Cubic Spherical Neutrosophic Geometric Weighted Bonferroni Mean Operator for Selecting a Machine using MCDM Techniques

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Abstract: Neutrosophic cubic fuzzy sets (NCFSs) involve interval-valued and single-valued neutrosophic sets and are used to describe uncertainty or fuzziness more efficiently. The aggregation of neutrosopic cubic fuzzy information is crucial and necessary in a decision-making theory. To get a better solution to decision-making problems under a neutrosophic cubic fuzzy environment. The main objective of this study propose the Cubic Spherical Neutrosophic Geometric Bonferroni Mean Operator (CSNGBM) and the Cubic Spherical Neutrosophic Geometric Weighted Bonferroni Mean Operator (CSNGWBM) to find the best alternatives. In this study considers the 15 alternatives D_1 to D_5 and criteria ζ_1 to ζ_5 and to make the decision matrix deep interviews and collected opinions from three experts are:1) Industry Professionals 2) production and sales 3) technical advisers. Finally, the outcome of the results to find the greatest similarity value are taken to be the best alternatives by using neutrosophic cubic FMCDM.

Keywords: Keywords: Cubic Spherical Neutrosophic set, Cubic Spherical Neutrosophic Bonferroni Mean operator, Cubic Spherical Neutrosophic Weighted Bonferroni Mean operator, multi-criteria decision making

1. Introduction

L.A. Zadeh [50], who also created several fuzzy set operators,initially introduced the concept of a fuzzy set. According to Zadeh’s precise definition, a set ”A” can be distinguished from other fuzzy sets in X by a membership function fA(x) that quantifies the quantity of x in ”A” When applied to crisp sets, fuzzy set operations permanently reduce to their corresponding counterparts. The logical extensions of crisp set operations are proposed by Atanassov [1] with membership and non-membership functions called intuitionistic fuzzy set (IFS). The Smarandache [30] proposed neutrosophic set(NS) now includes a deterministic independent membership function. The main distinction between NS and IFS is the actual membership function in NS as opposed to the falsity of membership and non-membership of IFS and the indeter-minacy of membership function in IFS. Truth membership, indeterminacy membership, and false membership are the three parts of NS, and they are all independent of one another. The decision-making research with various attributes based on FS, IFS and NS has made significant strides lately. Mardain et al. [24] and Kahrman et al. [19] evaluated the approaches and applications of fuzzy multiple criteria decision-making. Recently, there has been a lot of interest in NS because it is a generalization of IFS and FS and can be utilized to more effectively ex-plain ambiguous information [7, 26, 27, 32, 46]. By extending the indeterminacy membership, truth membership, and false membership to the interval numbers, Wang et al. [41] defined the interval Neutrosophic (INS). The Normal intuitionistic fuzzy numbers NIFNs to multi-criteria decision-making (MCDM) problems; meanwhile, some new aggregation operators are pro-posed by [36]. Finally, their proposed method is compared to the existing techniques under a numerical example to verify its feasibility and rationality. [8], the Bonferroni mean (BM), which combines the max and min operators with the logical ”or” and ”and” operators as opposed to this, when the input arguments are real, non-negative integers. The average mean (AM) and geometric mean (GM) of the aggregation operators are fundamental functions on which many extensions have been created. To rank the arguments before aggregation, Yager [40] developed the ordered weighted averaging (OWA) operator. This inspired other writers[11, 39] to research the ordered weighted geometric (OWG) operator. The argument given is a continuous interval with a value determined by comparison to a limited set of arguments.[23] discuss fuzzy relative knowledge distances using fuzzy granularity spaces with proper-ties corresponding to fuzzy knowledge distances. Further, several experimental analyses were conducted to show that precise knowledge distances (fuzzy) contain different structure information. [9]The new logic for homogeneous group DM was introduced to select robotic systems using extended TOPSIS. [15]] discussed the picture of fuzzy mean operators and their applica-tions in DM. [4]The new concept of the Spherical Distance Measurement Method for Solving MCDM Problems under PFS was recently introduced. [25]discusses q-rung CDNN weighted averaging (q-rung CDNNWA), q-rung CDNN weighted geometric (q-rung CDNNWG), q-runggeneralized CDNN weighted averaging (q-rung GCDNNWA) and q-rung generalized CDNN weighted geometric (q-rung GCDNNWG). Additionally,their develop an algorithm for solving MADM problems using these operators. Several real-world examples illustrate how en-hanced score values can be applied. Sensor robots are said to rely heavily on computer science and machine tool technology. [13]introduces the concept of cubic spherical neutrosophic sets (CSNSs), a geometric representation of neutrosophic sets, as well as a specification of its operational principles. In CSNs, two aggregation operators are investigated. The shape of CSNSs represents the evaluation values of alternatives for criteria in an MCDM strategy based on the two aggregation operators and cosine distance for CSNSs. The cosine distance between an alternative and the ideal alternative is used to rank them, and the best alternative(s) can be selected. Their outcome the result concludes by demonstrating the use of the suggested method. proves to be more effective than PFS by assuming higher values for the three degrees of membership. However, the CF set (CFS), proposed by[18]. [48] suggests a variety of novel Bonferroni Mean and Weighted Bonferroni Mean operators to aggregate the SR-Fuzzy values for the various decision-maker preferences. Finally, a comparative study of the developed and existing approaches has been discussed to evaluate the pertinency and practicality of the proposed DM technique. [21]discussed interval-valued picture fuzzy geometric Bonferroni mean (IVPFGBM) and interval-valued picture fuzzy weighted geometric Bonferroni mean (IVPFWGBM) under interval-valued picture fuzzy (IVPF) environments is studied.

Their problem of how to aggregate this interval-valued picture fuzzy data using the Bonferronimean is therefore an important one and it is the paper’s primary focus. [48] paper proposes anovel modified Delphibased spherical fuzzy analytical hierarchy process (SFAHP) integrated spherical fuzzy combinative distance-based assessment (SFCODAS) methodology to the vending machine location selection (VMLS) problem. To validate the applicability of the proposed methodology, comparison analysis is presented with the results of the spherical fuzzy weighted aggregated sum-product assessment (SFWASPAS) method.proposed the robust aggregation operators (AOs) of PFSs based on Dombi aggregation models,namely "picture fuzzy Dombi Bonferroni mean" (PFDBM), "picture fuzzy Dombi weighted Bonferroni mean" (PFDWBM), "picture fuzzy Dombi geometric Bonferroni mean" (PFDGBM),and picture fuzzy Dombi weighted geometric Bonferroni mean" (PFDWGBM) operators. To ratify the reliability and versatility of our current approaches, by contrasting the findings of existing approaches with the results of developed techniques. [14] developed multi attribute group decision-making ("MAGDM") method is presented to evaluate, on the basis of conflicting attributes (environmental, economic, and technical), the alternatives in a renewable energy technology (RET) selection problem in an intuitionistic fuzzy MCDM.

However, since the normal distribution cannot be explained by the IFS and INS, more and more people are getting interested in the study of normal fuzzy information. To understand the phenomenon of the normal distribution, Yang and Ko [41] first established the normal fuzzy numbers (NFNs). Finding aggregation functions that can be utilized to simulate the numerous possible correlations between the criteria in multi-criteria situations is quite exciting because there are so many potential correlations between the criteria. Here the capabilities of the aggregation operators known as the Bonferroni mean included in recent research on the Bonferroni mean are [5, 10]are discussed. In the recently, numerous innovative MADM techniques have emerged and found applications in various Decision Making (DM) scenarios with uncertainties[31],[2],[6],[29], [20], [49], [29],[3], [16]. Recently, many authors have used the idea of neutrosophic set in MCDM methods. The concept of the neutrosophic set was introduced by Smarandache [30], which is distinguished by the role of truth-membership function, indeterminacy-membership function, and falsity-membership function. Therefore the neutrosophic set theory can be used to rationalize the confusion associated with ambiguity in an analogous way to human thought. This handles vague data as distributions of possibilities in terms of membership functions. Using the concept of triangular neutrosophic additive reciprocal preference relations. [16] developed a novel method for the group decision-making problem under the neutrosophic environment. The objective of this paper is to measure the relative importance of the criteria and to select the best performence of the machine using cubic neutrosophic FMCDM.

Motivation and Contributions

Many researchers proposed different aggregation expressions, methodologies, and models by using several extensions of the triangular norms such as Frank aggregation, Einstein Aggregation tools, operations of Aczel Alsina aggregation tools, and Hamacher aggregation tools. To our knowledge, no work has been studied for a weighted bonferroni mean operator for selection performance machine by using cubic neutrosophic FMCDM. The contributions of the paper as
follows:

(I) Opinion has been collected from three experts to make decision matrix

(II) In this study considers the 15 alternatives D1 to D5 and criteria ζ1 to ζ5

(III) To determine the relative performance of the criteria.

(IV) The outcome of the results to find the most significant similarity value is taken to be
the best alternative by using neutrosophic cubic FMCDM.

2 Preliminaries

Here in this section the basic definition required for the study is discussed.

[20] Let's assume a fixed universe X and its subset csA. The set

$csA_p = < x, cs\mu(x), csv(x), cs\eta(x); \rho >: x \in X$

Where $cs\mu, csv, cs\eta : X \rightarrow [0,1]$ are functions such that $cs\mu_A + csv_A + cs\eta_A \leq 3$ and $\rho \in [0,1]$. The radius ρ of the sphere with center $(cs\mu(x), csv(x), cs\eta(x))$ inside the cube or cube inside the sphere is called cubic spherical neutrosophic set (CSNS) $csvA_\rho$. This sphere represents the membership degree, indeterminacy degree, and non-membership degree of x ∈ X.

Let $\{< cs\mu_i,1, csv_i,1, cs\eta_i,1 >, < cs\mu_i,2, csv_i,2, cs\eta_i,2 >, ..., < cs\mu_i,k_i, csv_i,k_i, cs\eta_i,k_i > \}$ be a collection of NSs assigned for any x_i in X. We construct the center of the sphere by

$$< cs\mu(x_i), csv(x_i), cs\eta(x_i) =< \frac{\sum_{j=1}^{k_i} cs\mu_{i,j}}{k_i}, \frac{\sum_{j=1}^{k_i} csv_{i,j}}{k_i}, \frac{\sum_{j=1}^{k_i} cs\eta_{i,j}}{k_i} >$$

and the radius using

$\rho i = min n max 1 \leq j \leq ki$ q$(cs\mu(xi - cs\mu 2i,j + (csv(xi) - csvi,j)2 + (cs\eta(xi) - cs\eta i,j)2,1$o

Then the spheres inside the cube or cube inside the sphere is

$$csA_\rho = \{< x_i, cs\mu(x_i), csv(x_i), cs\eta(x_i); \rho >: x_i \in X\}$$

Definition 2.1 *[8] Let* a_u $(u = 1,2,....,n)$ *be a collection of crisp data, where* $a_u \geq 0$*, for all u, and* $r,s \geq 0$*, then we call*

$$B^{r,s}(a_1, a_2, ..., a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{u,v=1 \\ u \neq v}}^{n} a_u^r a_v^s \right)^{\frac{1}{r+s}} \tag{1}$$

the Bonferroni mean(BM).

Especially, if s = 0 *then by Eq.(1), the BM reduces to the generalized mean operator[12] as follows:*

$$B^{r,0}(a_1, a_2, ..., a_n) = \left(\frac{1}{n} \sum_{u=1}^{n} a_u^r \left(\frac{1}{(n-1)} \sum_{\substack{v=1 \\ v \neq u}}^{n} a_v^0 \right) \right)^{\frac{1}{r+0}}$$

$$= \left(\frac{1}{n} \sum_{u=1}^{n} a_u^r \right)^{\frac{1}{r}} \tag{2}$$

If r = 1 *and s* = 0*, then Eq.(2) reduces to the well-known average mean(AM):*

$$B^{1,0}(a_1, a_2, ..., a_n) = \frac{1}{n} \sum_{u=1}^{n} a_u \tag{3}$$

Based on the usual geometric mean(GM) and the BM, we introduce the geometric Bonferroni mean such as:

Definition 2.2 *Let r,s* > 0*, and* a_u(u = 1,2,....,n) *be a collection of non-negative numbers. If*

$$GB^{r,s}(a_1,a_2,...,a_n)=\frac{1}{r+s}\prod_{\substack{u,v=1\\u\neq v}}^{n}(pa_u+qa_v)^{\frac{1}{n(n-1)}} \quad (4)$$

then we call $GB^{r,s}$ *the geometric Bonferroni mean(GBM).*

Obviously, the GBM has the following properties:

(1) $GB^{r,s}(0,0,...,0) = 0$.

(2) $GB^{r,s}(a,a,...,a) = a$, *if* $a_u = a$, *for all u.*

(3) $GB^{r,s}(a_1,a_2,...,a_n) \geq GB^{r,s}(d_1,d_2,...,d_n)$, *i.e.,*$GB^{r,s}$ *is monotonic, if* $a_u \geq d_u$, *for all u.*

(4) $min_u\{a_u\} \leq GB^{r,s}(a_1,a_2,....,a_n) \leq max_u(a_u)$.

Furthermore, if s = 0*, then by Eq.(1), it reduces to the geometric mean:*

$$GB^{r,0}(a_1,a_2,...,a_n)=\frac{1}{p}\prod_{\substack{u,v=1\\u\neq v}}^{n}(pa_u)^{\frac{1}{n(n-1)}}=\prod_{u=1}^{n}(a_u)^{\frac{1}{n}} \quad (5)$$

Definition 2.3 *[37, 38] Let* $\alpha_u = (CN\mu_{\alpha u},v_{\alpha u})(u = 1,2)$ *and* $\alpha = (CN\mu_\alpha,v_\alpha)$ *be three AIFNs, then we have*

(1) $\alpha_1 \bigoplus \alpha_2 = (CN\mu_{\alpha_1} + CN\mu_{\alpha_2} - CN\mu_{\alpha_1}CN\mu_{\alpha_2}, v_{\alpha_1}v_{\alpha_2})$

(2) $\alpha_1 \bigoplus \alpha_2 = (CN\mu_{\alpha_1}CN\mu_{\alpha_2}, v_{\alpha_1} + v_{\alpha_2} - v_{\alpha_1}v_{\alpha_2})$.

(3) $\lambda\alpha = \left(1-(1-CN\mu_\alpha)^\lambda, v_\alpha^\lambda\right), \lambda > 0$

(4) $\alpha^\lambda = \left(CN\mu_\alpha^\lambda, 1-(1-v_\alpha)^\lambda\right), \lambda > 0$.

Moreover, the relations of these operational laws are given as:

(5) $\alpha_2 \bigoplus \alpha_1 = \alpha_1 \bigoplus \alpha_2$.

(6) $\alpha_2 \bigotimes \alpha_1 = \alpha_1 \bigotimes \alpha_2$.

(7) $\lambda\alpha_1 \bigoplus \lambda\alpha_2 = \lambda(\alpha_1 \bigoplus \alpha_2)$.

(8) $\alpha_1^\lambda \bigotimes \alpha_2^\lambda = (\alpha_1 \bigotimes \alpha_2)^\lambda$.

(9) $(\lambda_1+\lambda_2)\alpha = \lambda_1\alpha \bigoplus \lambda_2\alpha$.

(10) $\alpha^{\lambda_1+\lambda_2} = \alpha^{\lambda_1} \bigotimes \alpha^{\lambda_2}$.

To rank any two AIFNs $\alpha_u = (CN\mu_{\alpha u},v_{\alpha u})(u = 1,2)$*, Xu and Yager[37] gave a straightforward method:*

Definition 2.4 *[47] Let* $\lambda =< cs\mu_\lambda,csv_\lambda,cs\eta_\lambda;\rho_\lambda >, \lambda 1 =< cs\mu_{\lambda 1},csv_{\lambda 1},cs\eta_{\lambda 1};\rho_{\lambda 1} >$ *and* $\lambda 2 =< cs\mu_{\lambda 2},csv_{\lambda 2},cs\eta_{\lambda 2};\rho_{\lambda 2} >$ *be three CSNS over the universal set X,* $\gamma \in \{MINI, MAXI\}$*and* $\alpha > 0$*. Then the following operations are defined as follows*

1. $\lambda_1 \cup_\gamma \lambda_2 = \langle MAXI\{cs\mu_{\lambda 1}, cs\mu_{\lambda 2}\}, MINI\{csv_{\lambda 1}, csv_{\lambda 2}\}, Mini\{cs\eta_{\lambda 1}, cs\eta_{\lambda 2}\}; \gamma\{\rho_{\lambda 1}, \rho_{\lambda 2}\}\rangle$*2.* $\lambda_1 \cup_\gamma \lambda_2 = \langle MAXI\{cs\mu_{\lambda 1}, cs\mu_{\lambda 2}\}, |csv_{\lambda 1}, csv_{\lambda 2}|, MAXI\{cs\eta_{\lambda 1}, cs\eta_{\lambda 2}\}; \gamma\{\rho_{\lambda 1}, \rho_{\lambda 2}\}\rangle$

3. $\lambda_1 = \lambda_2 \cdot$ *iff* $\rho_{\lambda 1} = \rho_{\lambda 2}$ *and* $cs\mu_{\lambda 1} = cs\mu_{\lambda 2}, csv_{\lambda 1} = csv_{\lambda 2}, cs\eta_{\lambda 1} = cs\eta_{\lambda 2}$

4. $\lambda_1 \subseteq \lambda_2 \cdot$ *iff* $\rho_{\lambda 1} \subseteq \rho_{\lambda 2}$ *and* $cs\mu\lambda_1 \subseteq cs\mu_{\lambda 2}, csv_{\lambda 1} \subseteq csv_{\lambda 2}, cs\eta_{\lambda 1} \supseteq cs\eta_{\lambda 2}$

5. $\lambda_1 \oplus \lambda_2 = \langle cs\mu_{\lambda 1} + cs\mu_{\lambda 2} - cs\mu_{\lambda 1} + cs\mu_{\lambda 2}, csv_{\lambda 1} + csv_{\lambda 2} - csv_{\lambda 1} + csv_{\lambda 2}, cs\eta_{\lambda 1} + cs\eta_{\lambda 2} - cs\eta\lambda 1 + cs\eta\lambda 2; \rho\lambda 1 + rho\lambda 2 - \rho\lambda 1 + rho\lambda 2\rangle$

6. $\lambda 1 \otimes \lambda 2 = \langle cs\mu\lambda 1 cs\mu\lambda 2, csv\lambda 1 csv\lambda 2, cs\eta\lambda 1 cs\eta\lambda 2; \rho\lambda 1 \rho\lambda 2\rangle$

7. $\alpha\lambda = \langle 1-(1-cs\mu_\lambda)^\alpha, 1-(1-csv_\lambda)^\alpha, 1-(1-cs\eta_\lambda)^\alpha; 1-(1-\rho_\lambda)^\alpha\rangle$

8. $\lambda^\alpha = \langle cs\mu^\alpha_\lambda, csv_\lambda{}^\alpha, cs\eta_\lambda{}^\alpha; \rho^\alpha{}_\lambda\rangle$.

3. Cubic Spherical Neutrosophic Geometric Bonferroni Mean Operator(CSNGBMO)

Here the aggregate value and properties of CSNGBM and CSNGWBM operator are discussed.

Theorem 3.1 *Let r,s > 0, and CSNGBM is* $\{\lambda_u = (CN\mu_{\lambda u}, CN\rho_{\lambda u}, CNv_{\lambda u}, CNr_{\lambda u})$ $(u = 1,2,3,...,n)$ *be a collection of CSNGM, then the aggregated value by using the CSNGBM is*

$$CSNGBM^{r,s}(\lambda_1,\lambda_2,...,\lambda_n) = \left(1-\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(1-CN\mu_{\lambda_u})^r(1-CN\mu_{\lambda_u})^s)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}},\right.$$
$$\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(CN\rho_{\lambda_u})^r(CN\rho_{\lambda_u})^{s^{\frac{1}{n(n-1)}}}\right)^{\frac{1}{r+s}},$$
$$\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-CNv^r_{\lambda_u}CNv^s_{\lambda_u})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}};$$
$$\left(1-\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(1-CNr_{\lambda_u})^r(1-CNr_{\lambda_u})^s)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}.$$

Proof: By the operational laws (1) and (3) described in Definition 3, We have $r_{\lambda u} = (1-(1-CN\mu_{\lambda u})^r, CN\rho^r{}_{\lambda u}, CNv_\lambda{}^r{}_u; 1-(1-CNr_{\lambda u})^r)$ $s_{\lambda u} = (1-(1-CN\mu_{\lambda u})^s,$

$CN\rho^r{}_{\lambda u}, CNv_\lambda{}^s{}_u; 1-(1-CNr_{\lambda u})^s)$ and then

$r_{\lambda u} \oplus s_{\lambda u} = 1-(1-CN\mu_{\lambda u})^r + 1-(1-CN\mu_{\lambda u})^s - (1-(1-CN\mu_{\lambda u})^r(1-(1-$

$CN\rho^p{}_{\lambda u}CN\rho^s{}_{\lambda u} + 1-(1-CNr_{\lambda u})^r + 1-(1-CNr_{\lambda u})^s - ((1-(1-CNr_{\lambda u})^s)(1-(1-CNr_{\lambda u})^s), CN$

Let,

$\beta_{uv} = r_{\lambda u} \oplus s_{\lambda u} = (1-(1-CN\mu_{\lambda u})^r(1-CN\mu_{\lambda u})^s, (CN\rho^r{}_{\lambda u}CN\rho^s{}_{\lambda u}), +(1-(1-CNr_{\lambda u})^r(1-$

$CNr\lambda u)s, CNv\lambda ru CNv\lambda su)$

$$CSNGBM^(r,s)(\lambda_1,\lambda_2,...,\lambda_n)=\frac{1}{r+s}\left(\prod_{\substack{u,v=1\\u\neq v}}^{n} CN\mu_{\beta_{uv}}^{\frac{1}{n(n-1}},1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-CN\rho_{\beta_{uv}})^{\frac{1}{n(n-1)}}\right.$$
$$\left.\prod_{\substack{u,v=1\\u\neq v}}^{n} CNv_{\beta_{uv}}^{\frac{1}{n(n-1}},1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-CNr_{\beta_{uv}})^{\frac{1}{n(n-1)}}\right),$$

Since,

$$\bigotimes_{\substack{u,v=1\\u\neq v}}^{n}(r_{\lambda_u}\oplus s_{\lambda_u})^{\frac{1}{n(n-1)}}=\left(\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(1-CN\mu_{\lambda_u})^r(1-CN\mu_{\lambda_u})^s)^{\frac{1}{n(n-1)}},1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-CN\rho_{\lambda_u}CN\rho_{\lambda_u})^{\frac{1}{n(n-}}\right.$$
$$\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-CNv_{\lambda_u}CNv_{\lambda_u})^{\frac{1}{n(n-1}},\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(1-CNr_{\lambda_u})^r(1-CNr_{\lambda_u})^s)^{\frac{1}{n(n-1)}}$$

$$\bigotimes_{\substack{u,v=1\\u\neq v}}^{n}\beta_{u_v}{}^{\frac{1}{n(n-1)}}=\left(\prod_{\substack{u,v=1\\u\neq v}}^{n} CN\mu_{\beta_{uv}}^{\frac{1}{n(n-1}},1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-CN\rho_{\beta_{uv}})^{\frac{1}{n(n-1)}}\right.$$
$$\left.\prod_{\substack{u,v=1\\u\neq v}}^{n} CNv_{\beta_{uv}}^{\frac{1}{n(n-1}},1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-CNr_{\beta_{uv}})^{\frac{1}{n(n-1)}}\right),$$

By using the operational law,

$$CSNGBM^(r,s)(\lambda_1,\lambda_2,...,\lambda_n)=\frac{1}{r+s}\bigotimes_{\substack{u,v=1\\u\neq v}}^{n}(r_{\lambda_u}\oplus s_{\lambda_u})^{\frac{1}{n(n-1)}}$$

$$=\left(1-\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(1-CN\mu_{\lambda_u})^r(1-CN\mu_{\lambda_u})^s\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}$$
$$\times\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-CN\rho_{\lambda_u}CN\rho_{\lambda_u})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}$$
$$\times\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-CNv_{\lambda_u}CNv_{\lambda_u})^{\frac{1}{n(n+1)}}\right)^{\frac{1}{r+s}}$$
$$\times\left(1-\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(1-CNr_{\lambda_u})^r(1-CNr_{\lambda_u})^s\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}$$

Where $0\leq 1$,

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$$0 \leq 1 - \left(1 - \left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} (1 - (1 - CN\mu_{\lambda_u})^r (1 - CN\mu_{\lambda_u})^s)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}} \leq 1,$$

$$\left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} (1 - CN\rho_{\lambda_u} CN\rho_{\lambda_u})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}} \leq 1,$$

$$\left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} (1 - CNv_{\lambda_u} CNv_{\lambda_u})^{\frac{1}{n(n+1)}}\right)^{\frac{1}{r+s}} \leq 1$$

and

$$\left(1 - \left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} (1 - (1 - CNr_{\lambda_u})^r (1 - CNr_{\lambda_u})^s)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}} \leq 1.$$

,

Then we have,

$$1 - \left(1 - \left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} (1 - (1 - CN\mu_{\lambda_u})^r (1 - CN\mu_{\lambda_u})^s)^{\frac{1}{n(n-1)}}\right.\right.$$

$$+\left(1 - \prod_{u,v=1 u \neq v}^{n} (1 - CN\rho_{\lambda_u} CN\rho_{\lambda_u})^{\frac{1}{n(n-1)}}\right.$$

$$+\left(1 - \prod_{u,v=1 u \neq v}^{n} (1 - CNv_{\lambda_u} CNv_{\lambda_u})^{\frac{1}{n(n+1)}}\right)^{\frac{1}{r+s}} + \left(1 - \left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} (1 - (1 - CNr_{\lambda_u})^r (1 - CNr_{\lambda_u})^s)^{\frac{1}{n(n-1)}}\right.\right.$$

$$\leq 1 - \left(1 - \left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} (1 - (1 - CN\mu_{\lambda_u})^r (1 - CN\mu_{\lambda_u})^s)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}\right.$$

$$+\left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} (1 - (1 - CN\mu_{\lambda_u})^r (1 - CN\mu_{\lambda_u})^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}$$

$$+1 - \left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} (1 - (1 - CNr_{\lambda_u})^r (1 - CNr_{\lambda_u})^s)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}$$

$$+\left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} ((1 - (1 - CNr_{\lambda_u})^r)(1 - (1 - CNr_{\lambda_u})^s)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}$$

Which completes the proof.

Properties of CSNGBM:

$$CSNGBM^{1,0}(\lambda_1,\lambda_2,...,\lambda_n)=\bigotimes_{u=1}^{n}{\lambda_u}^{\frac{1}{n}}$$
$$=\left\{\prod_{u=1}^{n}CN\mu_u^{\frac{1}{n}},(1-\prod_{u=1}^{n}(1-CN\rho_{\lambda_u})^{\frac{1}{n}}),(1-\prod_{u=1}^{n}(1-CNv_{\lambda_u})^{\frac{1}{n}}),\prod_{u=1}^{n}CNr_i^{\frac{1}{n}}\right.$$

1. Idempotency: If all α_i (i = 1,2,...,n) are equal. i.e., $\alpha_i = \alpha = (CN\mu_\alpha,CN\rho_\alpha,CNv_\alpha,CNr_\alpha)$ ∀ i, then

$$CSNGBM^{p,q}(\alpha_1,\alpha_2,...,\alpha_n)=CSNGBM^{p,q}(\alpha,\alpha,...,\alpha)$$
$$=\frac{1}{p+q}\left(\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}(p_\alpha\oplus q_\alpha)^{\frac{1}{n(n-1)}}\right)$$
$$=\frac{1}{p+q}\left(\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}((p+q)\alpha)^{\frac{1}{n(n-1)}}\right.$$
$$=\frac{1}{p+q}\left((p+q)\alpha\right)^{\frac{n(n-1)}{n(n-1)}}$$
$$=\frac{1}{p+q}(p+q)\alpha$$
$$=\alpha$$

2. Commutativity: If $\alpha_i = (CN\mu_{\alpha i},CN\rho_{\alpha i},CNv_{\alpha i},CNr_{\alpha i})$ (i = 1,2,...,n) be a collection of CSNGM. Then if $CSNGBM^{p,q}(\alpha_1,...,\alpha_n) = IFGM^{q,p}(\alpha_1,...,\alpha_n)$. Now,

$$CSNGBM^{p,q}(\alpha_1,...,\alpha_n)=\frac{1}{p+q}\left(\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}(p_{\alpha_i}\oplus q_{\alpha_j})^{\frac{1}{n(n-1)}}\right)$$
$$=\frac{1}{p+q}\left(\bigotimes_{\substack{i,j=1\\ i\neq j}}^{n}(q_{\alpha_j}\oplus p_{\alpha_i})^{\frac{1}{n(n-1)}}\right.$$
$$=CSNGM^{p,q}(\alpha_1,\alpha_2,...,\alpha_n).$$

3. Monotonicity: Let $\alpha_i = (CN\mu_{\alpha i},CN\rho_{\alpha i},CNv_{\alpha i},CNr_{\alpha i})$ (i = 1,2,...,n), and β_i = $(CN\mu_{\beta i},CN\rho_{\beta i},CNv_{\beta i},CNr_{\beta i})$ (i = 1,2,...,n) be two collections of CSNGBM, $(CN\mu_{\alpha i}\leq CN\mu_{\beta i},CN\rho_{\alpha i}\geq CN\rho_{\beta i},CNv_{\alpha i}\geq CNv_{\beta i},CNr_{\alpha i}\leq CNr_{\beta i})$ ∀ i, Then $CSNGBM^{p,q}(\alpha_1,...,\alpha_n)\leq CSNGBM^{p,q}(\beta_1,...,\beta_n)$

4.Boundedness: Let $\lambda_u = (CN\mu_{\lambda u},CN\rho_{\lambda u},CNv_{\lambda u},CNr_{\lambda u})$ (u = 1,2,...,n) be a collection of CSNGM and let

$$\lambda^-=(Min\{CN\mu_{\lambda_u}\},Max\{CN\rho_{\lambda_u}\},Max\{CNv_{\lambda_u}\},Min\{CNr_{\lambda_u}\})$$

$$\lambda^+=(Max\{CN\mu_{\lambda_u}\},Min\{CN\rho_{\lambda_u}\},Min\{CNv_{\lambda_u}\},Max\{CNr_{\lambda_u}\})$$

Then $\lambda^-\leq CSNGM^{r,s}(\lambda_1,\lambda_2,...,\lambda_n)\leq\lambda^+$.

Theorem 3.2 *Let r,s* > 0, *and CSNGWBM is* $\lambda_u(CN\mu_{\lambda u},CN\nu_{\lambda u},CN\rho_{\lambda u},CNr_{\lambda u})(u=1,2,3,...,n)$ *be a collection of CSNGWM, then the aggregated value by using the CSNGWBM*

$$CSNGWBM^{r,s}(\lambda_1,\lambda_2,...,\lambda_n)=\left(1-\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(1-(CN\mu_{\lambda_u})^{w_v})^r(1-(CN\mu_{\lambda_v})^{w_v})^s)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}},\right.$$

$$\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(CN\rho_{\lambda_u})^{w_v})^r((CN\rho_{\lambda_v})^{w_v})^s)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}},$$

$$\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(CNv_{\lambda_u})^{w_u})^r((CNv_{\lambda_v})^{w_v})^s)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}$$

$$\left(1-\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(1-(CNr_{\lambda_u})^{w_u})^r(1-(CNr_{\lambda_v})^{w_v})^s)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}.$$

,

Proof: By the operational laws (1) and (3) described in Definition 3, We have $r\lambda u=(1-(1-(CN\mu\lambda u)wu)r,((CN\rho\lambda u)wu)r,((CN\nu\lambda u)wu)r,1-(1-(CNr\lambda u)wu)r)$ $s\lambda v=(1-(1-(CN\mu\lambda v)wv)s,((CN\rho\lambda v)wv)r,((CN\nu\lambda v)wv)s,1-(1-(CNr\lambda v)wv)s)$ and then

$r_{\lambda u}\oplus s_{\lambda v}=1-(1-(CN\mu_{\lambda u})^{wu})^r+1-(1-(CN\mu_{\lambda v})^{wv})^s-(1-(1-(CN\mu_{\lambda u})^{wu})^r(1-(1-(CN\mu\lambda v)wv)s),((CN\rho\lambda u)wu)r((CN\rho\lambda v)wv)s+1-(1-(CNr\lambda v)wu)r+1-(1-(CNr_{\lambda v})^{wv})^s-((1-(1-(CNr_{\lambda u})^{wu})^s)(1-(1-(CNr_{\lambda v})^{wv})^s),((CN\nu\lambda u)wu)r((CN\nu\lambda v)wv)s)$

Let,

$\beta uv=r\lambda u\oplus s\lambda v=(1-(1-(CN\mu\lambda u)wu)r(1-(CN\mu\lambda v)wv)s,(((CN\rho\lambda u)wu)r((CN\rho\lambda v)wv)s),+(1-(1-(CNr\lambda u)wu)r(1-(CNr\lambda v)wv)s,((CN\nu\lambda u)wu)r((CN\nu\lambda v)wv)s)$

$$CSNGWBM^{r,s}(\lambda_1,\lambda_2,...,\lambda_n)=\frac{1}{r+s}\left(\prod_{\substack{u,v=1\\u\neq v}}^{n}CN\mu_{\beta_{u_v}}^{\frac{1}{n(n-1}},1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-CN\rho_{\beta_{u_v}})^{\frac{1}{n(n-1)}}\right.$$

$$\left.\prod_{\substack{u,v=1\\u\neq v}}^{n}CNv_{\beta_{u_v}}^{\frac{1}{n(n-1}},1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-CNr_{\beta_{u_v}}^{\frac{1}{n(n-1)}}\right),$$

Since,

$$\bigotimes_{\substack{u,v=1\\u\neq v}}^{n} \beta_{u_v}{}^{\frac{1}{n(n-1)}} = \left(\prod_{\substack{u,v=1\\u\neq v}}^{n} CN\mu_{\beta_{u_v}}^{\frac{1}{n(n-1}}, 1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-CN\rho_{\beta_{u_v}})^{\frac{1}{n(n-1)}}, \prod_{\substack{u,v=1\\u\neq v}}^{n} CNv_{\beta_{u_v}}^{\frac{1}{n(n-1}}, 1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-CNr_{\beta_{u_v}})^{\frac{1}{n(n-1)}} \right)$$

$$\bigotimes_{\substack{u,v=1\\u\neq v}}^{n} (r_{\lambda_u} \oplus s_{\lambda_v})^{\frac{1}{n(n-1)}} = \left(\prod_{\substack{u,v=1\\u\neq v}}^{n} (1-(1-(CN\mu_{\lambda_u})^{w_u})^r(1-(CN\mu_{\lambda_v})^{w_v})^s)^{\frac{1}{n(n-1)}} \right.$$
$$1-\prod_{\substack{u,v=1\\u\neq v}}^{n} (1-(CN\rho_{\lambda_u})^{w_u}(CN\rho_{\lambda_v})^{w_v})^{\frac{1}{n(n-1)}},$$
$$\prod_{\substack{u,v=1\\u\neq v}}^{n} (1-(CNv_{\lambda_u})^{w_u}(CNv_{\lambda_v})^{w_v})^{\frac{1}{n(n-1}},$$
$$\left. \prod_{\substack{u,v=1\\u\neq v}}^{n} (1-(1-(CNr_{\lambda_u})^{w_u})^r(1-(CNr_{\lambda_v})^{w_v})^s)^{\frac{1}{n(n-1)}} \right),$$

By using the operational law,

$$CSNGWBM^{r,s}(\lambda_1,\lambda_2,\dots,\lambda_n) = \frac{1}{r+s}\bigotimes_{\substack{u,v=1\\u\neq v}}^{n} (r_{\lambda_u} \oplus s_{\lambda_v})^{\frac{1}{n(n-1)}}$$

$$= \left(1-\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(1-(CN\mu_{\lambda_u})^{w_u})^r(1-(CN\mu_{\lambda_v})^{w_v})^s\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}$$
$$\times\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(CN\rho_{\lambda_u})^{w_u}(CN\rho_{\lambda_v})^{w_v})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}$$
$$\times\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(CNv_{\lambda_u})^{w_u}(CNv_{\lambda_v})^{w_v})^{\frac{1}{n(n+1)}}\right)^{\frac{1}{r+s}}$$
$$\times\left(1-\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(1-(CNr_{\lambda_u})^{w_u})^r(1-(CNr_{\lambda_v})^{w_v})^s\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}$$

Where $0 \leq 1$,

$$0 \leq 1 - \left(1 - \left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} (1 - (1 - (CN\mu_{\lambda_u})^{w_u})^r (1 - (CN\mu_{\lambda_v})^{w_v})^s)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}} \leq 1,$$

$$\left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} (1 - (CN\rho_{\lambda_u})^{w_u} (CN\rho_{\lambda_v})^{w_v})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}} \leq 1,$$

$$\left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} (1 - (CNv_{\lambda_u})^{w_u} (CNv_{\lambda_v})^{w_v})^{\frac{1}{n(n+1)}}\right)^{\frac{1}{r+s}} \leq 1$$

,

and

$$\left(1 - \left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} (1 - (1 - (CNr_{\lambda_u})^{w_u})^r (1 - (CNr_{\lambda_v})^{w_v})^s)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}} \leq 1$$

.

Then we have,

$$1 - \left(1 - \left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} (1 - (1 - (CN\mu_{\lambda_u})^{w_u})^r (1 - (CN\mu_{\lambda_v})^{w_v})^s)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}$$

$$+ \left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} (1 - (CN\rho_{\lambda_u})^{w_u} (CN\rho_{\lambda_v})^{w_v})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}$$

$$+ \left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} (1 - (CNv_{\lambda_u})^{w_u} (CNv_{\lambda_v})^{w_v})^{\frac{1}{n(n+1)}}\right)^{\frac{1}{r+s}}$$

$$+ \left(1 - \left(1 - \prod_{\substack{u,v=1 \\ u \neq v}}^{n} (1 - (1 - (CNr_{\lambda_u})^{w_u})^r (1 - (CNr_{\lambda_v})^{w_v})^s)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}$$

$$\leq 1-\left(1-\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(1-(CN\mu_{\lambda_u})^{w_u})^{p}(1-(CN\mu_{\lambda_v})^{w_v})^{s})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}\right.$$

$$+\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(1-(CN\mu_{\lambda_u})^{w_u})^{r}(1-(CN\mu_{\lambda_v})^{w_v})^{s})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}$$

$$+1-\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}(1-(1-(CNr_{\lambda_u})^{w_u})^{r}(1-(CNr_{\lambda_v})^{w_v})^{s})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}$$

$$+\left(1-\prod_{\substack{u,v=1\\u\neq v}}^{n}((1-(1-(CNr_{\lambda_u})^{w_u})^{r})(1-(1-(CNr_{\lambda_v})^{w_v})^{s})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{r+s}}$$

Which completes the proof.

Properties of CSNGWBM:

1. Idempotency: If all λ_u (u = 1,2,...,n) are equal. i.e., $\lambda_u = \lambda = (CN\mu_\lambda, CN\rho_\lambda, CNv_\lambda, CNr_\lambda)$ ∀ u, then

$$CSNGWBM^{r,s}(\lambda_1,\lambda_2,...,\lambda_n) = CSNGWBM^{r,s}(\lambda,\lambda,...,\lambda)$$

$$=\frac{1}{r+s}\left(\bigotimes_{\substack{u,v=1\\u\neq v}}^{n}(r_\lambda\oplus s_\lambda)^{\frac{1}{n(n-1)}}\right)$$

$$=\frac{1}{r+s}\left(\bigotimes_{\substack{u,v=1\\u\neq v}}^{n}((r+s)\lambda)^{\frac{1}{n(n-1)}}\right.$$

$$=\frac{1}{r+s}\left((r+s)\lambda\right)^{\frac{n(n-1)}{n(n-1)}}$$

$$=\frac{1}{r+s}(r+s)\lambda$$

$$=\lambda$$

2. Commutativity: If $\alpha_u = (CN\mu_{\alpha_u}^{w_u}, CN\rho_{\lambda_u}^{w_u}, CNv_{\lambda_u}^{w_u}, CNr_{\lambda_u}^{w_u})$ $(u=1,2,...,n)$ be a collection of CSNGWM. Then if $CSNGWBM^{r,s}(\lambda_1,...,\lambda_n) = CSNGWBM^{s,r}(\lambda_1,...,\lambda_n)$. Now,

$$CSNGWBM^{r,s}(\lambda_1,...,\lambda_n) = \frac{1}{r+s}\left(\bigotimes_{\substack{u,v=1\\u\neq v}}^{n}(r_{\lambda_u}\oplus s_{\lambda_v})^{\frac{1}{n(n-1)}}\right)$$

$$=\frac{1}{r+s}\left(\bigotimes_{\substack{u,v=1\\u\neq v}}^{n}(s_{\lambda_v}\oplus r_{\lambda_u})^{\frac{1}{n(n-1)}}\right)$$

$$= CSNGWBM^{s,r}(\Lambda_1,\lambda_2,...,\lambda_n).$$

3. Monotonicity: Let $\lambda_u = (CN\mu_{\lambda_u}^{w_u}, CN\rho_{\lambda_u}^{w_u}, CNv_{\lambda_u}^{w_u}, CNr_{\lambda_u}^{w_u})\ (u = 1, 2, ..., n$), and β_u = $(CN\mu_{\beta_u}^{w_u}, CN\rho_{\beta_u}^{w_u}, CNv_{\beta_u}^{w_u}, CNr_{\beta_u}^{w_u})\ (u = 1, 2, ..., n$) be two collections of CSNGWM, $(CN\mu_{\lambda_u}^{w_u} \leq CN\mu_{\beta_u}^{w_u}, CN\rho_{\lambda_u}^{w_u} \geq CN\rho_{\beta_u}^{w_u}, CNv_{\lambda_u}^{w_u} \geq CNv_{\beta_u}^{w_u}, CNr_{\lambda_u}^{w_u} \leq CNr_{\beta_u}^{w_u})\ \forall\ u$, Then

$CSNGWBM^{r,s}(\lambda_1,...,\lambda_n) \leq CSNGWBM^{r,s}(\beta_1,...,\beta_n)$

4. Boundedness: Let $\lambda_u = (CN\mu_{\lambda_u}^{w_u}, CN\rho_{\lambda_u}^{w_u}, CNv_{\lambda_u}^{w_u}, CNr_{\lambda_u}^{w_u})\ (u = 1, 2, ..., n$) be a collection of CSNGWM and let

$$\lambda^- = (Min\{CN\mu_{\lambda_u}^{w_u}\}, Max\{PFCN\rho_{\lambda_u}^{w_u}\}, Max\{CNv_{\lambda_u}^{w_u}\}, Min\{CNr_{\lambda_u}^{w_u}\})$$

$$\alpha^+ = (Max\{CN\mu_{\lambda_u}^{w_u}\}, Min\{PFCN\rho_{\lambda_u}^{w_u}\}, Min\{CNv_{\lambda_u}^{w_u}\}, Max\{CNr_{\lambda_u}^{w_u}\})$$

Then $\lambda^- \leq CSNGWBM^{r,s}(\lambda_1,\lambda_2,...,\lambda_n) \leq \lambda^+$. Which can be obtained easily by the monotonicity. If the values of the parameters r and s change in the CSNGWM,

3.1.A MCDM Method

In this section the proposed algorithm for solving MCDM by using Cubic Spherical Neutrosophic Geometric Weighted Bonferroni Mean operator is discussed.

In the cubic spherical neutrosophic environment, a *MCDM* method is suggested in this section. The suggested approach is used to solve a *MCDM* problem that has been taken from the literature to demonstrate its effectiveness in the following example. Following are the steps of the suggested method that we can present:

Step 1: Suppose there are k alternatives that $A = \{A_1,A_2,.......,A_k\}$ expert has evaluated in light of a list with j criteria as $C = \{c_1,c_2,......,c_j\}$. Step 2: For each criterion, the expert chooses the weight vector and converts the assessment results of the alternatives into *CSNV s*. Step 3: If there are any cost criteria, based on their values, the complement operation is used.

Step 4: Evaluation findings for each choice that are expressed as *CSNV s* are transformed,using suggested weighted aggregation operations.

Step 5: The variation in CSNV between each alternative's aggregate value and the ideal alternative's positive value < 1,0,0;1 > is determined.

Step 6: The alternative with the greatest similarity value is taken to be the best.

3.2Numerical Example: Selection of Best Machine using CSNVs

Machines are the tools that enable people to operate more efficiently and quickly. To simplify our daily lives, we use machines. Our tasks can be performed by machines more effectively. These devices typically shift the direction of the force, reduce the amount of force needed to perform a certain amount of work, or change one form of motion or energy into another. Modern power tools, automated machine tools, and power machinery that is operated by humans are all examples of tools and machines. Engines are devices that convert heat or other forms of energy into mechanical energy. Give 15 potential names for the machine you want to choose, such as Machine-o-matic, Techtronic, Mechmaster, Autowizard, Machinamate, robopro, Electra Tech, Turbo Engineer, Cyborginator, Prodroid, Automatrix, Megamech, Gear Guru, Electric Craft, and MachinaX. Additionally, there are three experts: E1, E2, and E3. These experts are:1) Industry Professionals 2) production and sales 3) technical advisers. The following variables should be analyzed to establish which of these classes the need for new machines fits into (A1) purpose; (A2) performance; (A3) dependability and durability; (A4) scalability and future-proofing; and (A5) cost-effectiveness.

Step-1: The alternatives' language terms are shown in Table 1.

Table 1: Linguistic scale

	Linguistic Scale		
	T(Truth Value)	(Indeterminacy Value)	F(False Value)
VVG(Very Very Good)	1	0	0
VG(Very Good)	0.9	0.1	0.1
G(Good)	0.8	0.15	0.2
P(Poor)	0.7	0.25	0.3
EX(Exemplary)	0.6	0.35	0.4
VP(Very Poor)	0.5	0.5	0.5
VVP(Very Very Poor)	0.4	0.65	0.6
MEX(Most Exemplary)	0.3	0.75	0.7
VEX(Very Exemplary)	0.2	0.85	0.8
O(Outstanding)	0.1	0.9	0.9

Step-2: The language choices made for the options are shown in Table 2.

Table 2: The decision matrix of each attribute

		$\zeta 1$	$\zeta 2$	$\zeta 3$	$\zeta 4$	$\zeta 5$
E1	D1	VVG	EX	P	G	G
	D2	G	G	VG	P	P
	D3	VG	P	MEX	VG	EX
	D4	P	VP	O	VP	VG
	D5	EX	VG	VEX	VEX	VVG
E2	D6	VG	O	VG	MEX	VP
	D7	VEX	VG	G	VG	VEX
	D8	MEX	G	VVP	VP	MEX
	D9	VVG	P	MEX	EX	O
	D10	VG	VVG	VP	VEX	VG
E3	D11	EX	VVG	VP	VEX	VG
	D12	P	P	VP	VG	P
	D13	O	VG	G	MEX	EX
	D14	VP	MEX	EX	VG	VG
	D15	G	VG	VVG	G	VP

Step-3: The decision matrix for each alternative is shown in Table 3.

Table 3: The cubic spherical decision matrix

		ζ1			ζ2			ζ3			ζ4			ζ5		
		T	I	F	T	I	F	T	I	F	T	I	F	T	I	F
	D1	1	0	0	0.6	0.35	0.4	0.7	0.25	0.3	0.8	0.15	0.2	0.8	0.15	0.2
	D2	0.8	0.15	0.2	0.8	0.15	0.2	0.9	0.1	0.1	0.7	0.25	0.3	0.7	0.25	0.3
E1	D3	0.9	0.1	0.1	0.7	0.25	0.3	0.3	0.75	0.7	0.9	0.1	0.1	0.6	0.35	0.4
	D4	0.7	0.25	0.3	0.5	0.5	0.5	0.1	0.9	0.9	0.5	0.5	0.5	0.9	0.1	0.1
	D5	0.6	0.35	0.4	0.9	0.1	0.1	0.2	0.85	0.8	0.2	0.85	0.8	1	0	0
	D6	0.9	0.1	0.1	0.1	0.9	0.9	0.9	0.1	0.1	0.3	0.75	0.7	0.5	0.5	0.5
	D7	0.2	0.85	0.8	0.9	0.1	0.1	0.8	0.15	0.2	0.9	0.1	0.1	0.2	0.85	0.8
E2	D8	0.3	0.75	0.7	0.1	0.9	0.9	0.4	0.65	0.6	0.5	0.5	0.5	0.3	0.75	0.7
	D9	1	0	0	0.7	0.25	0.3	0.3	0.75	0.7	0.6	0.35	0.4	0.1	0.9	0.9
	D10	0.9	0.1	0.1	1	0	0	0.5	0.5	0.5	0.2	0.85	0.8	0.9	0.1	0.1
	D11	0.6	0.35	0.4	0.9	0.1	0.1	0.8	0.15	0.2	0.1	0.9	0.9	0.5	0.5	0.5
	D12	0.7	0.25	0.3	0.7	0.25	0.3	0.5	0.5	0.5	0.9	0.1	0.1	0.7	0.25	0.3
E3	D13	0.1	0.9	0.9	0.9	0.1	0.1	0.8	0.15	0.2	0.3	0.75	0.7	0.6	0.35	0.4
	D14	0.5	0.5	0.5	0.3	0.75	0.7	0.6	0.35	0.4	0.9	0.1	0.1	0.9	0.1	0.1
	D15	0.8	0.15	0.2	0.9	0.1	0.1	1	0	0	0.8	0.15	0.2	0.5	0.5	0.5

Step-4: The cubic spherical decision matrix's arithmetic average is shown in Table 4.

Step-5:The maximum radius lengths are shown in Table 5 and are determined using a decision matrix.

Table 4: Arithmetic average of cubic spherical decision matrix

ζ1			ζ2			ζ3			ζ4			ζ5		
0.80	0.18	0.20	0.67	0.33	0.33	0.73	0.25	0.27	0.20	0.83	0.80	0.63	0.37	0.37
0.50	0.48	0.50	0.83	0.15	0.17	0.70	0.27	0.30	0.63	0.37	0.37	0.47	0.53	0.53
0.37	0.63	0.63	0.57	0.42	0.43	0.57	0.43	0.43	0.57	0.45	0.43	0.53	0.45	0.47
0.53	0.47	0.47	0.63	0.37	0.37	0.57	0.42	0.43	0.60	0.40	0.40	0.53	0.45	0.47
0.73	0.25	0.27	0.73	0.28	0.27	0.70	0.28	0.30	0.63	0.37	0.37	0.77	0.23	0.23

Table 5: Maximum radius lengths based on decision matrix

ζ1	ζ2	ζ3	ζ4	ζ5
0.34	0.99	0.28	1.09	0.33
0.56	0.21	0.36	0.47	0.5
0.92	0.82	0.5	0.58	0.44
0.81	0.6	0.82	0.52	0.76
0.28	0.47	0.91	0.77	0.47

Step-6: The score value attained by CSNGBM operators and the order of alternatives are shown in Table 6.

Table 6: The score value obtained by CSNGBM operators and the ranking of alternatives

	ζ1	ζ2	ζ3	ζ4	ζ5	**RANK**
p=5,q=5	0.5112	0.4701	0.4442	0.5249	0.4785	ζ2>ζ4>ζ5>ζ1>ζ3
p=1,q=0.5	0.5121	0.494	0.4441	0.5641	0.4819	ζ2>ζ3>ζ5>ζ1>ζ4
p=3,q=2	0.3418	0.315	0.2961	0.3565	0.3207	ζ2>ζ4>ζ5>ζ1>ζ3
p=4,q=1	0.4981	0.4512	0.4339	0.5414	0.4734	ζ2>ζ4>ζ5>ζ1>ζ3
p=2,q=3	0.5141	0.4103	0.3915	0.5108	0.4697	ζ2>ζ4>ζ5>ζ1>ζ3
p=1,q=1	0.5278	0.5	0.4576	0.5468	0.4896	ζ2>ζ3>ζ5>ζ1>ζ4
p=1,q=2	0.5377	0.5028	0.4755	0.5603	0.5168	ζ2>ζ4>ζ5>ζ1>ζ3

p=4,q=2	0.5074	0.4627	0.4394	0.5395	0.4777	ζ2>ζ4>ζ5>ζ1>ζ3
p=4,q=3	0.5113	0.4692	0.4434	0.531	0.4796	ζ2>ζ4>ζ5>ζ1>ζ3
p=1,q=4	0.5391	0.5011	0.4672	0.5147	0.4928	ζ1>ζ3>ζ5>ζ2>ζ4

Step-7: The score value attained by CSNGWBM operators and the order of choices are shown in Table 7.

	ζ1	ζ2	ζ3	ζ4	ζ5	**RANK**
p=5,q=5	0.8383	0.8284	0.8094	0.8409	0.8261	ζ2>ζ3>ζ5>ζ1>ζ4
p=1,q=0.5	0.8206	0.8029	0.7864	0.8500	0.8154	ζ2>ζ4>ζ5>ζ1>ζ3
p=3,q=2	0.8281	0.8131	0.7973	0.8387	0.8201	ζ2>ζ4>ζ5>ζ1>ζ3
p=4,q=1	0.8203	0.7077	0.7875	0.8272	0.8069	ζ2>ζ5>ζ4>ζ1>ζ3
p=2,q=3	0.8490	0.8021	0.8355	0.8313	0.6274	ζ1>ζ4>ζ2>ζ3>ζ5
p=1,q=1	0.8408	0.8263	0.8114	0.8545	0.8319	ζ2>ζ4>ζ5>ζ1>ζ3
p=1,q=2	0.8468	0.8350	0.8173	0.8591	0.8393	ζ2>ζ4>ζ5>ζ1>ζ3
p=4,q=2	0.8282	0.8041	0.7970	0.8339	0.8166	ζ2>ζ4>ζ5>ζ1>ζ3
p=4,q=3	0.8324	0.8162	0.8023	0.8383	0.8221	ζ2>ζ4>ζ5>ζ1>ζ3
p=1,q=4	0.8508	0.8424	0.8242	0.9055	0.8417	ζ2>ζ3>ζ5>ζ1>ζ4

Table 8: The ranking results with different methods

Method	Ranking
CSNGBM	ζ2>ζ^{l}>ζ5>ζ1>ζ3
CSNGWBM	ζ2>ζ4>ζ5>ζ1>ζ3

Conclusion

This study, we have developed the Cubic Spherical Neutrosophic Geometric Bonferroni Mean Operator (CSNGBM) and the Cubic Spherical Neutrosophic Geometric Weighted Bonferroni Mean Operator (CSNG-WBM) under the Neutrosophic fuzzy MCDM. The cubic neutrosophic environment, in which criterion values concerning alternatives are evaluated by the form of (CSNGBM) and CSNG-WBM) values and the criterion weights are known information and were used to apply the two aggregation operators to MCDM problems. To rank the alternatives and choose the best one(s) based on the measure values, we used the distance between an alternative and the ideal alternative. A numerical example is then given to show how the developed approach is applied. Because it can handle not only incomplete information but also indeterminate information and inconsistent information that frequently exist in real situations, the proposed (CSNGBM) and (CSNG-WBM) methods is suitable for real scientific and engineering applications. The methods suggested in this paper can give DM more useful options. We will address group DM issues involving incomplete decision contexts and preference relations in the selection process in the future. We will also apply Fuzzy pilothinic approach aggregation operators to resolve real-world problems in other domains, such as expert systems, information fusion systems, and medical diagnoses.

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