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# On Third-Order Differential Subordination and Superordination Properties of Analytic Functions Defined by Tayyah-Atshan Fractional Integral Operator

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**Abstract:**

The objective of this paper is to examine the outcomes of third-order differential subordination and superordination for analytic functions within the region $\mathcal{G} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, specifically focusing on the utilization of the operator $\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(z)$. The results are derived by analyzing relevant categories of allowable functions. New findings have been found about differential subordination and superordination, along with the discovery of several sandwich theorems. Furthermore, other specific instances are also seen. The qualities and outcomes of differential subordination exhibit symmetry with the properties of differential superordination, leading to the formulation of the sandwich theorems.

**Keywords:** Differential subordination, Superordination, Analytic function, Fractional operator, Sandwich theorem, Third-order.

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## 1-Introduction

Consider $\mathbb{H}(\mathcal{G})$ as a set of analytic functions within the open unit disk $\mathcal{G}$, defined as $\mathcal{G} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. For $n \in \mathbb{N} = \{1,2,3,\cdots\}$ such that $a \in \mathbb{C}$, and let $\mathbb{H}[a,n] = \{f : f \in \mathbb{H}(\mathcal{G}) \text{ and } f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}$ and suppose that $\mathbb{H}_0 = \mathbb{H}[0,1]$. Consider $\mathcal{A} \subset \mathbb{H}(\mathcal{G})$, which denotes the subset of functions in $\mathcal{G}$ that are both analytic and have been normalized. The Taylor-Maclaurin series is a mathematical series that takes the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathcal{G}). \tag{1.1}$$

Assume that both $f$ and $g$ belong to $\mathbb{H}(\mathcal{G})$. We assert that $f$ is a subordinate of $g$, (or $g$ is a superordinate of $f$), denoted

$$f \prec g \text{ in } \mathcal{G} \text{ or } f(z) \prec g(z), (z \in \mathbb{C}),$$

when there is a Schwarz function $\mathfrak{h} \in \mathbb{H}$, defined in the unit disk $\mathcal{G}$, that is analytic and satisfies $\mathfrak{h}(0) = 0$ with $|\mathfrak{h}(z)| < 1$ $(z \in \mathbb{C})$, it means $f(z) = g(\mathfrak{h}(z)), (z \in \mathbb{C})$. Furthermore, if the function $g$ is injective within the domain $\mathcal{G}$. Subsequently, we can construct the following equivalency based on the references provided in [1,2,3].

$$f(z) \prec g(z) \leftrightarrow f(0) = g(0) \text{ and } f(\mathcal{G}) \subset g(\mathcal{G}).$$

Tayyah and Atshan [4] introduced the following fractional operator:

$$\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(z) = \frac{(\mu+1)^{1-\frac{\sigma(\delta-1)+1}{\tau}}}{\sigma \Gamma_{\sigma,\tau}(\delta)} \int_{0}^{z} (z^{\mu+1} - v^{\mu+1})^{\frac{\sigma(\delta-1)+1}{\tau}-1} v^{\mu} f(v)\, dv, \qquad (1.2)$$

where $\sigma \in \mathbb{N}, \tau > 0, \delta \geq 1 - \frac{1}{\sigma}, \mu \geq 0,$ and

$$\Gamma_{\sigma,\tau}(\delta) = \tau^{\frac{\sigma(\delta-1)+1}{\tau}-1} \Gamma\left(\frac{\sigma(\delta-1)+1}{\tau}\right).$$

Define $\mathfrak{T}^{\delta}_{(\alpha,\beta,\mu)}: \mathcal{A} \to \mathcal{A}$ by

$$\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\frac{\sigma(\delta-1)+1}{\tau} + 1 + \frac{1}{\mu+1}) \Gamma(\frac{n}{\mu+1} + 1)}{\Gamma(\frac{1}{\mu+1} + 1) \Gamma(\frac{\sigma(\delta-1)+1}{\tau} + 1 + \frac{n}{\mu+1})} z^{n}. \qquad (1.3)$$

By (3), it is easy to verify the following identity:

$$z\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(z)\right)' =$$

$$\left[(\mu+1)\frac{\sigma(\delta-1)+1}{\tau} + 1\right] \mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(z) - \left[(\mu+1)\frac{\sigma(\delta-1)+1}{\tau}\right] \mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(z). \quad (1.4)$$

Antonino and Miller [1] have extended the idea of second-order differential subordination and superordination in the domain $\mathcal{G}$, as initially established by Miller and Mocanu [2,3,5], to the third-order scenario. This extension is also referenced in [6,7]. The features of functions p that meet the third-order differential subordination were determined:

$$\{\Theta(G(z), zG'(z), z^{2}G''(z), z^{3}G'''(z); z) : z \in \mathcal{G}\} \subset \Omega.$$

Additionally, this applies to third-order differential superordination:

$$\Omega \subset \{\Theta(G(z), zG'(z), z^{2}G''(z), z^{3}G'''(z); z) : z \in \mathcal{G}\},$$

where $\Omega$ be a set within $\mathbb{C}$, $G$ be an analytic function with $\Theta: \mathbb{C}^{4} \times \mathcal{G} \to \mathbb{C}$.

In recent studies, multiple writers have examined various implementations of the second-order differential subordination and superordination idea. They have also produced sandwich outcomes, as evidenced by references [8]. Additionally, third-order outcomes have been explored for diverse classes, as indicated by references [6,7,9]. To explore intriguing applications of differential subordination and superordination in other mathematical disciplines, we can consult references [10,11,12].

Ponnusamy and Juneja's work [13] built the concept of third-order differential subordination. Tang et al. introduced a recent study that is a good example of this (see [6,7]).

The second and third-order terms are used interchangeably. Uneven subordination and bi-univalent functions have garnered the attention of numerous scholars in this domain (see [8, 9, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]).

In this study, we analyze a collection of appropriate admissible functions related to the integral operator with established precise requirements on the normalized analytic function, referred to as the sandwich condition.

## 2- Preliminaries

The concepts and lemmas listed below are necessary for the demonstration of our findings.

**Definition 2.1. [1].** Consider the function $\Theta : \mathbb{C}^4 \times \mathcal{G} \to \mathbb{C}$ with assume that a function $\mathfrak{h}(\mathfrak{z})$ is univalent within $\mathcal{G}$. Given that the function $G(\mathfrak{z})$ is analytic in $\mathcal{G}$ with fulfilling the given third-order differential subordination:

$$\Theta(G(\mathfrak{z}), \mathfrak{z}G'(\mathfrak{z}), \mathfrak{z}^2 G''(\mathfrak{z}), \mathfrak{z}^3 G'''(\mathfrak{z}); \mathfrak{z}) \prec \mathfrak{h}(\mathfrak{z}). \tag{2.1}$$

thus $G(\mathfrak{z})$ is denoted as a solution of the differential subordination (2.1). Moreover, a given univalent function $\mathcal{T}(\mathfrak{z})$ is referred to as a dominant of the solutions of Equation (2.1). Alternatively, a dominant is defined as $G(\mathfrak{z}) \prec \mathcal{T}(\mathfrak{z})$ for any $G(\mathfrak{z})$ that meets Equation (2.1). The best dominant is defined as a dominant $\tilde{\mathcal{T}}(\mathfrak{z})$ which fulfills the condition $\tilde{\mathcal{T}}(\mathfrak{z}) \prec \mathcal{T}(\mathfrak{z})$ for every dominant $\mathcal{T}(\mathfrak{z})$ of (2.1).

**Definition 2.2. [1].** Consider the set $\mathbb{Q}$, which consists of every function $\mathcal{T}$ that is both univalent and analytic on $\bar{\mathcal{G}} \backslash E(\mathcal{T})$, where

$$E(\mathcal{T}) = \{\zeta : \zeta \in \partial\mathcal{G} : \lim_{\mathfrak{z} \to \zeta} \mathcal{T}(\mathfrak{z}) = \infty\},$$

with $\min|\mathcal{T}'(\zeta)| = \rho > 0$ for $\zeta \in \partial\mathcal{G} \backslash E(\mathcal{T})$. Additionally, we can represent the subclass of $\mathbb{Q}$, where $\mathcal{T}(0) = a$, as $Q(a)$, where $\mathbb{Q}(0) = \mathbb{Q}_0$ and $\mathbb{Q}(1) = \mathbb{Q}_1$.

The method of subordination is used for an appropriate class of admissible functions.

The following class of admissible functions was given by Antonino and Miller [1].

**Definition 2.3. [1].** Consider $\Omega$, a set of complex numbers, and let $\mathcal{T} \in \mathbb{Q}$ such that $n \in \mathbb{N}\ \{1\}$. The class of admissible functions $\Psi_n[\Omega, \mathcal{T}]$ has functions $\Theta : \mathbb{C}^4 \times \mathcal{G} \to \mathbb{C}$, which fulfills the following admissibility conditions:

$$\Theta(\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{u}; \mathfrak{z}) \notin \Omega,$$

whenever

$$\mathfrak{r} = \mathcal{T}(\zeta), \qquad \mathfrak{s} = k\zeta\mathcal{T}'(\zeta), \qquad Re\left(\frac{\mathfrak{t}}{\mathfrak{s}} + 1\right) \geq kRe\left(\frac{\zeta\mathcal{T}''(\zeta)}{\mathcal{T}'(\zeta)} + 1\right),$$

and

$$Re\left(\frac{\mathfrak{u}}{\mathfrak{s}}\right) \geq k^2 Re\left(\frac{\zeta^2\mathcal{T}'''(\zeta)}{\mathcal{T}'(\zeta)}\right),$$

where $\mathfrak{z} \in \mathcal{G}, \zeta \in \partial\mathcal{G} \backslash E(\mathcal{T})$, with $k \geq n$.

**Lemma 2.1. [1].** Suppose that $G \in \mathbb{H}[a,n]$ such that $n \geq 2$, and $\mathcal{T} \in \mathbb{Q}(a)$ satisfying the following conditions:

$$Re\left(\frac{\zeta\mathcal{T}''(\zeta)}{\mathcal{T}'(\zeta)}\right) \geq 0, \left|\frac{\mathfrak{z}G'(\mathfrak{z})}{\mathcal{T}'(\zeta)}\right| \leq k,$$

where $\mathfrak{z} \in \mathcal{G}, \zeta \in \partial\mathcal{G} \setminus E(\mathcal{T}), k \geq n.$ If $\Omega$ is a set within $\mathbb{C}, \Theta \in \Psi_n[\Omega, \mathcal{T}]$, with

$$\Theta(G(\mathfrak{z}), \mathfrak{z}G'(\mathfrak{z}), \mathfrak{z}^2G''(\mathfrak{z}), \mathfrak{z}^3G'''(\mathfrak{z}); \mathfrak{z}) \in \Omega,$$

then

$$G(\mathfrak{z}) \prec \mathcal{T}(\mathfrak{z}), \quad (\mathfrak{z} \in \mathcal{G}).$$

**Definition 2.4. [7].** Consider the function $\Theta: \mathbb{C}^4 \times \mathcal{G} \to \mathbb{C}$. Let $\mathfrak{h}(\mathfrak{z})$ be an analytic in $\mathcal{G}$. Given the function $G(\mathfrak{z})$ with

$$\Theta(G(\mathfrak{z}), \mathfrak{z}G'(\mathfrak{z}), \mathfrak{z}^2G''(\mathfrak{z}), \mathfrak{z}^3G'''(\mathfrak{z}); \mathfrak{z}),$$

are univalent in $\mathcal{G}$ and fulfill the given third-order differential superordination:

$$\mathfrak{h}(\mathfrak{z}) \prec \Theta(G(\mathfrak{z}), \mathfrak{z}G'(\mathfrak{z}), \mathfrak{z}^2G''(\mathfrak{z}), \mathfrak{z}^3G'''(\mathfrak{z}); \mathfrak{z}), \tag{2.2}$$

if $G(\mathfrak{z})$ satisfies differential superordination, it is considered a solution. Further, an analytic function $\mathcal{T}$ is referred to as a subordinant of the solutions of the differential superordination, or it's just a subordinant, if $\mathcal{T}(\mathfrak{z}) \prec G(\mathfrak{z})$ in each $G(\mathfrak{z})$ fulfilling Equation (2.2). A univalent subordnant $\tilde{\mathcal{T}}(\mathfrak{z})$ which fulfill $\mathcal{T}(\mathfrak{z}) \prec \tilde{\mathcal{T}}(\mathfrak{z})$ for every subordinants $\mathcal{T}(\mathfrak{z})$ of (2.2) is known to be the best subordinant.

**Definition 2.5. [7].** Consider Ω, a set of the complex numbers, and let $\mathcal{T} \in \mathbb{H}[a,n]$ such that $\mathcal{T}'(\mathfrak{z}) \neq 0$. The function class $\Psi'_n[\Omega, \mathcal{T}]$ is defined as the set of functions $\Theta : \mathbb{C}^4 \times \bar{\mathcal{G}} \to \mathbb{C}$ which fulfills the next admissibility conditions:

$$\Theta(\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{u};\ \zeta) \in \Omega,$$

whenever

$$\mathfrak{r} = \mathcal{T}(\mathfrak{z}), \mathfrak{s} = \frac{\mathfrak{z}\mathcal{T}'(\mathfrak{z})}{m}, Re\left(\frac{\mathfrak{t}}{\mathfrak{s}} + 1\right) \leq \frac{1}{m} Re\left(\frac{\mathfrak{z}\mathcal{T}''(\mathfrak{z})}{\mathcal{T}'(\mathfrak{z})} + 1\right),$$

and

$$Re\left(\frac{\mathfrak{u}}{\mathfrak{s}}\right) \leq \frac{1}{m^2} Re\left(\frac{\mathfrak{z}^2\mathcal{T}'''(\mathfrak{z})}{\mathcal{T}'(\mathfrak{z})}\right),$$

where $\mathfrak{z} \in \mathcal{G}, \zeta \in \partial\mathcal{G}$, with $m \geq n \geq 2$.

**Lemma 2.2. [7].** Consider $\mathcal{T} \in \mathbb{H}[a,n]$, and $\Theta \in \Psi'_n[\Omega, \mathcal{T}]$. Assuming that

$$\Theta(G(z), zG'(z), z^2G''(z), z^3G'''(z); z)$$

is a univalent within $\mathcal{G}$ with $G \in \mathbb{Q}(a)$ that fulfills the following conditions:

$$Re\left(\frac{\zeta\mathcal{T}''(z)}{\mathcal{T}'(z)}\right) \geq 0, \left|\frac{zG'(z)}{\mathcal{T}'(z)}\right| \leq m,$$

where $z \in \mathcal{G}, \zeta \in \partial\mathcal{G}$, with $m \geq n \geq 2$, thus

$$\Omega \subset \{\Theta(G(z), zG'(z), z^2G''(z), z^3G'''(z); z) : z \in \mathcal{G}\},$$

indicates that

$$\mathcal{T}(z) \prec G(z), \quad (z \in \mathcal{G}).$$

The present study applies the methods described in the works of Antonino and Miller [1], Jeyaraman and Suresh [23], and Tang et al. [6,25] to examine the third-order differential subordination and superordination outcomes. Various situations are considered, as documented in references [18,24]. This paper examines specific categories of permissible functions and presents novel findings about third-order differential subordination and superordination for analytic functions within the domain $\mathcal{G}$, concerning the operator $\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(z)$.

## 3- Third-Order Differential Subordination of the Operator $\mathfrak{T}^{\delta}_{(\alpha,\beta,\mu)}f(z)$

We present some differential subordination results using the operator $\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(z)$.

**Definition 3.1.** Let $\Omega$ be a set in $\mathbb{C}$ and $\mathcal{T} \in \mathbb{Q}_0 \cap \mathrm{H}_0$. The acceptable function class $\Im_j[\Omega, \mathcal{T}]$ consists of those functions $\Theta : \mathbb{C}^4 \times \mathcal{G} \to \mathbb{C}$ that fulfill the admissibility condition:

$$\Theta(a, b, c, d; z) \notin \Omega,$$

whenever

$$a = \mathcal{T}(\zeta), \qquad b = \frac{\zeta k\mathcal{T}'(\zeta) + (\eta + 1)\mathcal{T}(\zeta)}{\eta + 2},$$

$$Re\left(\frac{(\eta+2)[(\eta+2)c - 2(\eta+1)b] + (\eta+1)^2 a}{(\eta+2)b - (\eta+1)a}\right) \geq kRe\left(\frac{\zeta\mathcal{T}''(\zeta)}{\mathcal{T}'(\zeta)} + 1\right),$$

and

$$Re\left(\frac{(\eta+2)^3[d - 3(c-b)] - (\eta+2)[3(\eta+1)a + b] + [(\eta+1)a(1 - (\eta+1)^2)]}{(\eta+2)b - (\eta+1)a}\right) \geq k^2 Re\left(\frac{\zeta^2\mathcal{T}'''(\zeta)}{\mathcal{T}'(\zeta)}\right),$$

where $z \in \mathcal{G}, \zeta \in \partial\mathcal{G} \backslash E(\mathcal{T})$ and $k \in \mathbb{N} \backslash \{1\}$.

**Theorem 3.1.** Let $\Theta \in \Im_j[\Omega, \mathcal{T}]$. If the functions $f \in \mathcal{A}$ and $\mathcal{T} \in \mathbb{Q}_0 \cap \mathbb{H}_0$ satisfies the condition:

$$Re\left(\frac{\zeta\mathcal{T}''(\zeta)}{\mathcal{T}'(\zeta)}\right) \geq 0 \quad , \quad \left|\frac{\mathfrak{z}\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\right)'}{\mathcal{T}'(\zeta)}\right| \leq k, \tag{3.1}$$

and

$$\left\{\Theta\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}), \mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}), \mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}), \mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}); \mathfrak{z}\right) : \mathfrak{z} \in \mathcal{G}\right\} \subset \Omega, \tag{3.2}$$

then

$$\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}) \prec \mathcal{T}(\mathfrak{z}), \qquad (\mathfrak{z} \in \mathcal{G}).$$

**Proof.** Let $G(\mathfrak{z})$ be analytic function in $\mathcal{G}$ by

$$G(\mathfrak{z}) = \mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}). \tag{3.3}$$

From equation (1.4) and differentiating (3.3) with respect to $\mathfrak{z}$, we get

$$\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}) = \frac{\mathfrak{z}G'(\mathfrak{z}) + (\eta+1)G(\mathfrak{z})}{\eta+2}. \tag{3.4}$$

By a similar argument, yields

$$\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}) = \frac{\mathfrak{z}^2G''(\mathfrak{z}) + [2(\eta+1)+1]\mathfrak{z}G'(\mathfrak{z}) + (\eta+1)^2G(\mathfrak{z})}{(\eta+2)^2}, \tag{3.5}$$

and

$$\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}) = \frac{\mathfrak{z}^3G'''(\mathfrak{z}) + 3(\eta+2)\mathfrak{z}^2G''(\mathfrak{z}) + [3(\eta+1)(\eta+2)+1]\mathfrak{z}G'(\mathfrak{z}) + (\eta+1)^3G(\mathfrak{z})}{(\eta+2)^3}. \tag{3.6}$$

Define the transformation starting with $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(\mathfrak{r},\mathfrak{s},\mathfrak{t},u) = \mathfrak{r}, \qquad b(\mathfrak{r},\mathfrak{s},\mathfrak{t},u) = \frac{\mathfrak{s} + (\eta+1)\mathfrak{r}}{\eta+2},$$

$$c(\mathfrak{r},\mathfrak{s},\mathfrak{t},u) = \frac{\mathfrak{t} + (2\eta+3)\mathfrak{s} + (\eta+1)^2\mathfrak{r}}{(\eta+2)^2}, \tag{3.7}$$

and

$$d(\mathfrak{r},\mathfrak{s},\mathfrak{t},u) = \frac{u + 3(\eta+2)\mathfrak{t} + [3(\eta+1)(\eta+2)+1]\mathfrak{s} + (\eta+1)^3\mathfrak{r}}{(\eta+2)^3}. \tag{3.8}$$

Let $\varphi(\mathfrak{r},\mathfrak{s},\mathfrak{t},u) = \Theta(a,b,c,d) =$

$$\Theta\left(\begin{array}{c} \mathfrak{r}, \dfrac{\mathfrak{s} + (\eta+1)\mathfrak{r}}{\eta+2}, \dfrac{\mathfrak{t} + (2\eta+3)\mathfrak{s} + (\eta+1)^2\mathfrak{r}}{(\eta+2)^2}, \\ \dfrac{u + 3(\eta+2)\mathfrak{t} + [3(\eta+1)(\eta+2)+1]\mathfrak{s} + (\eta+1)^3\mathfrak{r}}{(\eta+2)^3}; \mathfrak{z} \end{array}\right). \tag{3.9}$$

The proof will be put to use by Lemma(2.1).Using equations (3.3) to (3.6), and from (3.9), we get

$$\varphi(G(\mathfrak{z}),\mathfrak{z}G'(\mathfrak{z}),\mathfrak{z}^2G''(\mathfrak{z}),\mathfrak{z}^3G'''(\mathfrak{z});\mathfrak{z}) = \Theta\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z});\mathfrak{z}\right). \quad (3.10)$$

Hence, (3.2) leads to

$$\varphi(G(\mathfrak{z}),\mathfrak{z}G'(\mathfrak{z}),\mathfrak{z}^2G''(\mathfrak{z}),\mathfrak{z}^3G'''(\mathfrak{z});\mathfrak{z}) \in \Omega,$$

note that

$$\frac{\mathfrak{t}}{\mathfrak{s}}+1=\frac{(\eta+2)[(\eta+2)c-2(\eta+1)b]+(\eta+1)^2a}{(\eta+2)b-(\eta+1)a}$$

and

$$\frac{u}{\mathfrak{s}}=\frac{(\eta+2)^3[d-3(c-b)]-(\eta+2)[3(\eta+1)a+b]+\left[(\eta+1)a(1-(\eta+1)^2)\right]}{(\eta+2)b-(\eta+1)a}.$$

As a result, the admissibility condition in Definition (3.1) for $\Theta \in \Im_j[\Omega,\mathcal{T}]$ is equivalent to the condition $\varphi \in \Psi_2[\Omega,\mathcal{T}]$ as stated in Definition (2.3) with $n=2$. As a result, using (3.1) and Lemma (2.1), we have

$$\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}) \prec \mathcal{T}(\mathfrak{z}).$$

The proof of the Theorem (3.1) is complete.

The following outcome is an extension of Theorem (3.1) to the case where the actions of $\mathcal{T}(\mathfrak{z})$ on $\partial\mathcal{G}$ is unknown.

**Corollary 3.1.** Let $\Omega\subset\mathbb{C}$ and the function $\mathcal{T}$ be univalent in $\mathcal{G}$ with $\mathcal{T}(0)=1$. Let $\Theta \in \Im_j[\Omega,\mathcal{T}_\rho]$ for some $\rho \in (0,1)$, where $\mathcal{T}_\rho(\mathfrak{z})=\mathcal{T}(\rho\mathfrak{z})$. If the function $f\in\mathcal{A}$ and $\mathcal{T}_\rho \in \mathbb{Q}_0$ satisfies the next conditions:

$$Re\left(\frac{\zeta\mathcal{T}_\rho''(\zeta)}{\mathcal{T}_\rho'(\zeta)}\right)\geq 0\,,\left|\frac{\mathfrak{z}\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\right)'}{\mathcal{T}_\rho'(\zeta)}\right|\leq k\,,(\mathfrak{z}\in\mathcal{G},\zeta\in\partial\mathcal{G}\backslash E(\mathcal{T}_\rho)\text{ and }k\geq 2)$$

and

$$\Theta\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z});\mathfrak{z}\right)\in\Omega,$$

then

$$\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}) \prec \mathcal{T}_\rho(\mathfrak{z}),(\mathfrak{z}\in\mathcal{G}).$$

**Proof.** Using the Theorem (3.1), we obtain

$$\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}) \prec \mathcal{T}_\rho(\mathfrak{z}),(\mathfrak{z}\in\mathcal{G}).$$

Corollary asserts the following conclusion (3.1) is now deduced from the subordination characteristic that follows: $\mathcal{T}_\rho(\mathfrak{z}) \prec \mathcal{T}(\mathfrak{z})$, $(\mathfrak{z}\in\mathcal{G})$.

If $\Omega \neq \mathbb{C}$ is a domain with only one connection, then $\Omega = \mathfrak{h}(\mathcal{G})$ for the purpose of conformal mapping $\mathfrak{h}(z)$ of $\mathcal{G}$ onto $\Omega$. The class in this situation is $\mathfrak{I}_j[\mathfrak{h}(\mathcal{G}), \mathcal{T}]$ is written as $\mathfrak{I}_j\ [\mathfrak{h}, \mathcal{T}]$.

This is a direct result of the Theorem.(3.1) and Corollary (3.1).

**Theorem 3.2.** Suppose that $\Theta \in \mathfrak{I}_j[\mathfrak{h}, \mathcal{T}]$. When the function $f \in \mathcal{A}$ with $\mathcal{T} \in \mathbb{Q}_0 \cap \mathrm{H}_0$ satisfy the following conditions:

$$Re\left(\frac{\zeta \mathcal{T}_\rho''(\zeta)}{\mathcal{T}_\rho'(\zeta)}\right) \geq 0\,, \left|\frac{z\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(z)\right)'}{\mathcal{T}_\rho'(\zeta)}\right| \leq k, \tag{3.11}$$

and

$$\Theta\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(z), \mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(z), \mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(z), \mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(z); z\right) \prec \mathfrak{h}(z). \tag{3.12}$$

Then

$$\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(z) \prec \mathcal{T}(z)\,, (z \in \mathcal{G}).$$

The following result is a direct result of Corollary. (3.1).

**Corollary 3.2.** Suppose that $\Omega \subset \mathbb{C}$ with $\mathcal{T}$ be univalent within $\mathcal{G}$ such tha $\mathcal{T}(0) = 1$. Let $\Theta \in \mathfrak{I}_j[\Omega, \mathcal{T}]$ for some $\rho \in (0,1)$, where $\mathcal{T}_\rho(z) = \mathcal{T}(\rho z)$. If the function $f \in \mathcal{A}$ and $\mathcal{T}_\rho$ satisfies the following conditions:

$$Re\left(\frac{\zeta \mathcal{T}_\rho''(\zeta)}{\mathcal{T}_\rho'(\zeta)}\right) \geq 0\,, \qquad \left|\frac{z\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(z)\right)'}{\mathcal{T}_\rho'(\zeta)}\right| \leq k\,, (z \in \mathcal{G}, \zeta \in \partial\mathcal{G} \backslash E(\mathcal{T}_\rho) \text{ and } k \geq 2)$$

and

$$\Theta\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(z), \mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(z), \mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(z), \mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(z); z\right) \prec \mathfrak{h}(z),$$

then

$$\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(z) \prec \mathcal{T}(z), \qquad (z \in \mathcal{G}).$$

The best dominant of the differential subordination is seen in the following result (3.12).

**Theorem 3.3.** Consider the function $\mathfrak{h}$ in $\mathcal{G}$ to be univalent, and consider $\Theta\colon \mathbb{C}^4 \times \mathcal{G} \to \mathbb{C}$ and $\varphi$ to be defined by (3.10). Consider the equation of differentiation:

$$\varphi(G(z), zG'(z), z^2G''(z), z^3G'''(z); z) = \mathfrak{h}(z) \tag{3.13}$$

possesses a solution $\mathcal{T}(z)$ with $\mathcal{T}(0) = 1$, which fulfils requirement (3.1). If $f \in \mathcal{A}$ meets the condition (3.12) and if

$$\Theta\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z});\mathfrak{z}\right),$$

is analytic in $\mathcal{G}$, then

$$\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\prec\mathcal{T}(\mathfrak{z}),\qquad(\mathfrak{z}\in\mathcal{G})$$

and $\mathcal{T}(\mathfrak{z})$ is the best dominant.

**Proof.** According to Theorem (3.1), $\mathcal{T}$ is a dominant of (3.12). Given that $\mathcal{T}$ satisfies (3.13), it follows that $\mathcal{T}$ is also a solution of (3.12). Consequently, $\mathcal{T}$ will be subjugated by all dominant entities. Consequently, $\mathcal{T}$ is the optimal dominator. The proof of the theorem is comprehensive.

In light of the Definition (3.1), and in the special case $\mathcal{T}(\mathfrak{z})=\mathrm{M}\mathfrak{z}$ ($M>0$), the class of admissible functions $\Im_j[\Omega,\mathcal{T}]$, denoted by $\Im_j[\Omega,M]$, expresses itself as follows.

**Definition 3.2.** Let $\Omega$ be set in $\mathbb{C}$ and $M>0$. The class of admissible functions $\Im_j[\Omega,M]$ includes those functions $\Theta:\mathbb{C}^4\times\mathcal{G}\to\mathbb{C}$ such that

$$\Theta\left(\begin{array}{c}Me^{i\theta},\frac{(k+(\eta+1))Me^{i\theta}}{\eta+2},\frac{L+[[2(\eta+1)+1]k+(\eta+1)^2]Me^{i\theta}}{(\eta+2)^2},\\ \frac{\mathbb{N}+3(\eta+2)L+[(3(\eta+1)(\eta+2)+1)k+(\eta+1)^3]Me^{i\theta}}{(\eta+2)^3};\mathfrak{z}\end{array}\right)\notin\Omega,\quad(3.14)$$

whenever $\mathfrak{z}\in\mathcal{G}$,

$$Re\left(Le^{-i\theta}\right)\geq(k-1)kM,$$

and

$$Re\left(\mathbb{N}e^{-i\theta}\right)\geq 0,\qquad\forall\theta\in\mathbb{R},k\geq 2.$$

**Corollary 3.3.** Consider $\Theta\in\Im_j[\Omega,M]$. If the function $f\in\mathcal{A}$ satisfies:

$$\left|\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\right|\leq kM,\qquad(\mathfrak{z}\in\mathrm{U},k\geq 2;M>0)$$

and

$$\Theta\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z});\mathfrak{z}\right)\in\Omega,$$

then

$$\left|\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\right|<M.$$

In the special case $\Omega=\mathcal{T}(\mathcal{G})=\{w:|w|<M\}$, the class $\Im_j[\Omega,\mathcal{T}]$ is simply referred as $\Im_j[M]$.Corollary (3.3) can now be used written as follows from.

**Corollary 3.4.** Suppose that $\Theta\in\Im_j[M]$. If the function $f\in\mathcal{A}$ fulfills the following criteria:

$$\left|\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\right|\leq kM,\qquad(\mathfrak{z}\in\mathrm{U},k\geq 2;M>0)$$

and

$$\left|\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z});\mathfrak{z}\right)\right| < M,$$

then

$$\left|\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\right| < M.$$

**Corollary 3.5.** Suppose that $k \geq 2$, and $M > 0$. If the function $f \in \mathcal{A}$ and meets the following criteria:

$$\left|\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\right| \leq kM,$$

and

$$\left|\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})-\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\right| \leq \frac{M}{\eta+2},$$

then

$$\left|\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\right| \leq M.$$

**Proof.** Letting $\Theta(a,b,c,d;\mathfrak{z}) = b-a,\ \Omega = \mathfrak{h}(\mathcal{G})$, such that $\mathfrak{h}(\mathfrak{z}) = \frac{M\mathfrak{z}}{\eta+2}, \mathfrak{z} \in \mathcal{G}, M > 0$.

Make use of the Corollary (3.3). We must prove it $\Theta \in \mathfrak{I}_j[\Omega, M]$, in other words, admissibility condition (3.14) is satisfied. This follows readily, since it is seen that

$$|\Theta(a,b,c,d;\mathfrak{z})| = \left|\frac{(k-1)}{\eta+2}Me^{i\theta}\right| = \frac{k-1}{\eta+2}M \geq \frac{M}{\eta+2},$$

whenever $\mathfrak{z} \in \mathcal{G}, \theta \in \mathbb{R}$ and $k \geq 2$.

**Definition 3.3.** Letting $\Omega$ be a set in $\mathbb{C}$, and $\mathcal{T}$ be an element of the intersection of $\mathbb{Q}_1$ and $\mathrm{H}_1$. The class $\mathfrak{I}_{j,1}[\Omega,\mathcal{T}]$ of admissible functions comprises those functions $\Theta: \mathbb{C}^4 \times \mathcal{G} \to \mathbb{C}$ that fulfill the specified admissibility criteria:

$$\Theta(a,b,c,d;\mathfrak{z}) \notin \Omega,$$

whenever

$$a = \mathcal{T}(\zeta), \qquad b = \frac{k\zeta\mathcal{T}'(\zeta)+(\eta+2)\mathcal{T}(\zeta)}{\eta+2},$$

$$Re\left(\frac{(\eta+2)[a+c-2b]}{b-a}\right) \geq kRe\left(\frac{\zeta\mathcal{T}''(\zeta)}{\mathcal{T}'(\zeta)}+1\right),$$

and

$$Re\left(\frac{(\eta+2)^2(d-a)-3(\eta+2)(\eta+3)(c-a)+(b-a)\left[3(\eta+3)^2-1\right]}{b-a}\right) \geq k^2Re\left(\frac{\zeta^2\mathcal{T}'''(\zeta)}{\mathcal{T}'(\zeta)}\right),$$

where $\mathfrak{z} \in \mathcal{G}, \zeta \in \partial\mathcal{G} \backslash E(\mathcal{T})$ with $k \geq 2$.

**Theorem 3.4.** Suppose that $\Theta \in \mathfrak{I}_{j,1}[\Omega, \mathcal{T}]$. If $f \in \mathcal{A}$ be a function and $\mathcal{T} \in \mathbb{Q}_1 \cap \mathbb{H}_1$ satisfy the next requirements:

$$Re\left(\frac{\zeta \mathcal{T}''(\zeta)}{\mathcal{T}'(\zeta)}\right) \geq 0, \qquad \left|\frac{\mathfrak{z}\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(\mathfrak{z})\right)' - \mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(\mathfrak{z})}{\mathfrak{z}\mathcal{T}'(\zeta)}\right| \leq k, \qquad (3.15)$$

and

$$\left\{\Theta\left(\frac{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(\mathfrak{z})}{\mathfrak{z}}; \mathfrak{z}\right) : \mathfrak{z} \in \mathcal{G}\right\} \subset \Omega, \qquad (3.16)$$

then

$$\frac{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(\mathfrak{z})}{\mathfrak{z}} \prec \mathcal{T}(\mathfrak{z}), \qquad (\mathfrak{z} \in \mathcal{G}).$$

**Proof.** The analytic function should be defined $G(\mathfrak{z})$ in $\mathcal{G}$ by

$$G(\mathfrak{z}) = \frac{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(\mathfrak{z})}{\mathfrak{z}}. \qquad (3.17)$$

Using the equation (1.4) and (3.17), we have

$$\frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(\mathfrak{z})}{\mathfrak{z}} = \frac{\mathfrak{z}G'(\mathfrak{z}) + (\eta+2)G(\mathfrak{z})}{\eta+2}. \qquad (3.18)$$

By a similar argument, we get

$$\frac{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(\mathfrak{z})}{\mathfrak{z}} = \frac{\mathfrak{z}^2 G''(\mathfrak{z}) + [2(\eta+2)+1]\mathfrak{z}G'(\mathfrak{z}) + (\eta+2)^2 G(\mathfrak{z})}{(\eta+2)^2}, (3.19)$$

And

$$\frac{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(\mathfrak{z})}{\mathfrak{z}} =$$

$$\frac{\mathfrak{z}^3 G'''(\mathfrak{z}) + 3(\eta+3)\mathfrak{z}^2 G''(\mathfrak{z}) + [3(\eta+2)(\eta+3)+1]\mathfrak{z}G'(\mathfrak{z}) + (\eta+2)^3 G(\mathfrak{z})}{(\eta+2)^3}. \qquad (3.20)$$

Define the transformation starting with $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, u) = \mathfrak{r}, \qquad b(\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, u) = \frac{\mathfrak{s} + (\eta+2)\mathfrak{r}}{\eta+2},$$

$$c(\mathfrak{r},\mathfrak{s},\mathfrak{t},u)=\frac{\mathfrak{t}+(2\eta+5)\mathfrak{s}+(\eta+2)^2\mathfrak{r}}{(\eta+2)^2}, \tag{3.21}$$

and

$$d(\mathfrak{r},\mathfrak{s},\mathfrak{t},u)=\frac{u+3(\eta+2)\mathfrak{t}+[3(\eta+2)(\eta+3)+1]\mathfrak{s}+(\eta+2)^3\mathfrak{r}}{(\eta+2)^3}. \tag{3.22}$$

Let

$$\varphi(\mathfrak{r},\mathfrak{s},\mathfrak{t},u)=\Theta(a,b,c,d;\mathfrak{z})=$$

$$\Theta\left(\begin{array}{c}\mathfrak{r},\dfrac{\mathfrak{s}+(\eta+2)\mathfrak{r}}{\eta+2},\dfrac{\mathfrak{t}+[2(\eta+2)+1]\mathfrak{s}+(\eta+2)^2\mathfrak{r}}{(\eta+2)^2},\\ \dfrac{u+3(\eta+2)\mathfrak{t}+[3(\eta+2)(\eta+3)+1]\mathfrak{s}+(\eta+2)^3\mathfrak{r}}{(\eta+2)^3};\mathfrak{z}\end{array}\right). \tag{3.23}$$

The proof will make use of Lemma (2.1). Equations are used (3.17) to (3.20), and from (3.23), we have

$$\varphi(G(\mathfrak{z}),\mathfrak{z}G'(\mathfrak{z}),\mathfrak{z}^2G''(\mathfrak{z}),\mathfrak{z}^3G'''(\mathfrak{z});\mathfrak{z})=$$

$$\Theta\left(\frac{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}},\frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}},\frac{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}},\frac{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}};\mathfrak{z}\right). \tag{3.24}$$

Hence, clearly, (3.16) becomes

$$\varphi(G(\mathfrak{z}),\mathfrak{z}G'(\mathfrak{z}),\mathfrak{z}^2G''(\mathfrak{z}),\mathfrak{z}^3G'''(\mathfrak{z});\mathfrak{z})\in\Omega,$$

Note that

$$\frac{\mathfrak{t}}{\mathfrak{s}}+1=\frac{(\eta+2)[a+c-2b]}{b-a},$$

and

$$\frac{u}{\mathfrak{s}}=\frac{(\eta+2)^2(d-a)-3(\eta+2)(\eta+3)(c-a)+(b-a)[3(\eta+3)^2-1]}{b-a}.$$

As a result, the admissibility condition for $\Theta\in\Im_{j,1}[\Omega,\mathcal{T}]$ in Definition (3.3) is the same as the admissibility criterion for $\varphi\in\Psi_2[\Omega,\mathcal{T}]$ as stated in the Definition (2.3) with $n=2$. As a result, using (3.13) and Lemma (2.1), we have

$$\frac{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}}\prec\mathcal{T}(\mathfrak{z}).$$

Now completes the proof of theorem (3.4).

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=\mathfrak{h}(\mathcal{G})$ for some conformal mapping $\mathfrak{h}(\mathfrak{z})$ of $\mathcal{G}$ onto $\Omega$. In this situation, the class $\Im_{j,1}[\mathfrak{h}(\mathcal{G}),\mathcal{T}]$ is written as $\Im_{j,1}[\Omega,\mathcal{T}]$. This follows immediate consequence of Theorem (3.4), as follows:

**Theorem 3.5.** Let $\Theta \in \mathfrak{I}_{j,1}[\Omega,\mathcal{T}]$. If the functions $f \in \mathcal{A}$ and $\mathcal{T} \in \mathbb{Q}_1$ satisfy the following conditions:

$$Re\left(\frac{\zeta\mathcal{T}_\rho''(\zeta)}{\mathcal{T}_\rho'(\zeta)}\right) \geq 0, \qquad \left|\frac{\mathfrak{z}\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\right)' - \mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}\mathcal{T}_\rho'(\zeta)}\right| \leq k, \tag{3.25}$$

and

$$\Theta\left(\frac{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}}; \mathfrak{z}\right) \prec \mathfrak{h}(\mathfrak{z}), \tag{3.26}$$

then

$$\frac{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}} \prec \mathcal{T}(\mathfrak{z}), \qquad (\mathfrak{z} \in \mathcal{G}).$$

In light of the Definition (3.3) and in the special case $\mathcal{T}(\mathfrak{z}) = M\mathfrak{z}$, $M > 0$, the class functions that are admissible $\mathfrak{I}_{j,1}[\Omega,\mathcal{T}]$, denoted by $\mathfrak{I}_{j,1}[\Omega, M]$ is expressed as follows.

**Definition 3.4.** Let $\Omega$ be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\mathfrak{I}_{j,1}[\Omega,\mathcal{T}]$ consists of those functions $\Theta: \mathbb{C}^4 \times \mathcal{G} \to \mathbb{C}$ such that:

$$\Theta\left(\begin{array}{c} Me^{i\theta}, \frac{k+(\eta+2)Me^{i\theta}}{\eta+2}, \frac{L+([(2\eta+5)]k+(\eta+2)^2)Me^{i\theta}}{(\eta+2)^2}, \\ \frac{\mathbb{N}+3(\eta+3)L+([3(\eta+2)(\eta+3)+1]k+(\eta+2)^3)Me^{i\theta}}{(\eta+2)^3}; \mathfrak{z} \end{array}\right) \notin \Omega, \tag{3.27}$$

whenever

$$\mathfrak{z} \in \mathcal{G}, \qquad Re\left(Le^{-i\theta}\right) \geq (k-1)kM,$$

and

$$Re\left(\mathbb{N}e^{-i\theta}\right) \geq 0, \qquad \forall\theta \in \mathbb{R}; k \geq 2.$$

**Corollary 3.6.** Let $\Theta \in \mathfrak{I}_{j,1}[\Omega,\mathcal{T}]$. If the function $f \in \mathcal{A}$ satisfies the following conditions:

$$\left|\frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}}\right| \leq kM, \qquad (\mathfrak{z} \in \mathcal{G}, k \geq 2; M > 0),$$

and

$$\Theta\left(\frac{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}}; \mathfrak{z}\right) \in \Omega,$$

then

$$\left|\frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}}\right| < M.$$

In the special case, when $\Omega = \mathcal{T}(\mathcal{G}) = \{w : |w| < M\}$, the class $\Im_{j,1}[\Omega, \mathcal{T}]$ is simply denoted by $\Im_{j,1}[M]$. Corollary (3.6) can now be expressed as follows:

**Corollary 3.7.** Let $\Theta \in \Im_{j,1}[\Omega, \mathcal{T}]$. If the function $f \in \mathcal{A}$ satisfies what follows circumstances:

$$\left|\frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}}\right| \leq kM, \qquad (\mathfrak{z} \in \mathcal{G}, k \geq 2; M > 0),$$

and

$$\left|\Theta\left(\frac{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}}, \frac{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}}; \mathfrak{z}\right)\right| < M,$$

then

$$\left|\frac{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{z}}\right| < M.$$

**Definition 3.5.** Let $\mathcal{T} \in \mathbb{Q}_1 \cap \mathbb{H}_1$ and $\Omega$ be a set in $\mathbb{C}$. The class $\Im_{j,2}[\Omega, \mathcal{T}]$ of admissible functions consists of those functions $\Theta: \mathbb{C}^4 \times \mathcal{G} \to \mathbb{C}$, which satisfy the following admissibility requirements:

$$\Theta(a, b, c, d; \mathfrak{z}) \notin \Omega,$$

whenever

$$a = \mathcal{T}(\zeta), \qquad b = \frac{1}{\eta + 2}\left[\frac{k\zeta\mathcal{T}'(\zeta) + (\eta + 2)\big(\mathcal{T}(\zeta)\big)^2}{\mathcal{T}(\zeta)}\right],$$

$$Re\left(\frac{(\eta + 2)[2a^2 + cb - 3ab]}{b - a}\right) \geq k\ Re\left(\frac{\zeta\mathcal{T}''(\zeta)}{\mathcal{T}'(\zeta)} + 1\right),$$

and

$$Re\Big(\big[bc(d - c)(\eta + 2)^2 - b(\eta + 2)^2(c - b)(1 - b - c + 3a) - 3(\eta + 2)(c - b)b + 2(b - a) + 3a(\eta + 2)(b - a) + (b - a)^2(\eta + 2)\big((b - c)(\eta + 2) - 3 - 4a(\eta + 2)\big) + a^2(\eta + 2)^2(b - a)\big](b - a)^{-1}\Big) \geq k^2 Re\left(\frac{\zeta^2\mathcal{T}'''(\zeta)}{\mathcal{T}'(\zeta)}\right),$$

where $\mathfrak{z} \in \mathcal{G},\ \zeta \in \partial\mathcal{G} \backslash E(\mathcal{T})$ and $k \geq 2$.

**Theorem 3.6.** Consider $\Theta \in \Im_{j,2}[\Omega, \mathcal{T}]$. If the functions $f \in \mathcal{A}$ and $\mathcal{T} \in \mathbb{Q}_1 \cap \mathbb{H}_1$ provided they meet the subsequent criteria:

$$Re\left(\frac{\zeta\mathcal{T}''(\zeta)}{\mathcal{T}'(\zeta)}\right)\geq 0,\ \left|\frac{z\left(\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(z)\right)'\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(z)-z\left(\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(z)\right)'\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(z)}{\left(\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(z)\right)^{2}\mathcal{T}'(\zeta)}\right|\leq k,\quad (3.28)$$

and

$$\left\{\Theta\left(\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(z)}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(z)},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{2\tau}{\sigma}}f(z)}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(z)},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{3\tau}{\sigma}}f(z)}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{2\tau}{\sigma}}f(z)},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{4\tau}{\sigma}}f(z)}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{3\tau}{\sigma}}f(z)};z\right):z\in\mathcal{G}\right\}\subset\Omega,\qquad (3.29)$$

then

$$\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(z)}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(z)}\prec\mathcal{T}(z),\qquad (z\in\mathcal{G}).$$

**Proof.** The analytic function should be defined $G(z)$ in $\mathcal{G}$ by

$$G(z)=\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(z)}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(z)}.\qquad (3.30)$$

From equation (1.4) and (3.30), we have

$$\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{2\tau}{\sigma}}f(z)}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(z)}=\frac{1}{\eta+2}\left[\frac{zG'(z)+(\eta+2)G^{2}(z)}{G(z)}\right]=\frac{A}{\eta+2}.\qquad (3.31)$$

By a similar argument, we have

$$\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{3\tau}{\sigma}}f(z)}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{2\tau}{\sigma}}f(z)}=\frac{B}{\eta+2},\qquad (3.32)$$

and

$$\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{4\tau}{\sigma}}f(z)}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{3\tau}{\sigma}}f(z)}=\frac{1}{\eta+2}[B+B^{-1}(C+A^{-1}D-A^{-2}C^{2})],\qquad (3.33)$$

where

$$B=\frac{zG'(z)}{G(z)}+(\eta+2)G(z)+\frac{\frac{z^{2}G''(z)+zG'(z)}{G(z)}-\left(\frac{zG'(z)}{G(z)}\right)^{2}+(\eta+2)zG'(z)}{\frac{zG'(z)}{G(z)}+(\eta+2)G(z)},$$

$$C=\frac{\mathfrak{z}^2G''(\mathfrak{z})+\mathfrak{z}G'(\mathfrak{z})}{G(\mathfrak{z})}-\left(\frac{\mathfrak{z}G'(\mathfrak{z})}{G(\mathfrak{z})}\right)^2+(\eta+2)\mathfrak{z}G'(\mathfrak{z}),$$

and

$$D=\frac{\mathfrak{z}^3G'''(\mathfrak{z})+3\mathfrak{z}^2G''(\mathfrak{z})+\mathfrak{z}G'(\mathfrak{z})}{G(\mathfrak{z})}-\frac{3\mathfrak{z}^2\left(G'(\mathfrak{z})\right)^2+3\mathfrak{z}^3G''(\mathfrak{z})G'(\mathfrak{z})}{(G(\mathfrak{z}))^2}+2\left(\frac{\mathfrak{z}G'(\mathfrak{z})}{G(\mathfrak{z})}\right)^3+(\eta+2)\mathfrak{z}^2G''(\mathfrak{z})+(\eta+2)\mathfrak{z}G'(\mathfrak{z}).$$

Define the transformation starting with $\mathbb{C}^4$ to $\mathbb{C}$ b$y$

$$a(\mathfrak{r},\mathfrak{s},\mathfrak{t},u)=\mathfrak{r},b(\mathfrak{r},\mathfrak{s},\mathfrak{t},u)=\frac{1}{\eta+2}\left[\frac{\mathfrak{s}+(\eta+2)\mathfrak{r}^2}{\sigma}\right]=\frac{E}{\eta+2},$$

$$c(\mathfrak{r},\mathfrak{s},\mathfrak{t},u)=\frac{1}{\eta+2}\left[\frac{\mathfrak{s}+(\eta+2)\mathfrak{r}^2}{\mathfrak{r}}+\frac{\frac{\mathfrak{t}+\mathfrak{s}}{\mathfrak{r}}-\left(\frac{\mathfrak{s}}{\mathfrak{r}}\right)^2+(\eta+2)\mathfrak{s}}{\frac{\mathfrak{s}}{\mathfrak{r}}+(\eta+2)\mathfrak{r}}\right]=\frac{F}{\eta+2}, \qquad (3.34)$$

and

$$d(\mathfrak{r},\mathfrak{s},\mathfrak{t},u)=\frac{1}{\eta+2}\left[F+F^{-1}\left(L+\mathbb{H}E^{-1}-E^{-2}L^2\right)\right], \qquad (3.35)$$

where

$$L=\frac{\mathfrak{t}+\mathfrak{s}}{\mathfrak{r}}-\left(\frac{\mathfrak{s}}{\mathfrak{r}}\right)^2+(\eta+2)\mathfrak{s},$$

and

$$\mathbb{H}=\frac{u+3\mathfrak{t}+\mathfrak{s}}{\mathfrak{r}}-3\left(\frac{\mathfrak{s}}{\mathfrak{r}}\right)^2-3\frac{\mathfrak{s}\mathfrak{t}}{\mathfrak{r}^2}+2\left(\frac{\mathfrak{s}}{\mathfrak{r}}\right)^3+(\eta+2)(\mathfrak{s}+\mathfrak{t}).$$

Let

$$\varphi(\mathfrak{r},\mathfrak{s},\mathfrak{t},u)=\Theta(a,b,c,d)=$$

$$\Theta\left(\mathfrak{r},\frac{E}{\eta+2},\frac{F}{\eta+2},\frac{1}{\eta+2}\left[F+F^{-1}\left(L+\mathbb{H}E^{-1}-E^{-2}L^2\right)\right]\right). \qquad (3.36)$$

The proof will make use of Lemma (2.1). Using the equations (3.30) to (3.33), and from (3.36), we have

$$\varphi(G(\mathfrak{z}),\mathfrak{z}G'(\mathfrak{z}),\mathfrak{z}^2G''(\mathfrak{z}),\mathfrak{z}^3G'''(\mathfrak{z});\mathfrak{z})=$$

$$\Theta\left(\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(\mathfrak{z})},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{2\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(\mathfrak{z})},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{3\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{2\tau}{\sigma}}f(\mathfrak{z})},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{4\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{3\tau}{\sigma}}f(\mathfrak{z})};\mathfrak{z}\right). \qquad (3.37)$$

Hence, clearly (3.29) leads to

$$\varphi(G(\mathfrak{z}),\mathfrak{z}G'(\mathfrak{z}),\mathfrak{z}^2G''(\mathfrak{z}),\mathfrak{z}^3G'''(\mathfrak{z});\mathfrak{z})\in\Omega.$$

We note that

$$\frac{\mathfrak{t}}{\mathfrak{s}}+1=\frac{(\eta+2)[2a^2+cb-3ab]}{b-a},$$

and

$\frac{u}{\mathfrak{s}}=\big[bc(d-c)(\eta+2)^2-b(\eta+2)^2(c-b)(1-b-c+3a)-3(\eta+2)(c-b)b+2(b-a)+3a(\eta+2)(b-a)+(b-a)^2(\eta+2)\big((b-c)(\eta+2)-3-4a(\eta+2)\big)+a^2(\eta+2)^2(b-a)\big](b-a)^{-1}.$

Thus, the admissibility condition for $\Theta\in\mathfrak{I}_{j,2}[\Omega,\mathcal{T}]$ in Definition (3.5) is the same as the criteria of admissibility for $\Theta\in\Psi_2[\Omega,\mathrm{q}]$ as stated in the Definition (2.3) with $\mathrm{n}=2$. As a result, using (3.30) and Lemma (2.1), we have

$$\frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}\prec\mathcal{T}(\mathfrak{z}),$$

this completes the proof of the Theorem (3.6).

When $\Omega\neq\mathbb{C}$ is a simply connected domain, therefore $\Omega$ is the image of a conformal mapping $\mathfrak{h}(\mathcal{G})$ for some domain $\mathcal{G}$. In this context, the class $\mathfrak{I}_{j,2}[\mathfrak{h}(\mathcal{G}),\mathcal{T}]$ is denoted as $\mathfrak{I}_{j,2}[\Omega,\mathcal{T}]$. The immediate conclusion of Theorem (3.6) is presented below without proof.

**Theorem 3.7.** Consider $\Theta\in\mathfrak{I}_{j,2}[\Omega,\mathcal{T}]$. If the functions $f\in\mathcal{A}$ with $\mathcal{T}\in\mathbb{Q}_1$ and they fulfill the constraints specified in (3.29) and

$$\Theta\left(\frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})},\frac{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})},\frac{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})},\frac{\mathfrak{T}^{\delta-\frac{4\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})};\mathfrak{z}\right)\prec\mathfrak{h}(\mathfrak{z}),$$

then

$$\frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}\prec\mathcal{T}(\mathfrak{z}),\qquad(\mathfrak{z}\in\mathcal{G}).$$

## 4- Third-Order Differential Superordination of the Operator $\mathfrak{T}^{\delta}_{(\alpha,\beta,\mu)}f(\mathfrak{z})$

**Definition 4.1.** Consider $\Omega$ be a subset of $\mathbb{C}$, and $\mathcal{T}$ be an element of $\mathbb{Q}_0\cap\mathbb{H}_0$ such that $\mathcal{T}'(\mathfrak{z})\neq 0$. The class of admissible functions $\mathfrak{I}_j'[\Omega,\mathcal{T}]$ comprises functions $\Theta:\mathbb{C}^4\times\mathcal{G}\to\mathbb{C}$ that fulfill the specified admissibility criteria:

$$\Theta(a,b,c,d;\zeta)\in\Omega,$$

whenever

$$a=\mathcal{T}(\mathfrak{z}),\qquad b=\frac{\mathfrak{z}\mathcal{T}'(\mathfrak{z})+m\,(\eta+1)\mathcal{T}(\mathfrak{z})}{m\,(\eta+2)},$$

$$Re\left(\frac{(\eta+2)[(\eta+2)c-2(\eta+1)b]+(\eta+1)^2a]}{(\eta+2)b-(\eta+1)a}\right)\leq\frac{1}{m}Re\left(\frac{\mathfrak{z}\mathcal{T}''(\mathfrak{z})}{\mathcal{T}'(\mathfrak{z})}+1\right),$$

and

$$Re\left(\frac{(\eta+2)^3[d-3(c-b)]-(\eta+2)[3(\eta+1)a+b]+\left[(\eta+1)a(1-(\eta+1)^2)\right]}{(\eta+2)b-(\eta+1)a}\right)\leq\frac{1}{m^2}Re\left(\frac{\mathfrak{z}^2\mathcal{T}'''(\mathfrak{z})}{\mathcal{T}'(\mathfrak{z})}\right),$$

where $\mathfrak{z}\in\mathcal{G},\zeta\in\partial\mathcal{G}$ and $m\geq 2$.

**Theorem 4.1.** Let $\Theta\in\Im_j'[\Omega,\mathcal{T}]$. If the functions $f\in\mathcal{A}$, with $\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\in\mathbb{Q}_0$ and if $\mathcal{T}\in\mathbb{H}_0$ with $\mathcal{T}'(\mathfrak{z})\neq 0$, satisfying the following conditions:

$$Re\left(\frac{\zeta\mathcal{T}''(\mathfrak{z})}{\mathcal{T}'(\mathfrak{z})}\right)\geq 0,\qquad \left|\frac{\mathfrak{z}\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\right)'}{\mathcal{T}'(\mathfrak{z})}\right|\leq m,\tag{4.1}$$

and the function

$$\Theta\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z});\mathfrak{z}\right),$$

is univalent in $\mathcal{G}$, then

$$\Omega\subset\left\{\Theta(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z});\mathfrak{z}\in\mathcal{G}\right\},\tag{4.2}$$

implies that

$$\mathcal{T}(\mathfrak{z})\prec\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\qquad(\mathfrak{z}\in\mathcal{G}).$$

**Proof.** Let the function $G(\mathfrak{z})$ be defined by (3.3) and $\varphi$ by (3.8). Since $\Theta\in\Im_j'[\Omega,\mathcal{T}]$. From (3.10) and (4.2), we have

$$\Omega\subset\{\Theta(G(\mathfrak{z}),\mathfrak{z}G'(\mathfrak{z}),\mathfrak{z}^2G''(\mathfrak{z}),\mathfrak{z}^3G'''(\mathfrak{z});\mathfrak{z}):\mathfrak{z}\in\mathcal{G}\}.$$

From (3.7) and (3.8), We can observe that the admissibility condition for $\Theta\in\Im_j'[\Omega,\mathcal{T}]$ in Definition (4.1) is the same as the admissibility criterion for $\varphi\in\Psi_n'[\Omega,\mathcal{T}]$ as stated in the Definition (2.5) with $n=2$. Hence $\varphi\in\Psi_2'[\Omega,\mathcal{T}]$ as well as (4.2) and Lemma (2.2), we have

$$\mathcal{T}(\mathfrak{z})\prec\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\qquad(\mathfrak{z}\in\mathcal{G}).$$

This completes the proof of the Theorem (4.1).

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=\mathfrak{h}(\mathcal{G})$ for some conformal mapping $\mathfrak{h}(\mathfrak{z})$ of $\mathcal{G}$ onto $\Omega$. In this case, the class $\Im_j'[\mathfrak{h}(\mathcal{G}),\mathcal{T}]$ is written as $\Im_j'[\mathfrak{h},\mathcal{T}]$. This follows an immediate repercussion of Theorem (4.1) is stated below.

**Theorem 4.2.** Suppose that $\Theta\in\Im_j'[\mathfrak{h},\mathcal{T}]$ and $\mathfrak{h}$ be analytic in $\mathcal{G}$. If the function $f\in\mathcal{A}$, with $\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\in\mathbb{Q}_0$ and $\mathcal{T}\in\mathbb{H}_0$ with $\mathcal{T}'(\mathfrak{z})\neq 0$, satisfying the following conditions (4.1) and the function

$$\Theta\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z});\mathfrak{z}\right),$$

is univalent in $\mathcal{G}$, then

$$\mathfrak{h}(\mathfrak{z}) \prec \Theta\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z});\mathfrak{z}\right), \qquad (4.3)$$

implies that

$$\mathcal{T}(\mathfrak{z}) \prec \mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}), \qquad (\mathfrak{z}\in\mathcal{G}).$$

Theorem (4.1) and (4.2) may only be utilized to get third-order differential superordination of the forms' subordination (4.2) or (4.3).

The following theorem gives the existence of the best subordinant of (4.3) for suitable Θ.

**Theorem 4.3.** Let $\mathfrak{h}$ be univalent function in $\mathcal{G}$ and $\Theta\colon\mathbb{C}^4\times\bar{\mathcal{G}}\to\mathbb{C}$ and $\varphi$ be given by (3.9). Assume the differential equation:

$$\varphi(\mathcal{T}(\mathfrak{z}),\mathfrak{z}\mathcal{T}'(\mathfrak{z}),\mathfrak{z}^2\mathcal{T}''(\mathfrak{z}),\mathfrak{z}^3\mathcal{T}'''(\mathfrak{z});\mathfrak{z}) = \mathfrak{h}(\mathfrak{z}), \qquad (4.4)$$

has a solution $\mathcal{T}(\mathfrak{z})\in\mathbb{Q}_0$. If the functions $f\in\mathcal{A}$, and $\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\in\mathbb{Q}_0$ and if $\mathcal{T}\in\mathrm{H}_0$ with $\mathcal{T}'(\mathfrak{z})\neq 0$, which satisfy the following criteria (4.1) and the function

$$\Theta\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z});\mathfrak{z}\right),$$

is analytic in $\mathcal{G}$, then

$$\mathfrak{h}(\mathfrak{z}) \prec \Theta\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z}),\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z});\mathfrak{z}\right),$$

implies that

$$\mathcal{T}(\mathfrak{z}) \prec \mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}), \qquad (\mathfrak{z}\in\mathcal{G})$$

and $\mathcal{T}(\mathfrak{z})$ is the best subordinant.

**Proof**. According to Theorems (4.1) and (4.2), $\mathcal{T}$ is a subordinate of (4.3). Given that $\mathcal{T}$ fulfills (4.4), it is consequently a solution of (4.3), and so, $\mathcal{T}$ will be subordinate to all subordinants. Therefore, $\mathcal{T}$ is the optimal subordinate. The demonstration of Theorem (4.3) is concluded.

**Definition 4.2.** Suppose that Ω be a set in $\mathbb{C}$ and $\mathcal{T}\in\mathrm{H}_1$ such that $\mathcal{T}'(\mathfrak{z})\neq 0$. The class of admissible functions $\Im'_{j,1}[\Omega,\mathcal{T}]$ includes those functions $\Theta\colon\mathbb{C}^4\times\bar{\mathcal{G}}\to\mathbb{C}$ , that satisfy the following admission requirements:

$$\Theta(a,b,c,d;\zeta)\in\Omega,$$

whenever

$$a = \mathcal{T}(z), \qquad b = \frac{z\mathcal{T}'(z) + m\,(\eta+2)\mathcal{T}(z)}{m\,(\eta+2)},$$

$$Re\left(\frac{(\eta+2)[c+a-2b]}{b-a}\right) \le \frac{1}{m} Re\left(\frac{z\mathcal{T}''(z)}{\mathcal{T}'(z)} + 1\right),$$

and

$$Re\left(\frac{(\eta+2)^2(d-a)-3(\eta+2)(\eta+3)(c-a)+(b-a)\left[3(\eta+3)^2-1\right]}{b-a}\right) \le \frac{1}{m^2} Re\left(\frac{z^2\mathcal{T}'''(z)}{\mathcal{T}'(z)}\right),$$

where $z \in \mathcal{G}, \zeta \in \partial\mathcal{G}$ and $m \ge 2$.

**Theorem 4.4.** Assume $\Theta$ belongs to $\mathfrak{I}'_{j,1}[\Omega, \mathcal{T}]$. If the function $f \in \mathcal{A}$ and $\frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(z)}{z} \in \mathbb{Q}_1$, and if $\mathcal{T} \in \mathrm{H}_1$with $\mathcal{T}'(z) \neq 0$, satisfying the following conditions:

$$Re\left(\frac{\zeta\mathcal{T}''(\zeta)}{\mathcal{T}'(\zeta)}\right) \ge 0, \qquad \left|\frac{z\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(z)\right)' - \mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(z)}{z\mathcal{T}'(z)}\right| \le m, \tag{4.5}$$

and the function

$$\Theta\left(\frac{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(z)}{z}, \frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(z)}{z}, \frac{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(z)}{z}, \frac{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(z)}{z}; z\right),$$

is univalent in $\mathcal{G}$, thus

$$\Omega \subset \left\{\Theta\left(\frac{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(z)}{z}, \frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(z)}{z}, \frac{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(z)}{z}, \frac{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)} f(z)}{z}; z\right) : z \in \mathcal{G}\right\}, \tag{4.6}$$

implies that

$$\mathcal{T}(z) \prec \frac{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(z)}{z}, \qquad (z \in \mathcal{G}).$$

**Proof**. Consider the function $G(z)$ be defined by (3.17) and $\Theta$ by (3.23). Since $\Theta \in \mathfrak{I}'_{j,1}[\Omega, \mathcal{T}]$, from (3.24) and (4.6) yield

$$\Omega \subset \{\varphi(G(z), zG'(z), z^2G''(z), z^3G'''(z); z) : z \in \mathcal{G}\}$$

From the equations (3.21) and (3.22), it is clear that admissibility is a requirement $\Theta \in \mathfrak{I}'_{j,1}[\Omega, \mathcal{T}]$ in Definition (4.1) is equivalent to the admissibility condition for $\varphi$ as stated in Definition (2.3) with $n = 2$. Hence $\varphi \in \Psi'_2[\Omega, \mathcal{T}]$ and by using (4.6) and Lemma (2.2), we have

$$\mathcal{T}(z) \prec \frac{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)} f(z)}{z}, \qquad (z \in \mathcal{G}).$$

The proof of Theorem (4.4) is complete.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = \mathfrak{h}(\mathcal{G})$ for some conformal mapping $\mathfrak{h}(\mathfrak{z})$ of $\mathcal{G}$ onto $\Omega$. In this case, the class $\Im_{j,1}^{'}[\mathfrak{h}(\mathcal{G}),\mathcal{T}]$ is written as $\Im_{j,1}^{'}[\mathfrak{h},\mathcal{T}]$. This is a direct consequence of the Theorem. (4.4).

**Theorem 4.5.** Let $\Theta$ belong to $\Im_{j,1}^{'}[\mathfrak{h},\mathcal{T}]$ and $\mathfrak{h}$ be analytic in $\mathcal{G}$. If the functions $f$ belong to $\mathcal{A}$, with $\mathcal{T}$ in $\mathrm{H}_1$ and $\mathcal{T}^{'}(\mathfrak{z})$ not equal to zero, meeting the requirements (4.5) and the function

$$\Theta\left(\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(\mathfrak{z})}{\mathfrak{z}},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{z}},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{2\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{z}},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{3\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{z}};\mathfrak{z}\right),$$

is univalent in $\mathcal{G}$, then

$$\mathfrak{h}(\mathfrak{z}) \prec \Theta\left(\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(\mathfrak{z})}{\mathfrak{z}},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{z}},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{2\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{z}},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{3\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{z}};\mathfrak{z}\right),$$

implies that

$$\mathcal{T}(\mathfrak{z}) \prec \frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(\mathfrak{z})}{\mathfrak{z}}, \qquad (\mathfrak{z} \in \mathcal{G}).$$

**Definition 4.3.** Consider $\Omega$ be a set in $\mathbb{C}$ and $\mathcal{T} \in \mathbb{H}_1$ with $\mathcal{T}^{'}(\mathfrak{z}) \neq 0$. The class $\Im_{j,2}^{'}[\Omega,\mathcal{T}]$ of admissible functions $\Im_{j,2}^{'}[\Omega,\mathcal{T}]$ comprises functions $\Theta\colon \mathbb{C}^4 \times \bar{\mathcal{G}} \to \mathbb{C}$ that fulfil the specified admissibility criteria:

$$\Theta(a,b,c,d;\zeta) \in \Omega,$$

whenever

$$a = \mathcal{T}(\mathfrak{z}), \qquad b = \frac{1}{\eta+2}\left[\frac{\mathfrak{z}\mathcal{T}^{'}(\mathfrak{z}) + m\,(\eta+2)\big(\mathcal{T}(\mathfrak{z})\big)^2}{m\,\mathcal{T}(\mathfrak{z})}\right],$$

$$Re\left(\frac{(\eta+2)[cb+2a^2-3ab]}{b-a}\right) \leq \frac{1}{m}\,Re\left(\frac{\mathfrak{z}\mathcal{T}^{''}(\mathfrak{z})}{\mathcal{T}^{'}(\mathfrak{z})} + 1\right),$$

and

$$Re\Big(\big[bc(d-c)(\eta+2)^2 - b(\eta+2)^2(c-b)(1-b-c+3a) - 3(\eta+2)(c-b)b + 2(b-a) + 3a(\eta+2)(b-a) + (b-a)^2(\eta+2)\big((b-c)(\eta+2) - 3 - 4a(\eta+2)\big) + a^2(\eta+2)^2(b-a)\big](b-a)^{-1}\Big) \leq \frac{1}{m^2}Re\left(\frac{\mathfrak{z}^2\mathcal{T}^{'''}(\mathfrak{z})}{\mathcal{T}^{'}(\mathfrak{z})}\right),$$

where $\mathfrak{z} \in \mathcal{G},\ \zeta \in \partial\mathcal{G}\backslash E(\mathcal{T})$ and $m \geq 2$.

**Theorem 4.6.** Suppose that $\Theta \in \mathfrak{I}'_{j,2}[\Omega,\mathcal{T}]$. If the function $f \in \mathcal{A}$ with $\frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})} \in \mathbb{Q}_1$ and if $\mathcal{T} \in \mathbb{H}_1$ with $\mathcal{T}'(\mathfrak{z}) \neq 0$, satisfying the following conditions:

$$Re\left(\frac{\zeta\mathcal{T}''(\zeta)}{\mathcal{T}'(\zeta)}\right) \geq 0, \qquad \left|\frac{\mathfrak{z}\left(\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\right)'\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z}) - \mathfrak{z}\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\right)'\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\left(\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})\right)^2\mathcal{T}'(\mathfrak{z})}\right| \leq m, \quad (4.7)$$

and the function

$$\Theta\left(\frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}, \frac{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}, \frac{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}, \frac{\mathfrak{T}^{\delta-\frac{4\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}; \mathfrak{z}\right),$$

is univalent in $\mathcal{G}$, then

$$\Omega \subset \left\{\Theta\left(\frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}, \frac{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}, \frac{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}, \frac{\mathfrak{T}^{\delta-\frac{4\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}; \mathfrak{z}\right) : \mathfrak{z} \in \mathcal{G}\right\}, \quad (4.8)$$

implies that

$$\mathcal{T}(\mathfrak{z}) \prec \frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}, \qquad (\mathfrak{z} \in \mathcal{G}).$$

**Proof**. Define the function $G(\mathfrak{z})$ as specified in (3.30) and $\Theta$ as outlined in (3.36). Given that $\Theta \in \mathfrak{I}'_{j,2}[\Omega,\mathcal{T}]$, we deduce from (3.37) and (4.8) that

$$\Omega \subset \{\varphi(G(\mathfrak{z}), \mathfrak{z}G'(\mathfrak{z}), \mathfrak{z}^2G''(\mathfrak{z}), \mathfrak{z}^3G'''(\mathfrak{z}); \mathfrak{z}) : \mathfrak{z} \in \mathcal{G}\}.$$

From equations (3.34) and (3.35), we can see that the requirement for admissibility is $\Theta \in \mathfrak{I}'_{j,2}[\Omega,\mathcal{T}]$ in Definition (4.3) is the same as the admissibility condition for $\varphi$ as given in Definition (2.5), when $n = 2$. Hence $\varphi \in \Psi'_2[\Omega,\mathcal{T}]$ and by using (4.7) and Lemma (2.2), we have

$$\mathcal{T}(\mathfrak{z}) \prec \frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}, \qquad (\mathfrak{z} \in \mathcal{G}).$$

The proof of Theorem (4.6) is complete.

**Theorem 4.7.** Let $\Theta \in \mathfrak{I}'_{j,2}[\Omega,\mathcal{T}]$. If the function $f \in \mathcal{A}$ and $\frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(\mathfrak{z})}{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(\mathfrak{z})} \in \mathbb{Q}_1$, and if $\mathcal{T} \in \mathbb{H}_1$ with $\mathcal{T}'(\mathfrak{z}) \neq 0$, satisfying the following conditions (4.7) and the function

$$\Theta\left(\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(\mathfrak{z})},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{2\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(\mathfrak{z})},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{3\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{2\tau}{\sigma}}f(\mathfrak{z})},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{4\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{3\tau}{\sigma}}f(\mathfrak{z})};\mathfrak{z}\right),$$

is univalent in $\mathcal{G}$, then

$$\mathfrak{h}(\mathfrak{z}) \prec \Theta\left(\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(\mathfrak{z})},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{2\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(\mathfrak{z})},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{3\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{2\tau}{\sigma}}f(\mathfrak{z})},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{4\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{3\tau}{\sigma}}f(\mathfrak{z})};\mathfrak{z}\right),$$

implies that

$$\mathcal{T}(\mathfrak{z}) \prec \frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(\mathfrak{z})}, \qquad (\mathfrak{z} \in \mathcal{G}).$$

## 5- Third-Order Sandwich of the Operator $\mathfrak{T}_{(\alpha,\beta,\mu)}^{\delta}f(\mathfrak{z})$

By combining Theorem (3.2) and (4.2), we obtain the following sandwich-type theorem.

**Theorem 5.1.** Let $\mathfrak{h}_1$ and $\mathcal{T}_1$ be analytic functions in $\mathcal{G}$. Also let $\mathfrak{h}_2$ be univalent function in $\mathcal{G}$ and $\mathcal{T}_2 \in \mathbb{Q}_0$ with $\mathcal{T}_1(0) = \mathcal{T}_2(0) = 1$ and $\Theta \in \mathfrak{I}_j[\mathfrak{h}_2,\mathcal{T}_2] \cap \mathfrak{I}_j'[\mathfrak{h}_1,\mathcal{T}_1]$. If the function $f \in \mathcal{A}$ with $\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(\mathfrak{z}) \in \mathbb{Q}_0 \cap \mathbb{H}_0$ and the function

$$\Theta\left(\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(\mathfrak{z}),\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(\mathfrak{z}),\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{2\tau}{\sigma}}f(\mathfrak{z}),\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{3\tau}{\sigma}}f(\mathfrak{z});\mathfrak{z}\right),$$

is univalent in $\mathcal{G}$, and if the conditions (3.1) and (4.1) are satisfied, then

$$\mathfrak{h}_1(\mathfrak{z}) \prec \Theta\left(\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(\mathfrak{z}),\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(\mathfrak{z}),\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{2\tau}{\sigma}}f(\mathfrak{z}),\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{3\tau}{\sigma}}f(\mathfrak{z});\mathfrak{z}\right) \prec \mathfrak{h}_2(\mathfrak{z})$$

implies that

$$\mathcal{T}_1(\mathfrak{z}) \prec \mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(\mathfrak{z}) \prec \mathcal{T}_2(\mathfrak{z}), \qquad (\mathfrak{z} \in \mathcal{G}). \tag{5.1}$$

Combining Theorems (3.5) and (4.5), The following sandwich-type theorem is obtained.

**Theorem 5.2.** Let $\mathfrak{h}_1$ and $\mathcal{T}_1$ be analytic functions in $\mathcal{G}$, and let $\mathfrak{h}_2$ be univalent function in $\mathcal{G}$ and $\mathcal{T}_2 \in \mathbb{Q}_1$ such that $\mathcal{T}_1(0) = \mathcal{T}_2(0) = 1$ with $\Theta \in \mathfrak{I}_{j,1}[\mathfrak{h}_2,\mathcal{T}_2] \cap \mathfrak{I}_{j,1}'[\mathfrak{h}_1,\mathcal{T}_1]$. If the function $f \in \mathcal{A}$ with $\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(\mathfrak{z})}{\mathfrak{z}} \in \mathbb{Q}_1 \cap \mathbb{H}_1$ and the function

$$\Theta\left(\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta}f(\mathfrak{z})}{\mathfrak{z}},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{z}},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{2\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{z}},\frac{\mathfrak{T}_{(\sigma,\tau,\mu)}^{\delta-\frac{3\tau}{\sigma}}f(\mathfrak{z})}{\mathfrak{z}};\mathfrak{z}\right),$$

is univalent in $\mathcal{G}$, and the conditions (3.15) and (4.5) are contented, then

$$\mathfrak{h}_1(z) \prec \Theta\left(\frac{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(z)}{z}, \frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}{z}, \frac{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}{z}, \frac{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}{z}; z\right) \prec \mathfrak{h}_2(z),$$

implies that

$$\mathcal{T}_1(z) \prec \frac{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(z)}{z} \prec \mathcal{T}_2(z), \qquad (z \in \mathcal{G}). \tag{5.2}$$

**Theorem 5.3.** Let $\mathfrak{h}_1$ and $\mathcal{T}_1$ be analytic functions in $\mathcal{G}$, and let $\mathfrak{h}_2$ be univalent function in $\mathcal{G}$ and $\mathcal{T}_2 \in \mathbb{Q}_1$ such that $\mathcal{T}_1(0) = \mathcal{T}_2(0) = 1$ with $\Theta \in \mathfrak{I}_{j,2}[\mathfrak{h}_2, \mathcal{T}_2] \cap \mathfrak{I}'_{j,2}[\mathfrak{h}_1, \mathcal{T}_1]$. If the function $f \in \mathcal{A}$ with $\frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(z)} \in \mathbb{Q}_1 \cap \mathbb{H}_1$ and the function

$$\Theta\left(\frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(z)}, \frac{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}, \frac{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}, \frac{\mathfrak{T}^{\delta-\frac{4\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}; z\right),$$

is univalent in $\mathcal{G}$, and the conditions (3.28) and (4.7) are contented, then

$$\mathfrak{h}_1(z) \prec \Theta\left(\frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(z)}, \frac{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}, \frac{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}{\mathfrak{T}^{\delta-\frac{2\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}, \frac{\mathfrak{T}^{\delta-\frac{4\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}{\mathfrak{T}^{\delta-\frac{3\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}; z\right) \prec \mathfrak{h}_2(z),$$

implies that

$$\mathcal{T}_1(z) \prec \frac{\mathfrak{T}^{\delta-\frac{\tau}{\sigma}}_{(\sigma,\tau,\mu)}f(z)}{\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(z)} \prec \mathcal{T}_2(z), \qquad (z \in \mathcal{G}). \tag{5.3}$$

### Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

### Concluding Remarks and Observations

The study of third-order differential subordination and differential superordination for analytic functions using fractional differential operators is a specialized and significant area within mathematics. The amalgamation of fractional differential operators with third-order differential subordination and superordination constitutes a specialized and advanced domain of mathematical enquiry. It necessitates a profound comprehension of complex analysis, fractional calculus, and the characteristics of analytic functions. In our present investigation, we have derived several third order differential subordination and superordination results for univalent functions in the unit disk involving the fractional operator $\mathfrak{T}^{\delta}_{(\sigma,\tau,\mu)}f(z)$. In this research, we sought to build relationships that facilitate the analysis and comparison of the behavior of analytic functions in complicated domains, taking into account higher-order derivatives, which may have substantial ramifications in many scientific and practical applications.

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