \times C(K(X))_V$ which converges to $(\bar{f}, \bar{g})$.

The greatest lower bound of $\bar{f}_\lambda$ and $\bar{g}_\lambda$, $\bar{f}_\lambda \wedge \bar{g}_\lambda$ for each λ, we hold $\bar{f}_\lambda \wedge \bar{g}_\lambda = \bar{f}_\lambda$.

As X acts as Hausdorff , $(K(X))_V$ is Hausdorff and hence $C(K(X))_V$ is Hausdorff. So, a net converges to at most one limit.

ie, $\bar{f} \wedge \bar{g} = \bar{f}$ which gives $\bar{f} \leq \bar{g}$.

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3.Continuity of ψ and ϕ

Proposition 2. *If X is Hausdorff, then* $\psi : C(K(X))_V \to (K(C(K(X))_V)_F$ *defined as* $\psi(\bar{f}) = (\bar{f})^*$ *is continuous.*

Proof. Let $\bar{f} \in C(K(X))_V$.

Consider U as an accessible set in $C(K(X))_V$ with $(\bar{f})^* \in \langle U \rangle^-$.

That is $(\bar{f})^* \cap U \neq \emptyset$. Let $\bar{g} \in (\bar{f})^* \cap U$.

As $\bar{g} \in (\bar{f})^*$, $\bar{g} \leq \bar{f}$, and hence $\bar{f} \wedge \bar{g} = \bar{g}$. U is an open neighbourhood of g in $C(K(X))_V$.

Due to Continuity of $\wedge$, there exist open sets V and W of $C(K(X))_V$ so that

$(\bar{f}, \bar{g}) \in V \times W$ and $(V \cap W) \subseteq U$.

Hence, $\bar{h} \wedge \bar{g} \in (\bar{h})^* \cap U$ for all $\bar{h} \in U$.

Hence, for each $\bar{h} \in U, (\bar{h})^* \cap U \neq \emptyset$.

ie, $(\bar{h})^* \in \langle U \rangle^-$. So $\{(\bar{h})^* : \bar{h} \in U\} \subseteq \langle U \rangle^-$.

Consider K as a compact subset of $C(K(X))_V$ with $(\bar{f})^* \in (K^c)^+$.ie, $(\bar{f})^* \subseteq K^c$. We will

find an open neighbourhood V of $\bar{f}$ in $C(K(X))_V$ as to $\{(\bar{h})^* : \bar{h} \in V\} \subseteq (K^c)^+$.

Suppose there is no such V. Let $\mathcal{U}(\bar{f})$ be the family of all open neighbourhood of $\bar{f}$ in $C(K(X))_V$.

Then for each $U \in \mathcal{U}(\bar{f})$, there exist $\bar{f} \in U$ such that $(\bar{f}_U)^* \nsubseteq K^c$.

That is, $(\bar{f}_U)^* \cap K \neq \emptyset$. Let $\bar{g}_U \in (\bar{f}_U)^* \cap K$ for each $U \in \mathcal{U}(\bar{f})$.

$\{\bar{g}_U\}$ is a net with directed set $(\mathcal{U}(\bar{f}), \supseteq)$.

Since K is compact, this net has a cluster point, say $\bar{g}$, in K.

So, $(\bar{g}, \bar{f}) \in \overline{\{(\bar{g}_U, \bar{f}_U) : U \in \mathcal{U}(\bar{f})\}}$, as $\bar{g}_U \leq \bar{f}_U$ for each $U \in \mathcal{U}(\bar{f})$, we have $\bar{g} \leq \bar{f}$. But then,

$\bar{g} \in (\bar{f})^* \cap K$, contradicting the fact that $(\bar{f})^* \subseteq K^c$.

Hence ψ is continuous.

Proposition 3. *If X acts as Hausdorff, locally compact, and* $C(K(X))_V$ *is also connected, then*

$\psi^{-1} : (\psi(C(K(X))_V))_F \to C(K(X))_V$ *is continuous.*

Proof. Let $\{\bar{f}_\wedge\}$ be a net in $C(K(X))_V$ with directed set $(\urcorner, \Delta)$ and $\bar{f} \in C(K(X))_V$ where $(\bar{f}_\wedge)^* \to (\bar{f})^*$

in $(\psi(C(K(X))_V))_F$.

We will prove that $\bar{f}_\wedge \to \bar{f}$ in $C(K(X))_V$.

If not, suppose $\bar{f}_\wedge \nrightarrow \bar{f}$ in $C(K(X))_V$.

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X satisfies both conditions of being locally compact and Hausdorff, $C\big(K(X)\big)_V$ *is also compact locally and Hausdorff. Therefore, there be a local base of compact neighbourhoods, denoted by* $\overline{K}$*, around* $\bar{f}$ *in* $C(K(X))_V$ so that $\overline{f_\Lambda} \in K^c$, $\forall \Lambda \in ㄱ$.

Let $U_{\overline{K}}(\bar{f}) = \{U \subseteq C\big(K(X)\big)_V : U \text{ is open and } U \in (\overline{K})^o\}$

Let $\Omega = \{(\lambda, U): \lambda \in ㄱ, U \in U_{\overline{K}}(\bar{f}) \text{ and } \bar{f_\lambda} \cap U \neq \emptyset\}$

Let '≪' be the partial order in Ω Identified by $(\alpha, U) \ll (\beta, V)$ if $\alpha \Delta \beta$ and $U \subseteq V$.

Then $(\Omega, \ll)$ is a directed poset.

For each $(\lambda, U) \in \Omega$, let $W_{\lambda,U} \in \left(\bar{f_\lambda}\right)^* \cap U$.

As $\left(\overline{f_\Lambda}\right)^* \to \left(\bar{f}\right)^*$ in $(\psi(C(K(X)))_V)_F$, every open neighbourhood V of $\bar{f}$ in

$C(K(X))_V$, there exist, $\lambda_V \in ㄱ$ As a result of, $\left(\bar{f_\lambda}\right)^* \in \langle V \cap (\overline{K})^o \rangle^-$ whenever $\lambda_V \Delta \lambda$.

Hence, $W_{\lambda, U} \in U \subseteq V$ whenever $(\lambda_V, V \cap ((\overline{K})^\circ) \ll (\lambda, U)$.

Thus $W_{\lambda,U} \to \bar{f}$.

As $C(K(X))_V$ is connected, for any $\lambda \in ㄱ$, there exists $\bar{h}_{\lambda,U} \in \partial(\overline{K}) \cap \left(\overline{W}_{\lambda,U}\right)_* \cap \left(\bar{f_\lambda}\right)^*$.

Otherwise, $(\overline{K})^o$ and $(\overline{K})^c$ would determine a separation of $\left(\overline{W}_{\lambda,U}\right)_* \cap \left(\bar{f_\lambda}\right)^*$.

As $\partial\overline{K}$ is compact, the net $\{\bar{h}_{\lambda,U}\}$ has a cluster point, say $\bar{h}$ in ∂K.

Claim: $\bar{h} = \bar{f}$.

First, we prove that $\bar{h} \leq \bar{f}$.

Suppose not, ie, $\bar{h} \in \left(\left(\bar{f}\right)^*\right)^c$.

Since $C(K(X))_V$ is locally compact, $\bar{h}$ must have a compact neighbourhood W

which is disjoint from $\left(\bar{f}\right)^*$.

As $\left(\overline{f_\Lambda}\right)^* \to \left(\bar{f}\right)^*$ in $(\psi(C(K(X))_V))_F$, there exist $\lambda_0 \in ㄱ$ such that

$\left(\bar{f_\lambda}\right)^* \cap W = \emptyset, \forall \lambda \in ㄱ$ with $\lambda_0 \Delta \lambda$. Since $\bar{h}$ is a cluster point of $\{\bar{h}_{\lambda,U}\}$.

We have $\{\bar{h}_{\lambda,U}\} \in W^o$ for cofinal set of indices $(\lambda, U) \in \Omega$.

As $\bar{h}_{\lambda,U} \in \left(\bar{f_\lambda}\right)^*, \forall \lambda$

This result is a contradiction. Hence $\bar{h} \leq \bar{f}$.

Now we prove that $\bar{f} \leq \bar{h}$.

take $V \in U_{\overline{K}}(\overline{f})$ and any open neighbourhood U of $\bar{h}$. Since $\left(\overline{f_\Lambda}\right)^* \to \left(\bar{f}\right)^*$ there exist $\lambda_0 \in ㄱ$ such that $\left(\bar{f_\lambda}\right)^* \cap V \neq \emptyset$. ie, $(\lambda, V) \in \Omega$, $\forall \lambda \in ㄱ$ with $\lambda_0 \Delta \lambda$.

As $\bar{h}$ is a cluster point of $\{\bar{h}_{\lambda,U}\}$, there exists a $(\lambda, G) \in \Omega$ with $\lambda_0 \Delta \lambda$ where both

$G \subseteq V$ and $\bar{h}_{\lambda,U} \in U$ hold.

Since $W_{\lambda, G} \in G$ we have, $\left(W_{\lambda,G}, \bar{h}_{\lambda,U}\right) \in G \times U \subseteq V \times U$.

As $W_{\lambda,G} \leq \bar{h}_{\lambda,U}$ we have $(\bar{f}, \bar{g})$ belong to the closure of $V \times U$ in $C(K(X))_V \times C(K(X))_V$.

As $\leq$ closed, $\bar{f} \leq \bar{h}$.

∴ The claim is proved.

So, ψ^{-1} is continuous. □

Proposition 4. *Let X be regular. Then* ϕ: $C(X) \to C(K(X))_V$, *defined as*

$\phi(f) = \bar{f}$ *is continuous*

Proof: To demonstrate this proposition, we need a result proved in [1], which we quote as a lemma given below.

Lemma 3.1 (1)**.** *Let X be regular. Though* $\overline{K} \subseteq K(X)$, *then* $\bigcup_{k\in\bar{K}} K \in K(X)$

Proof of Proposition 4:

Let V exist as a sub-basis neighbourhood of $\phi(f)$. that is to say, one can find a sub-basic neighbourhood U_i of f such that $\phi(\bigcap_i U_i) \subset V$.

Let $\bar{K}$ be a compact subset of $K(X)$, after by lemma 3.1, $\bigcup_{k\in\bar{K}} K$ is compact. Hence

sub basis open sets of $C(K(X))$ are given by

$\langle \bar{K}: \langle U_i \rangle \rangle = \{\psi \in C\big(K(X)\big): \psi(\bar{K}) \subset \langle U_i \rangle\}$ where $\langle U_i \rangle = \{K \in K(X): K \subseteq \cup_i U_i, K \cap U_i \neq \emptyset, \forall i\}$

Let $\langle \bar{K}: \langle U_i \rangle \rangle$ be a sub basis neighbourhood of $\phi(f)$.Then,

$$\phi^{-1}(\langle \bar{K}: \langle U_i \rangle \rangle) = [\langle \cup_{K\in\bar{K}} K : \cup_i U_i \rangle] \bigcap [\{f \in C(X): f(K) \cap U_i \neq \emptyset, \forall K \in \bar{K}\}]$$

Claim: For each $f^{-1}(U_i)$ there is an open sub set V_i of X such that $\bar{V} \subset f^{-1}(U_i)$ and for each

(3.1) $K \in \bar{K}, K \cap V_i \neq \emptyset$

If not, for each V_i , there exist $K_{V_i} \in \bar{K}$ such that $K_{V_i} \cap V_i = \emptyset$

Let $O(U) = \{U': U' \subset U\}$.Then $(O(U), \subseteq)$ is a poset. Hence, $N: V_i \to K_{V_i}$ is a net. N possesses a convergent cofinal sub net since $\bar{K}$ is compact. Let K_0 be the limit of this subnet.

Then, $K_0 \cap V_i = \emptyset, \forall V_i \subset \bar{V}_i \subset f^{-1}(U_i)$.

Since X is regular, $\cup_i V_i = U$.There fore, $K_0 \cap f^{-1}(U_i) = \emptyset$

This is a contradiction to the fact $K_0 \cap f^{-1}(U_i) \neq \emptyset$.

Hence the claim holds:

Choose $V_i \subset f^{-1}(U_i)$ with the property (3.1).

Let $x_k^i \in V_i \cap K$ for each V_i and K.

Define the set K_i is compact because it acts as a closed subset $\bigcup_{K \in \overline{K}} K$ of which is compact.

Therefore, $f(K_i) \subset \overline{V}_i \subset U_i$. So $f \in \langle K_i : U_i \rangle$

$\phi(\langle K_i : U_i \rangle) \subset \{f \in C(X) : f(K) \cap U_i \neq \emptyset, \forall K \in \overline{K}\}$.

Therefore, $[\langle \cup_{K \in \overline{K}} K : \cup_i U_i \rangle] \cap [\cap_i (\langle \overline{K}_i : \langle U_i \rangle \rangle]) \subset \phi^{-1}(\langle \overline{K}_i : \langle U_i \rangle \rangle)$

Hence $\phi^{-1}(\langle K_i : U_i \rangle)$ is an open set in $C(X)$.

So ϕ is continuous.

Proposition 5. *If X is regular, $\phi^{-1} : (\phi(C(X))_V \to C(X)$ is continuous.*

Proof: Since X is regular,C(K(X)) is regular. So $\phi(C(X))_V \subset C(K(X))$ is regular. Since ϕ is one-one. Hence by similar argument in Proposition.4, ϕ^{-1} is continuous.

Main Theorem

Theorem: *If X is Hausdorff, Locally compact, regular, and $C(K(X))_V$ is connected ,then $C(X)$ is embedded in $K(C(K(X))_V)_F$*

Proof. From proposition 2 and 3,

$$C\big(K(X)\big)_V \cong \Big(\phi\Big(C\big(K(X)\big)_V\Big)\Big)_F \subset \Big(K(C\big(K(X)\big)_V\Big)_F$$

From proposition 4 and 5,

$$C(X) \cong \phi(C(K(X))_V \subset C\big(K(X)\big)_V$$

ie, $f \overset{\phi}{\to} \bar{f} \overset{\psi}{\to} (\bar{f})^*$

Therefore

$$C(X) \cong (\psi(\phi(C(K(X))_V)_F \subseteq \Big(K(C\big(K(X)\big)_V\Big)_F.$$

Competing interests: The authors declare none

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