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On a Type of Generalized GS_β-Closed Set in Grill Topology

N Kalaivani[a*] K Fayaz Ur Rahman[b]
[ab]Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R & D Institute of Science and Technology, Chennai, India
E -Mail: [a]kalaivani.rajam@gmail.com, [a]drkalaivani@veltech.edu.in,[b]vtd675@veltech.edu.in,

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Abstract: This work intends to propose the concept of generalized GS_β-closed set-in grill topological space. Further, some of the properties of such generalized GS_β-closed set are evaluated and provided with suitable illustrations. Many theorems based on generalized GS_β-closed set have also been proved in this work. Furthermore, the ideas of β-regular space and β-normal space are also studied. Also, the relationship between GS_β-regular and GS_β- normal spaces is discussed. The properties of β-regular space and β-normal space concepts have also been discussed.

Introduction: Alo, R. A., and Shapiro, H. L defined the semi- normal spaces and considered some of their important inheritance. The authors further studied the compactness in grill topological spaces and studied some of their properties. Data granulation serves as a vital decision-making tool in numerous real-life applications. Mashhour, A. S and Abd, M. E introduced certain new topological tools employing rough set approximations for the purpose of data granulation. The introduction of GS_β-normality helps to separate closed subsets, continuous functions. It also extends the support to define the concepts of complete regularity and Tietze extension theorem. The GS_β-regular space studied in this article is regular, that is completely regular, which is a stronger condition.

Objectives: I.Arockiarani and A. Karthika introduced a new concept such as generalized B-closed set in grill topological space. The above paper motivates me to do the research in generalized GS_β-closed set. In this paper a new concept of "generalized GS_β-closed set in grill topological space" are introduced. In this article we are using stronger condition for proving some theorems.

Methods: The present task ambitioned at formation of the extension of topological system applying the idea of grill. The importance of the theory of grill between the topological spaces have been elaborated. A set is countable collection based on elements, without integrity of form. At the same time some kinds of algebraic actions are applied on this set, indefinitely the elements of this set are correlated into a whole, so ultimately it grows into a space. So ultimately in the whole paper we are applying the concept of grill topological space.

Results: In this article a new concept of generalized GS_β-closed set in a grill topological space are introduced. Using this definition we are proving many results based on operators on grill topological space. Furthermore, the implication such as "all closed set are generalized GS_β-closed sets" is proved and the converse may not be true are also proved with proper illustration. In addition the relationship between β-normal space, GS_β-normal space, β-regular space and GS_β-regular space are also studied. Further many equivalent conditions are also proved using β-normal space and β-regular space.

Conclusions: The work proposes and investigates the concept of generalized GS_β-closed sets within the framework of grill topological spaces. It explores the properties of these sets, provides illustrative examples, and proves theorems related to their behavior. Additionally, the study delves into the concepts of β-regular space and β-normal space, discussing their properties and implications. The research contributes to the understanding of these concepts and their applications in the context of grill topological spaces.

Keywords: GS_β-open sets, $GS_\beta O(X)$, G-GS_β-open set, β-Regular, β-Normal.

1. Introduction

Alo, R. A., and Shapiro, H. L. [1] defined the semi- normal spaces and considered some of their important properties. The authors further studied the compactness in grill topological spaces and inspected some of their properties. Chattopadhyay, K. C., and Thron, W. J. [2] introduced some special operators and studied some of their important properties. Further, the researchers evaluated the separation properties of extension, which are concerned with compact extension. Choquet in 1947 pioneered the grill concept [3]. It came out to be a very helpful device in investigating some topological ideas. Its use in the niches of proximity spaces and compactifications are reliable.

El-Deeb, N.et al [4, 5] introduced a different class of b-open sets referred to as Bc-open sets and considered a few of their important properties. Further, the authors also instigated and evaluated a new class of space called Bc-compact. Kalaivani, N et.al [6] introduced the GS_{β} open sets in grill topological spaces and studied certain vital properties of theirs. Levine, N., and Maki, H. [7, 8] presented a generalized closed set-in topology and premeditated some of its important properties. Mandal, D and Mukherjee, M. N [9] presented and evaluated another class of generalized closed set in a topological space X described as grill G on X. Further, crystal-clear characterizing such sets and their properties were done.

Data granulation serves as a vital decision-making tool in numerous real-life applications. Mashhour, A. S and Abd, M. E [10] introduced certain new topological tools employing rough set approximations for the purpose of data granulation. The authors also discussed certain topological measures for data granulation. Noiri, T [11] announced a different class of functions referred to as αg-closed set and studied some of its fundamental properties. The researcher proved some preservation theorems of normality, regularity and studied some of their generalizations.

Palaniappan, Y.*et.al* [12] established and analyzed another generalized semi closed sets class described as grill G on X. The authors also characterized those sets and their properties. Roy, B., Mukherjee, M. N., and Ghosh, S. K. [13,14-15] announced a class of sets in a topological space X, referred to as G-open, which makes up a subclass of the class of all preopen set of X. Further, the authors analyzed the favorable circumstances under which there would be a coincidence between the class of preopen sets and that of G-open sets.

Saif, A. *et al.* [16] found a different class of G_N pre-open sets in grill topological space and analyzed its important characteristics. Suliman, S. S., and Esmaeel, R. B. [17] introduced α-open sets in grill topological space. Additionally, the authors studied certain properties of this set and the relationship between them. Thron, W. J. [18] introduced a new concept of proximity space in the grill topological space and studied some of its basic properties. Vibin Salim Raj, S [19] presented and analyzed another class of contra- continuous function in the grill topological space. Woods, R. G. [20] studied the relationship between pairs of extension properties and their maximal extension. The author also studied and discussed several means of extension properties [21-24].

2. Objectives

Grill topology have been investigated by many researchers due to their wide spread applications in various fields such as artificial intelligence, information technology, robotics and so on. In this article a new concept of generalized GS_β-closed set in grill topological space is introduced with proper illustration. Many theorems based on generalized GS_β-closed set have been proved. In addition a new concepts of β-regular and β-normal space are studied and discussed. Furthermore, the theorems based on β-regular and β-normal space are derived and compared with GS_β-regular and normal space. In addition, the relationship connecting β-regular, β-normal, GS_β-regular and GS_β-normal space are studied and discussed with suitable illustrations.

3. Methods

Throughout this article, (Y, θ) or Y symbolizes a topological space that does not take into consideration any separation axiom, lest indicated differently. For a subset B of a space X, $int(B)$ and $cl(B)$, correspondingly represent the interior and closure of B. A union G of non-empty subsets in a space X is referred to as a grill on "X" if,

(1) $\emptyset \notin G$

(2) $A \in G$ along with $A \subseteq C \Rightarrow C \in G$

(3) $A, C \subseteq X$ along with $A \cup C \in G \ \Rightarrow A \in G$ or $C \in G$.

For every point x in a topological space (X, θ), $\theta(x)$, indicate the union of all open neighborhoods of x.

[The given statements describe a set G and its properties within a topological space (X, θ). Let's analyze each statement and its implications:

(1) $\emptyset \notin G$: This statement states that the empty set ($\emptyset$) is not an element of the set G. In other words, G does not contain the empty set as one of its elements.

(2) $A \in G$ along with $A \subseteq C \Rightarrow C \in G$: This statement describes a property of the set G. It states that if a set A is an element of G and A is a subset of another set C, then C is also an element of G. In other words, G is closed under the operation of taking supersets.

(3) $A, C \subseteq X$ along with $A \cup C \in G \Rightarrow A \in G$ or $C \in G$: This statement also describes a property of the set G. It states that if sets A and C are subsets of the topological space X, and their union $A \cup C$ is an element of G, then either A or C (or both) must be elements of G. In other words, G is closed under the operation of taking unions.]

Note : Here inside [ ] denotes the explanation of the definition.

If (Y, θ) is a topological space and G, a grill on Y, the mapping $\varphi: \wp(B) \longrightarrow \wp(B)$ indicates $\varphi_G(Y, \theta)$ for each $B \in \wp(B)$ where $\wp(B)$ indicates the unison of all feasible subsets of X. This is represented by $\varphi(B)$, the operator of the grill along with the topology τ. It is denoted by

$\varphi(B) = \{x \in X/B \cap U \in G,$ for every $U \in \theta(x)\}$.

If G is a grill on the space Y, a map $\omega: \wp(B) \to \wp(B)$ is denoted as $\omega(B) = \tau_G$-$cl(B) = B \cup \varphi(B)$, for every $B \in \wp(B)$, only a single topology exists. It is calculated by-

θ_G=$\{U \subseteq Y/\omega(Y\backslash U) = Y\backslash U\}$.

4. Results

Proposition 1

The following statements for a grill topological space (Y, θ, G) are obtained.

(a) All closed sets in Y are G-GS_β-closed sets.

(b) It is known that $A \notin G$ such that $\varphi(A) = \emptyset$. Thus, any non-member based on G is a G-GS_β-closed set.

Theorem 1

If G is a grill on a space (Y, θ), $B \subseteq X$ is a G-GS_β-open set. But this is the case only when $F \subseteq \tau_G$-$int(B)$, and in cases when $F \subseteq B$ and F is a β- closed set.

Theorem 2

For any subset B based on a space (Y, θ) and for a grill G on Y, the below-given statements are identical:

1) B is a G-GS_β-closed set.

2) $B \cup (Y\backslash\varphi(B))$ is a G-GS_β-closed set.

3) $\varphi(B)\backslash B$ is a G-GS_β-open set.

Theorem 3

In case G is a grill on a space (Y, θ), the below-given conditions are identical:

1) Y is a β-regular space.

2) Every closed set F and each $x \in Y\backslash F$, disjoint τ_G-GS_β-open sets U and V occur in a way that $x \in U$ and $F \subseteq V$.

3) Every GS_β-open set V based on (Y, θ) and every point $x \in V$, a τ_G-open set U becomes available such that $x \in U \subseteq \tau_G$-$cl(U) \subseteq V$.

Theorem 4

For a grill G on a space (Y, θ), the below-given conditions are identical:

1) Y is a GS_β-normal space.

2) For each pair based on disjoint GS_{β}-closed sets A and B, disjoint G-GS_{β} open sets U and V are available, in a way that $A \subseteq U$ and $B \subseteq V$.

3) For each GS_{β}-closed set A and any GS_{β}-open sets V composing F, a G-GS_{β} open set U becomes available so that $A \subseteq U \subseteq \tau_G$-$cl(U) \subseteq V$.

Theorem 5

If G is a grill on a T_1-space (Y, θ) such that $PO(Y)\backslash\{\emptyset\} \subseteq G$, then the below-given statements are identical:

a) Y is a GS_{β}- regular space.

b) Each GS_{β}-closed set A and each $x \in X\backslash A$, disjoint G-GS_{β}-open sets- U and V occur in a way that $x \in U$ and $A \subseteq V$.

c) Each GS_{β}-open set V based on (Y, θ) and each point $x \in V$, a G-GS_{β} open set U occurs in a way that $x \in U \subseteq \tau_G$-$cl(U) \subseteq V$.

5. Discussion

3. Generalized GS_{β}-closed set

Definition 3.1.

If (Y, θ) is a topological space with a grill G on Y, a subset B based on Y is denoted as the generalized GS_{β}-closed set with respect to the grill G(G-GS_{β}-closed set) in the case that $\varphi(B) \subseteq U$ in cases when $B \subseteq U$ along with U is GS_{β}-open set in Y.

Proposition 3.1

The following statements for a grill topological space (Y, θ, G) are obtained.

(a) All closed sets in Y are G-GS_{β}-closed sets.

(b) It is known that $A \notin G$ such that $\varphi(A) = \emptyset$. Thus, any non-member based on G is a G-GS_{β}-closed set.

Proof:

(a)$\Rightarrow$ $(\boldsymbol{b})$**:** Presume that A is a closed set in Y such that $cl(A) \subseteq U$, in cases when $A \subseteq U$ and U is a β-open set in Y. Then, for any subset A based on X, $\varphi(A) = \emptyset$.

(b)$\Rightarrow$ $(\boldsymbol{a})$**:** Presume that $A \notin G$, given a condition $\varphi(A) = \emptyset$.Then, there occurs a mapping $\varphi(A) \subseteq U$, in cases when $A \subseteq U$ and U is a GS_{β}-open set in Y. Hence, every closed set in Y is a G-GS_{β}-closed set.

Example 3.1. Accredit $X = \{a, b, c\}$, $\tau = \{\varphi, \{b\}, \{b, c\}, X\}$, $G = \{\{a\}, \{c\}, \{a, c\}, X\}$.

Then G-GS_{β}-$C(X) = \{\{a\}, \{c\}, \{a, c\}\}$.

Theorem 3.1

If (Y, θ) is a topological space and G is a grill on Y, for a subset B based on Y, the below-given affirmations are identical:

1) B is a G-GS_β closed set.

2) τ_G-$cl(B) \subseteq U$, in cases when $B \subseteq U$ and U is a β open set.

3) For every $x \in \tau_G$ -$cl(B)$, $cl\{x\} \cap B \neq \emptyset$

4) τ_G -$cl(B) \backslash B$ will not have any non-empty β-closed set based on (X, θ).

5) $\varphi(B) \backslash B$ has a non-empty β-closed set based on (X, θ).

Proof:

(1) $\Rightarrow$ $(\mathbf{2})$**:** Presume that B is a G-GS_β closed set along with $B \subseteq U$, in which U is a GS_β-open set in (Y, θ). Then $\varphi(B) \subseteq U$, so that $B \cup \varphi(B) \subseteq U$ along with τ_G-$cl(B) \subseteq U$.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$**:** Presume that there is a $x \in \tau_G$-$cl(A)$. If $cl(x) \cap B = \emptyset$, $B \subseteq X \backslash cl(x)$ is a conflict. That is because $x \in \tau_G$-$cl(B) \Rightarrow cl(x) \cap B = \emptyset$.

(3) $\Rightarrow$ $(\mathbf{4})$**:** Presume that F is a β-closed set based on (Y, θ) composition in τ_G -$cl(B) \backslash B$ along with $x \in F \Rightarrow F \cap B = \emptyset$. There is $cl(x) \cap B = \emptyset$. Contrarily, $cl(x) \cap B \neq \emptyset$ according to (3). Hence, $F = \emptyset$.

(4)$\Rightarrow$ $(\mathbf{5})$**:** It is derived from the equation τ_G-$cl(B) \backslash B = \varphi(B) \backslash B$.

(5)⇒(1): Presume that $B \subseteq U$ along with U is an β-open set in (Y, θ). As $\varphi(B)$ is a closed set and $\varphi(B) \cap (Y \backslash U) \subseteq \varphi(B) \backslash B$ is true. $\varphi(B) \cap (Y \backslash U)$ is a β-closed set in (Y, θ) composed in $\varphi(B) \backslash B$. So, $\varphi(B) \cap (Y \backslash U) = \emptyset$ and $\varphi(B) \subseteq U$. So, B is a G GS_β- closed set.

Preposition: 3.2

Assume a grill G on a space (Y, θ). If $B \subseteq X$ is a G-GS_β-closed set, the below-given statements are identical:

1) B is a τ_G-closed set.

2) τ_G -$cl(B) \backslash B$ is a β- closed set in (Y, θ).

3) $\varphi(B) \backslash B$ is a β- closed set in (Y, θ).

Proof:

Assume that B is a τ_G-closed set in (Y, θ). It is a fact that every closed set is a β-closed set. By definition of β-closed set, $cl(int(cl(B))) \subseteq B$. So, τ_G -$cl(B) \backslash B$ is a β- closed set in (Y, θ) also $\varphi(B) \backslash B$ is a β- closed set in (Y, θ).

Theorem: 3.2

If G is a grill on a space (Y,θ), $B \subseteq X$ is a G-GS_{β}-open set. But this is the case only when $F \subseteq \tau_G$-$int(B)$, and in cases when $F \subseteq B$ and F is a β- closed set.

Proof:

If B is an G-GS_{β} open set and $F \subseteq B$, in which F is a β- closed set in (Y,θ), then, $Y \backslash B \subseteq Y \backslash F$, in which $Y \backslash F$ is a β-open set. And, $Y \backslash B \subseteq Y \backslash F$. So, $\varphi(Y \backslash B) \subseteq Y \backslash F \Rightarrow \tau_G$-$cl(Y \backslash B) \subseteq Y \backslash F$. Hence, $F \subseteq \tau_G$-$int(B)$. Contradictorily, $Y \backslash B \subseteq U$, in which U is a β-open set in (Y,θ). So, $Y \backslash U$ is a β-closed set in (Y,θ). Hence, $Y \backslash U \subseteq B$, where $Y \backslash U = F$ is a β-closed set. $F \subseteq B \Rightarrow F \subseteq \tau_G$-$int(B)$, given B is a G-GS_{β}-open set. Therefore, $Y \backslash B$ is a G-GS_{β}-closed set.

Theorem: 3.3

Any grill G on a space (Y,θ) has the same set of conditions,

1) Every subset of Y is a G-GS_{β}-closed set.

2) Every β-open subset based on (Y,θ) is a τ_G -closed set.

Proof:

(1)$\Rightarrow$ **(2):** Presume that B is a β-open set in (Y,θ). Then by (1), B is a G-GS_{β}-closed set implies that $B \subseteq GS_{\beta}$-$O(Y)$, but B is a G-GS_{β}-closed set$\Rightarrow$ $\varphi(B) \subseteq B$. τ_G-$cl(B) = \varphi(B) \cup B = B$. Hence B is a τ_G-closed set.

(2)$\Rightarrow$ **(1):** Given an β-open subset based on (Y,θ) is τ_G-closed set. Accredit $B \subseteq X$ along with U being an β-open set in (Y,θ), in cases when $B \subseteq Y$. Now, as per (1), $\varphi(U) \subseteq U$. Again, $B \subseteq U \Rightarrow \varphi(B) \subseteq \varphi(U) \subseteq U \Rightarrow \varphi(B) \subseteq U$, in which U is a β-open set. So, B is a G-GS_{β}-closed set in (Y,θ).

Theorem: 3.4

For any subset B based on a space (Y,θ) and for a grill G on Y, the below-given statements are identical:

1) B is a G-GS_{β}-closed set.

2) $B \cup (Y \backslash \varphi(B))$ is a G-GS_{β}-closed set.

3) $\varphi(B) \backslash B$ is a G-GS_{β}-open set.

Proof:

(1)$\Rightarrow$**(2):** Presume that $B \cup (Y \backslash \varphi(B)) \subseteq U$, where U is an β-open set in Y. Then, $Y \backslash U \subseteq Y(B \cup \left(Y \backslash \varphi(B)\right) = \varphi(B) \backslash B$. Presume that B is a G-GS_{β}- closed set. Then, according to theorem 3.1, $Y\backslash U=\emptyset$. That is $Y=U$, presume that Y is the sole β-open set composing $(B\cup(Y\backslash\varphi(B))\Rightarrow(B\cup(Y\backslash\varphi(B)))$ is a G-GS_β-closed set.

(2)⇒ (1): Presume that $F\subseteq\varphi(B)\backslash B$ in which F is a β- closed set in (Y,θ). All at once $(B\cup(Y\backslash\varphi(B)))\subseteq Y\backslash F$ is a β-open set $\Rightarrow\varphi((B\cup(Y\backslash\varphi(B)))\subseteq Y\backslash F\Rightarrow F\subseteq Y\backslash$ $\varphi(\left(B\cup(Y\backslash\varphi(B))\right)\subseteq Y\backslash[\varphi(B)\cup\varphi(Y\backslash\varphi(B))]\subseteq[X\backslash\varphi(B)]\cup[Y\backslash\varphi(Y)\backslash\varphi(B)]\Rightarrow F\subseteq$ $Y\backslash\varphi(B)$. According to (1), $F\subseteq\varphi(B)$. But it is also true that $F\subseteq Y\backslash\varphi(B)$. So, there is a conflict. Therefore, $F=\emptyset$. And according to theorem 3.1, A is a G-GS_β-closed set.

(2)⇒ (3): It can be derived that $X(\varphi(B)\backslash B)=B\cup(Y\backslash\varphi(B))$

4. Characterization of β-Regular and β-Normal Spaces

This part of the paper shows the derivations of certain applications and characterizations of β-regular and β-normal spaces in relation to the novel idea of G-GS_β-closed set.

Definition 4.1

A space A is referred to as β-normal space if, for every pair of disjoint β-closed sets F along with K, disjoint open sets- U and V are available, in a way that $F\subseteq U$ and $K\subseteq V$.

Definition 4.2

A space A is referred to as β-regular if, for every β-closed set F and each $x\in X\backslash F$, disjoint open sets U and V are available, when $x\in U$ and $F\subseteq V$.

Theorem 4.1

For a grill G on a space (Y,θ), the below-given conditions are identical:

1) Y is a β-normal space.

2) For each pair based on disjoint β-closed sets A and B, disjoint G-GS_β open sets U and V are available, in a way that $A\subseteq U$ and $B\subseteq V$.

3) For each β closed set A and any β-open sets V composing F, a G-GS_β open set U becomes available so that $A\subseteq U\subseteq\tau_G$-$cl(U)\subseteq V$.

Proof

(1)⇒(2): Each open set is G-GS_β open set.

(2)⇒(3): In case A is a β-closed set and V is a β-open set in (Y,θ) in a way that $A\subseteq U$, then, A and $Y\backslash V$ are disjoint β-closed sets. According to (2), disjoint G-GS_β open sets U and W occur, in a way that $A\subseteq U$ and $Y\backslash V\subseteq W$. Therefore, W is a G-GS_β open set along with $Y\backslash V\subseteq W$, where $Y\backslash V$ is β closed set. According to theorem 3.2, $Y\backslash V\subseteq\theta_G$-$int(W)$ and

$Y\backslash\theta_G\text{-}int(W) \subseteq V$. Again, $U \cap W = \emptyset \Rightarrow U \cap \theta_G\text{-}int(W) = \emptyset$ and $\theta_G\text{-}cl(U) \subseteq X\backslash\theta_G\text{-}int(W) \subseteq V$. So, $A \subseteq U \subseteq \theta_G\text{-}cl(U) \subseteq V$, where U is a $G\text{-}GS_\beta$ open set.

(3)⇒(1): If A and B are any two disjoint β-closed sets in Y, as per (3), a $G\text{-}GS_\beta$-open set U exists such that $A \subseteq U \subseteq \tau_G\text{-}cl(U) \subseteq Y\backslash B$. Therefore U is a $G\text{-}GS_\beta$-open set along with $A \subseteq U$ where A is a β- closed set. Then, as per theorem 3.2, $A \subseteq \tau_G\text{-}int(U)$. Again, $PO(X)\backslash\{\emptyset\} \subseteq G \Rightarrow \tau_G \subseteq \tau^\alpha$ (by theorem 3.3 in [7]) $\Rightarrow \tau_G\text{-}int(U)$, $X\backslash\tau_G\text{-}cl(U) \in \tau^\alpha$. So $A \subseteq \tau_G\text{-}int(U) \subseteq int(cl(int(\tau_G\text{-}int(U))) = G$(say), along with $B \subseteq X\backslash\tau_G\text{-}cl(U) \subseteq int(cl(int(X\backslash\tau_G\text{-}cl(U))) = H$(say). In this, G and H are open sets.

For proving that (Y,θ) is a β- normal space, it suffices proving that $G \cap H = \emptyset$. Presume that $x \in G \cap H \Rightarrow x \in int(cl(int(\tau_G\text{-}int(U)))$ along with $x \in H \Rightarrow x \in cl(int(\tau_G\text{-}int(U)))$ and $x \in H \in \tau \Rightarrow$ there occurs a $y \in int(\tau_G\text{-}int(U)))$ and $y \in H \subseteq cl(int(X\backslash\tau_G\text{-}cl(U))) \Rightarrow int(\tau_G\text{-}int(U)) \cap int(X\backslash\tau_G\text{-}cl(U)) \neq \emptyset \Rightarrow \tau_G\text{-}int(U) \cap (X\backslash\tau_G\text{-}cl(U)) \neq \emptyset$, a conflict. So, we have $G \cap H = \emptyset$.

Theorem 4.2

In case G is a grill on a space (Y,θ), the below-given conditions are identical:

1) Y is a β-regular space.

2) Every closed set F and each $x \in Y\backslash F$, disjoint $\tau_G\text{-}GS_\beta$-open sets U and V occur in a way that $x \in U$ and $F \subseteq V$.

3) Every GS_β-open set V based on (Y,θ) and every point $x \in V$, a τ_G-open set U becomes available such that $x \in U \subseteq \tau_G\text{-}cl(U) \subseteq V$.

Proof

(1)⇒(2): It becomes evident that $\tau \subseteq \tau_G$.

(2)⇒(3): In case V is any open set in (Y,θ) which has a point x of X, then, by assumption, disjoint $\tau_G\text{-}GS_\beta$-open sets V and W occur in a way that $x \in U$ along with $Y\backslash V \subseteq W$. Now $U \cap W = \emptyset \Rightarrow \tau_G\text{-}cl(U) \subseteq Y\backslash W \subseteq V$. So $x \in U \subseteq \tau_G\text{-}cl(U) \subseteq V$.

(3)⇒(1): In case F is a closed set and $x \notin F$, then, as per the hypothesis, a τ_G- GS_β open set U becomes available in a way that $x \in U \subseteq \tau_G\text{-}cl(U) \subseteq Y\backslash F$.Accredit $V = Y\backslash\tau_G\text{-}cl(U)$. Then U and V are disjoint $\tau_G\text{-}GS_\beta$ open sets. As per theorem 3.3 in [7], there is $\tau_G \subseteq \tau^\alpha$. So, $x \in U \subseteq int\left(cl\big(int(U)\big)\right) = G$(say) along with $F \subseteq V \subseteq int\left(cl\big(int(V)\big)\right) = H$(say). As $U \cap V = \emptyset$ in theorem 3.4 in [7], it can be proved that $G \cap H = \emptyset$. Thus, it is a β- regular space.

Overall, the proof establishes the equivalence between the three statements and shows that the given conditions lead to a β-regular space.

Theorem 4.3

If G is a grill on a T_1-space (Y,θ) such that $PO(Y)\backslash\{\emptyset\} \subseteq G$, then the below-given statements are identical:

a) Y is a β- regular space.

b) Each β-closed set A and each $x \in X\backslash A$, disjoint τ_G-GS_β-open sets- U and V occur in a way that $x \in U$ and $A \subseteq V$.

c) Each β-open set V based on (Y,θ) and each point $x \in V$, a G-GS_β open set U occurs in a way that $x \in U \subseteq \tau_G$-$cl(U) \subseteq V$.

Proof

(a)⇒(b): Presume that Y is a β- regular space. By definition of β- regular space, for each β-closed set A and every $x \in X\backslash A$, disjoint τ_G-GS_β-open sets U and V occur, in a way that $x \in U$ and $A \subseteq V$.

(b)⇒(c): In case V is a β- open set in (Y,θ) and $x \in V$, then according to (b), a disjoint G-GS_β open sets U occurs along with W such that $x \in U$ along with $Y\backslash V \subseteq W$. As W is a G-GS_β-open set and $Y\backslash V$ is a β- closed set with $Y\backslash V \subseteq W$, it can be derived that $Y\backslash V \subseteq \tau_G$-$int(W)$. That is, $Y\backslash\tau_G$-$int(W) \subseteq V$. Now, $U \cap W = \emptyset \Rightarrow U \cap \tau_G$-$int(W) = \emptyset \Rightarrow \tau_G$-$cl(U) \subseteq Y\backslash\tau_G$-$int(W) \subseteq V$. So $x \in U \subseteq \tau_G$-$cl(U) \subseteq V$.

(c)⇒ $(\boldsymbol{a})$: If A is any β-closed set which does not have a point $x \in X$, then as per (c), a G-GS_β-open set U occurs such that $x \in U \subseteq \tau_G$-$cl(U) \subseteq X\backslash F$. Therefore, (Y,θ) is a T_1 space, and there is $\{x\} \subseteq \tau_G$-$int(U)$. Therefore, $PO(Y)\backslash\{\emptyset\} \subseteq G$, by Theorem 3.3 in [7], and τ_G-$int(U)$ and $Y\backslash\tau_G$-$cl(U)$ are β-open set. Now, $x \in \tau_G$-$int(U) \subseteq cl(int(cl)(\tau_G$-$int(U))) = G$(say). Also, $F \subseteq X\backslash\tau_G$-$cl(U) \subseteq cl(int(cl(X\backslash\tau_G$-$cl(U))) = H$. Therefore, G and H are disjoint β-open sets, and Y is a β- regular space.

5.Certain characterizations of GS_β-regular and GS_β-normal spaces

In this section we derive certain characterizations of GS_β-regular and GS_β-normal spaces which are derived in terms of the generalized GS_β-closed set.

Definition 5.1

A space A is said to be a GS_β-normal space if for each pair of disjoint GS_β-closed set F and K, there exists disjoint open sets U and V such that $F \subseteq U$ and $K \subseteq V$.

Definition 5.2

A space A is said to be GS_β-regular if, for every GS_β-closed set F and each $x \in X\backslash F$, there exists disjoint open sets U and V , such that $x \in U$ and $F \subseteq V$.

Theorem 5.1

For a grill G on a space (Y, θ), the below-given conditions are identical:

1) Y is a GS_β-normal space.

2) For each pair based on disjoint GS_β-closed sets A and B, disjoint G-GS_β open sets U and V are available, in a way that $A \subseteq U$ and $B \subseteq V$.

3) For each GS_β-closed set A and any GS_β-open sets V composing F, a G-GS_β open set U becomes available so that $A \subseteq U \subseteq \tau_G$-$cl(U) \subseteq V$.

Proof

(1)⇒(2): Each open set is G-GS_β open set.

(2)⇒(3): In case A is a GS_β-closed set and V is a GS_β-open set in (Y, θ) in a way that $A \subseteq U$, then, A and $Y \backslash V$ are disjoint GS_β-closed sets. According to (2), disjoint G-GS_β open sets U and W occur, in a way that $A \subseteq U$ and $Y \backslash V \subseteq W$. Therefore, W is a G-GS_β open set along with $Y \backslash V \subseteq W$, where $Y \backslash V$ is GS_β closed set. According to theorem 3.2, $Y \backslash V \subseteq \theta_G$-$int(W)$ and $Y \backslash \theta_G$-$int(W) \subseteq V$. Again, $U \cap W = \emptyset \Rightarrow U \cap \theta_G$-$int(W) = \emptyset$ and θ_G-$cl(U) \subseteq X \backslash \theta_G$-$int(W) \subseteq V$. So, $A \subseteq U \subseteq \theta_G$-$cl(U) \subseteq V$, where U is a G-GS_β open set.

(3)⇒(1): If A and B are any two disjoint GS_β-closed sets in Y, as per (3), a G-GS_β-open set U exists such that $A \subseteq U \subseteq \tau_G$-$cl(U) \subseteq Y \backslash B$. Therefore U is a G-GS_β-open set along with $A \subseteq U$ where A is a GS_β- closed set. Then, as per theorem 3.2, $A \subseteq \tau_G$-$int(U)$. Again, $PO(X) \backslash \{\emptyset\} \subseteq G \Rightarrow \tau_G \subseteq \tau^\alpha$ (by theorem 3.3 in [7]) $\Rightarrow \tau_G$-$int(U)$, $X \backslash \tau_G$-$cl(U) \in \tau^\alpha$. So $A \subseteq \tau_G$-$int(U) \subseteq int(cl(int(\tau_G\text{-}int(U)))) = G$(say), along with $B \subseteq X \backslash \tau_G$-$cl(U) \subseteq int(cl(int(X \backslash \tau_G\text{-}cl(U)))) = H$(say). In this, G and H are open sets.

For proving that (Y, θ) is a GS_β- normal space, it suffices proving that $G \cap H = \emptyset$. Presume that $x \in G \cap H \Rightarrow x \in int(cl(int(\tau_G\text{-}int(U))))$ along with $x \in H \Rightarrow x \in cl(int(\tau_G\text{-}int(U)))$ and $x \in H \in \tau \Rightarrow$ there occurs a $y \in int(\tau_G\text{-}int(U))$ and $y \in H \subseteq cl(int(X \backslash \tau_G\text{-}cl(U))) \Rightarrow int(\tau_G\text{-}int(U)) \cap int(X \backslash \tau_G\text{-}cl(U)) \neq \emptyset \Rightarrow \tau_G\text{-}int(U) \cap (X \backslash \tau_G\text{-}cl(U)) \neq \emptyset$, a conflict. So, we have $G \cap H = \emptyset$.

Theorem 5.2

In case G is a grill on a space (Y, θ), the below-given conditions are identical:

1) Y is a GS_β-regular space.

2) Every closed set F and each $x \in Y \backslash F$, disjoint G-GS_β-open sets U and V occur in a way that $x \in U$ and $F \subseteq V$.

3) Every GS_β-open set V based on (Y, θ) and every point $x \in V$, a τ_G-open set U becomes available such that $x \in U \subseteq \tau_G$-$cl(U) \subseteq V$.

Proof

(1)⇒(2): It becomes evident that $\tau \subseteq \tau_G$.

(2)⇒(3): In case V is any open set in (Y, θ) which has a point x of X, then, by assumption, disjoint τ_G-GS_β-open sets V and W occur in a way that $x \in U$ along with $Y \backslash V \subseteq W$. Now $U \cap W = \emptyset \Rightarrow \tau_G$-$cl(U) \subseteq Y \backslash W \subseteq V$. So $x \in U \subseteq \tau_G$-$cl(U) \subseteq V$.

(3)⇒(1): In case F is a closed set and $x \notin F$, then, as per the hypothesis, a τ_G- GS_β open set U becomes available in a way that $x \in U \subseteq \tau_G$-$cl(U) \subseteq Y \backslash F$.Accredit $V = Y \backslash \tau_G$-$cl(U)$. Then U and V are disjoint τ_G-GS_β open sets. As per theorem 3.3 in [7], there is $\tau_G \subseteq \tau^\alpha$. So, $x \in U \subseteq int\left(cl\big(int(U)\big)\right) = G$(say) along with $F \subseteq V \subseteq int\left(cl\big(int(V)\big)\right) = H$(say). As $U \cap V = \emptyset$ in theorem 3.4 in [7], it can be proved that $G \cap H = \emptyset$. Thus, it is a GS_β- regular space.

Theorem 5.3

If G is a grill on a T_1-space (Y, θ) such that $PO(Y) \backslash \{\emptyset\} \subseteq G$, then the below-given statements are identical:

a) Y is a GS_β- regular space.

b) Each GS_β-closed set A and each $x \in X \backslash A$, disjoint G-GS_β-open sets- U and V occur in a way that $x \in U$ and $A \subseteq V$.

c) Each GS_β-open set V based on (Y, θ) and each point $x \in V$, a G-GS_β open set U occurs in a way that $x \in U \subseteq \tau_G$-$cl(U) \subseteq V$.

Proof

(a)⇒(b): Presume that Y is a GS_β- regular space. By definition of GS_β- regular space, for each GS_β-closed set A and every $x \in X \backslash A$, disjoint G-GS_β-open sets U and V occur, in a way that $x \in U$ and $A \subseteq V$.

(b)⇒(c): In case V is a GS_β- open set in (Y, θ) and $x \in V$, then according to (b), a disjoint G-GS_β open sets U occurs along with W such that $x \in U$ along with $Y \backslash V \subseteq W$. As W is a G-GS_β-open set and $Y \backslash V$ is a GS_β- closed set with $Y \backslash V \subseteq W$, it can be derived that $Y \backslash V \subseteq \tau_G$-$int(W)$. That is, $Y \backslash \tau_G$-$int(W) \subseteq V$. Now, $U \cap W = \emptyset \Rightarrow U \cap \tau_G$-$int(W) = \emptyset \Rightarrow \tau_G$-$cl(U) \subseteq Y \backslash \tau_G$-$int(W) \subseteq V$. So $x \in U \subseteq \tau_G$-$cl(U) \subseteq V$.

(c)⇒ $(\boldsymbol{a})$: If A is any GS_β-closed set which does not have a point $x \in X$, then as per (c), a G-GS_β-open set U occurs such that $x \in U \subseteq \tau_G$-$cl(U) \subseteq X \backslash F$. Therefore, (Y, θ) is a T_1 space, and there is $\{x\} \subseteq \tau_G$-$int(U)$. Therefore, $PO(Y) \backslash \{\emptyset\} \subseteq G$, by Theorem 3.3 in [7], and τ_G-$int(U)$ and $Y \backslash \tau_G$-$cl(U)$ are GS_β-open set. Now, $x \in \tau_G$-$int(U) \subseteq cl(int(cl)(\tau_G$-$int(U))) = G$(say). Also, $F \subseteq X \backslash \tau_G$-$cl(U) \subseteq cl(int(cl(X \backslash \tau_G$-$cl(U))) = H$. Therefore, G and H are disjoint GS_β-open sets, and Y is a GS_β- regular space.

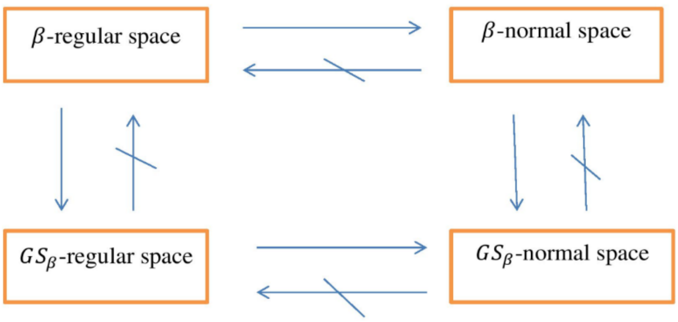

Figure 1: Relationship between β regular and normal space

Data availability

Data will be made available on request

Declaration of Competing Interest

The authors state that they do not have any known competing financial interests or personal relationships that could have influenced the work presented in this paper.

Credit authorship contribution statement

N Kalaivani: Conceptualization, Data analysis, Methodology, Writing of the manuscript.

K Fayaz Ur Rahman: Overall critical revision and final approval of the manuscript.

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