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Critical Analyzing on Some New Application of Almost Decreasing Sequence to Legendre Series Associated with [B] Sum

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Abstract: In this study, we present a novel application of a nearly decreasing sequence connected to [B] summation. Additionally, a novel and well-known arbitrary result was attained by applying the main theorem. The current findings are validated by taking into account the paper's circumstance of a prior result.

Keywords: Borel, summability, sequences, etc.

1. Introduction

It is important to note that Cauchy's "Course d'Analyses algebrique" in 1821 and Abel's research [1] on the Binomial series in 1826 gave the dated, hazy notations of convergence of infinite series a notable place on a solid foundation. On the other hand, it was noted that a few non-convergent series, particularly in Dynamical Astronomy, accoutered results that were remarkably close to the mark. When Cesaro [11] published a paper on the multiplication of series in the year 1890, the theory of divergent series was first introduced. Most eminent mathematical analysts have focused their creative energy on the theory of series. The late 19th and early 20th centuries are credited with being the eras when adequate methods were able to associate with them certain values that could be called their sums in a reasonable manner through a process closely associated with Cauchy's concept of convergence (see [13-20]).

1.1 Definition (see [11]); Consider a sequence $\{ U_m\}$ of m^{th} partial sums corresponding to the infinite series $\sum u_m$. If

$$\lim_{m\to\infty} e^{-a} \sum_{m=0}^{\infty} \frac{a^m}{m!} U_m \qquad (1)$$

exists and is equal to a finite limit U, then the sequence $\{ U_m\}$ is known as Borel exponential method to the finite limit U.

1.2 Definition (see [10]); Let $\{ U_m\}$ denotes the sequence of m^{th} partial sum of the given infinite series, $\sum u_m$.

$$\text{If } \sum_{m=0}^{\infty} \frac{a^m}{m!} | U_m - U | = O(e^a), \text{ as } a \to \infty \qquad (2)$$

then $\{ U_m\}$ is known as strongly [B] summable**.**

1.3 Definition(see [1]): The Lebesgue integral of the function g(y) over the interval [-1, 1],

$$g(y) \sim \sum_{m=0}^{\infty} b_m Q_m(y), \qquad (3)$$

is known as Legendre series, where

$$b_m = (m + \frac{1}{2}) \int_{-1}^{1} g(y)\, Q_m(y) \tag{4}$$

And

Legendre polynomial (see [1]) defined by the following relations

$$\frac{1}{\sqrt{(1-2yz+z^2)}} = \sum_{m=0}^{\infty} z^m\, Q_m(y). \tag{5}$$

2. Known Results

Let us write the following notations
$\chi(t) = g\{\cos(\theta - t) - g(\cos\theta)\}$,
$\tau = [\frac{1}{t}]$.
In 2012, Diekema and Koornminder [2] defined generalization of an integral for Legendre polynomials by Persson and Strong, In 2011, Kamimoto et al defined on Borel summability of WKB-theoretic transformation. Working there are several results in the same direction which have been studied by the authors ([4]-[9]).
Theorem: If

$$\int_0^t |\, g(y \pm v) - g(y)\,|\, dy = o\left[\frac{t}{\log\frac{1}{t}}\right], \text{ as } t \to 0 \tag{6}$$

Then, the given series is summable Borel summable to g(y) at y in (-1,1).

With the help of the above theorem, we established the following theorem:

3. Main Results

Theorem: Let us suppose that a positive and monotonic decreasing sequence of constant $\{\alpha_m\}$ be such that the m^{th} partial sum $\alpha_m \to 0$ as $m \to \infty$. If

$$\int_0^l |g(y \pm v) - g(v)|\, dv = o\left(\frac{\eta\left(\frac{1}{l}\right)\alpha_\tau}{\xi(\tau)}\right), \text{ as } l \to \infty, \tag{7}$$

Where, $\eta(l)$ and $\xi(l)$ are two positive functions l such that $\eta(l)$, $\xi(l)$ and $\frac{t.\eta(l)}{\xi(l)}$ increases monotonically with l and

$$\eta(m^\beta)\alpha_m^\beta = o\left[\xi\left(\alpha_m^\beta\right)\right], \text{ as } m \to \infty \text{ for } 0 < \beta \le 1, \tag{8}$$

Then, the equation (3) is strongly [B] summable to the sum g(y) at y ∈ (-1,1).

Lemma (see in [12]); Let $\{\alpha_m\}$ be sequence such that $\alpha_m \to \infty$ as $m \to \infty$, we have

$$\int_0^l |g(\cos\theta - z) - g(\cos\ \theta)|\, dz = o\left[\frac{\eta\left(\frac{1}{l}\right).\alpha_\tau}{\xi(\alpha_\tau)}\right] \quad \text{as } l \to \infty \tag{9}$$

where, $z = \cos\theta,\ z + v = \cos\phi \quad \theta - \phi = z.$

Proof: Using well-known result after (see [9]), we have the m^{th} partial sum $U_m(y)$ of the series (3) at $y \in (-1, 1)$, is given by

$$U_m(y) - g(y) = \frac{1}{\pi.\sqrt{\sin\theta}} \int_0^{\frac{1}{\alpha^\beta}} [g(\cos\theta - z) - g(\cos\theta)].\frac{\sin(m+1)l}{\sin\frac{l}{2}}.\sqrt{\sin(\theta - l)}\ \ dl + O(1) \tag{10}$$

where, $o < \frac{1}{\alpha^\beta} \le \sigma < 1, y = \cos\theta,\ z = \cos\phi,\ \ o < \theta\ \xi, o < \phi < \pi,\ \ \theta - p = l \quad etc.$

Therefore,

$$\sum_{m=0}^{\infty}\frac{a^m}{m\,!}|U_m-g(y)|=\sum_{m=0}^{\infty}\frac{a^m}{m\,!}\left|\int_0^{\frac{1}{a^\beta}}\frac{\chi(l)}{sin\frac{l}{2}}\frac{\sqrt{sin(\theta-l)}}{\pi.\sqrt{sin\,\theta}}sin(m+1)l\,dl\right|+\left(\sum_{m=0}^{\infty}\frac{a^m}{m!}\right)$$

$$=\left[\int_0^{\frac{1}{a^\beta}}\frac{|\chi(l)|}{l}\left|e^{a\,cos\,l}.sin(sin\,l+l)dl\right|\right]+O(e^a)$$

$$=\left[\left(\int_0^{\frac{1}{a}}+\int_{\frac{1}{a}}^{\frac{1}{a^\beta}}\right)\frac{\chi(l)}{l}.e^{a\,cos\,l}.sin(a\,sin\,l+l)dl+(e^a)\right]$$

$$=O(M_1)+O(M_2)+O(e^a)\text{ (say).}\qquad(11)$$

Now, we have to show that,

$M_1=O(\theta.e^a),\qquad M_2=O(e^a)\;as\;\;a\to\infty.$

Let us first consider M_1,

$$M_1=\int_0^{\frac{1}{a}}\frac{|\chi(l)|}{l}\left|e^{a\,cos\,l}.sin(a\,sin\,l+l)\right|dl$$

$$=\mathrm{O}[(a+l)e^a]\int_0^{\frac{1}{a}}|\,\chi(l)|dl$$

$$=\mathrm{O}\big[[(a+l)e^a]\big].O\left(\frac{\eta(a)\alpha_\tau}{\xi(\alpha_\tau)}\right),(by\;9)$$

$$=(e^a).\mathrm{O}\left[\frac{\eta(a)\alpha_\tau}{\xi(\alpha_\tau)}\right]$$

$\therefore\quad (a+1)\alpha_\tau$ (by the condition $\{\alpha_m\}$) $=O(e^a)o(1)\quad(\,by\;\;9\,).$

$$=O(e^a)\;\text{ as } a\to\infty.\qquad(12)$$

Again, considering $M_{2,}$

We have

$$M_2=\int_{\frac{1}{a}}^{\frac{1}{a^\beta}}\frac{|\chi(l)|}{l}\left|e^{a\cos l}.sin\,(a\,sin\,l+l)\right|dl$$

$$=\mathrm{O}(e^a)\int_{\frac{1}{a}}^{\frac{1}{a^\beta}}\frac{|\chi(l)|}{l}dl$$

Now,

$$\int_{\frac{1}{a}}^{\frac{1}{a^\beta}}\frac{|\chi(l)|}{l}dl\;=\;\frac{1}{l}\;\mathrm{O}\left[\frac{\eta(\frac{1}{l})\alpha_\tau}{\xi(\alpha_\tau)}\right].\eta^2dl$$

$$=\mathrm{O}\left[\frac{a^\beta\eta(a^\beta)\alpha_\tau^\beta}{\xi\alpha_\tau^\beta}\right]-\mathrm{O}\left[\frac{a\,\eta(a)\alpha_\tau}{\xi(\alpha_\tau)}\right]+O\left[\frac{\eta(a)\alpha_\tau}{\xi(\alpha_\tau)}\right]\int_{\frac{1}{a}}^{\frac{1}{a^\beta}}\eta^2dl$$

$$=\left[\frac{\eta(a^\beta)\alpha_\tau^\beta}{\xi\alpha_\tau^\beta}\right]-\mathrm{O}\left[\frac{\eta(a)\alpha_\tau}{\xi(\alpha_\tau)}\right]+\mathrm{O}\left[\frac{\eta(a)\alpha_\tau}{\xi(\alpha_\tau)}\right]\mathrm{O(a)}$$

$$= O\left[\frac{\eta(\alpha^{\beta})\alpha_{\tau}^{\beta}}{\xi\alpha_{\tau}^{\beta}}\right] - O\left[\frac{\eta(\alpha).\alpha_{\tau}}{\xi(\alpha_{\tau})}\right] + \left[\frac{\eta(\alpha).\alpha_{\tau}}{\xi(\alpha_{\tau})}\right]$$

$$= O(1) \quad \text{(by 8).}$$

Thus,

$$M_2 = O(e^{a}) \int_{\frac{1}{a}}^{\frac{1}{a^{\beta}}} \frac{|\chi(l)|}{l} dl$$

$$= O(e^{a})\, O(1) \tag{13}$$

$$= O(e^{a}) \text{ as } a \to \infty.$$

Combining (12) and (13) together with (10) and (11), we get the required result that

$\sum_{m=0}^{\infty} \frac{a^m}{m!} |U_m(y) - g(y)| = o(e^{a})$ as $n \to \infty$.

4. Conclusion

Borel method is used to overcome divergence problem, which in field theory seem to be present even for arbitrary small value of the interaction parameter.

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