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Connections Between K- Hausdorffness in First Order Interval Valued Fuzzy and K- Hausdorffness in Second Order Interval Valued Fuzzy Topological Spaces

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Abstract:

In this study, the notion of second order interval valued fuzzy topological space is introduced with suitable examples. Connections between first order interval valued fuzzy topological spaces and second order interval valued fuzzy topological spaces are established. K-Hausdorff separation axiom are introduced in both first and second order interval valued fuzzy topological spaces. Further, studied the connections between K- Hausdorffness in first order interval valued fuzzy and K- Hausdorffness in second order interval valued fuzzy topological spaces.

Keywords: Interval valued fuzzy topology, second order interval valued fuzzy topology, interval valued fuzzy K-Hausdorff space, second order interval valued fuzzy K-Hausdorff space.

1.Introduction

Uncertainty is a constant problem when making decisions and cannot be completely avoided in any situation. In order to address the issues surrounding ambiguity, Zadeh [29] proposed the idea of a fuzzy set, which has a wide range of applications and is used to articulate imprecise and unclear concepts in everyday language. As an extension of fuzzy sets, Zadeh [30] presented the idea of interval valued fuzzy sets, in which membership degree values are intervals of numbers rather than integers. Compared to regular fuzzy sets, interval valued fuzzy sets offer a more accurate portrayal of uncertainty.

Chang [6] defined the fundamental terms, including open and closed fuzzy sets, neighborhood of a fuzzy set, interior fuzzy sets, fuzzy continuity, and fuzzy compactness. Chang [6] also introduced the idea of fuzzy topological spaces based on fuzzy sets of Zadeh [29]. In order to conduct a thorough analysis of the structures of fuzzy topological spaces, Lowen [19] modified Chang 's definition.

By adopting Chang's and Lowen's fuzzy topology definition, Mondal and Samanta [22] introduced the idea of interval valued fuzzy topology. Additional research was done on structural characteristics like the closure of an interval valued fuzzy set, the base and subbase of an interval valued fuzzy set, the interior of an interval valued fuzzy set, and the neighborhood of an interval valued fuzzy point. Interval type 2 fuzzy sets, also known as second order interval valued fuzzy sets, are a specific kind of type 2 fuzzy sets that were introduced by Liang and Mendel [18]

Chang and Lowen's first order fuzzy topology were extended to second order fuzzy topology by Kalaichelvi [11] using second order fuzzy sets. In the context of fuzzy topological spaces, various iterations of the Hausdorff separation axiom have been defined and examined by ([2], [8],[12],[14],[21],[24],[28]).Kalaichelvi ([12],[14],[16]) extended three distinct Hausdorff separation axioms in fuzzy topological spaces, which were first proposed by Gantner et al. [9] , Srivastava et al. [25] and Katsaras[16] to second order fuzzy topological spaces. These axioms were designated as W Hausdorff, S-Hausdorff axioms and K-Hausdorff axioms respectively.

We extend Mondal and Samantha's first order interval valued fuzzy topology to second order interval valued fuzzy topology using second order interval valued fuzzy sets and studied the connections between first order interval valued fuzzy and second order interval valued fuzzy topological spaces. In addition to studying some of its fundamental properties, the goal of this work is to develop first order interval valued fuzzy K-Hausdorff space and second order interval valued fuzzy K-Hausdorff in first order interval valued fuzzy and second order interval valued fuzzy topological spaces. Also, connections between K-Hausdorffness in first order interval valued fuzzy topological spaces and K-Hausdorffness in second order interval valued fuzzy topological spaces are established.

2.Preliminaries

In this section, fundamental definitions essential for deriving the main outcomes are presented.

Definition: 2.1 [29]

Let X be an arbitrary nonempty set. Let I = [0, 1]. A fuzzy set in X is a mapping from X into I that is a fuzzy set is an element of I^X.

Definition: 2.2 [6]

Let X be a nonempty set. A subset $\tau \subset I^X$ is called a fuzzy topology on X iff τ satisfies the following requirements:

(i) The constant fuzzy sets 0 and 1 belong to τ

(ii) $\mu_j \in \tau$ for each $j \in J$ implies $\vee_{j\in J} \mu_j \in \tau$

(iii) $\mu, \lambda \in \tau$ implies $\mu \wedge \lambda \in \tau$

The pair (X, τ) is called a fuzzy topological space.

Definition: 2.3 [19]

Let X be a nonempty set. A subset $\tau \subset I^X$ is called a fuzzy topology on X iff the following conditions are satisfied:

(i) All constant fuzzy sets belong to τ

(ii) $\mu_j \in \tau$ for each $j \in J$ implies $\vee_{j\in J} \mu_j \in \tau$

(iii) $\mu, \lambda \in \tau$ implies $\mu \wedge \lambda \in \tau$

Definition: 2.4 [10]

Let X be a nonempty set. A function $\hat{\mu} : X \to [I]$ is called an interval valued fuzzy set in X, where [I] is the set of all closed subintervals of [0, 1]. The collection of all interval valued fuzzy sets on X is denoted by IVF(X).

For every $\hat{\mu} \in$ IVF(X) and $x \in X$, $\hat{\mu}(x) = [\mu^-(x), \mu^+(x)]$ is called the degree of membership of an element x to $\hat{\mu}$, where $\mu^- : X \to I$ and $\mu^+ : X \to I$ are called a lower fuzzy set and an upper fuzzy set in X respectively. $\hat{\mu}$ can also be represented as $\hat{\mu} = [\mu^-, \mu^+]$

For any two-interval valued fuzzy sets $\hat{\mu}$, $\hat{\lambda}$ in IVF(X)

(i) $\hat{\mu} \mathrel{\hat{\subseteq}} \hat{\lambda}$ iff $\mu^-(x) \leq \lambda^-(x)$ and

$\mu^+(x) \leq \lambda^+(x)$ for every $x \in X$

(ii) $\hat{\mu} = \hat{\lambda}$ iff $\hat{\mu} \mathrel{\hat{\subseteq}} \hat{\lambda}$ and $\hat{\lambda} \mathrel{\hat{\subseteq}} \hat{\mu}$

(iii) The union $\hat{\mu} \mathbin{\hat{\cup}} \hat{\lambda}$ and intersection $\hat{\mu} \mathbin{\hat{\cap}} \hat{\lambda}$ are defined respectively as

$$\hat{\mu} \mathbin{\hat{\cup}} \hat{\lambda} = [\mu^- \vee \lambda^-, \mu^+ \vee \lambda^+]$$

$$\hat{\mu} \mathbin{\hat{\cap}} \hat{\lambda} = [\mu^- \wedge \lambda^-, \mu^+ \wedge \lambda^+]$$

(iv) The complement of $\hat{\mu}$, denoted by $\hat{\mu}^c$ is defined as $\hat{\mu}^c(x) = [1 - \mu^+(x), 1 - \mu^-(x)]$, for every $x \in X$

(v) For a family $\{\hat{\mu}_j / j \in J\}$ of interval valued fuzzy sets on a set X, the union $\hat{\bigcup}_{j \in J} \hat{\mu}_j$ and the intersection $\hat{\bigcap}_{j \in J} \hat{\mu}_j$ are defined, respectively, as

$$\hat{\bigcup}_{j \in J} \hat{\mu}_j = [\bigvee_{j \in J} (\mu_j^-), \bigvee_{j \in J} (\mu_j^+)]$$

$$\hat{\bigcap}_{j \in J} \hat{\mu}_j = [\bigwedge_{j \in J} (\mu_j^-), \bigwedge_{j \in J} (\mu_j^+)]$$

(vi) The constant interval valued fuzzy sets zero, one and alpha are denoted as $\hat{0}$, $\hat{1}$ and $\hat{\alpha}$ which are defined, respectively, by

$$\hat{0} = [\mathbf{0}, \mathbf{0}],\ \hat{1} = [\mathbf{1}, \mathbf{1}],\ \hat{\alpha} = [\boldsymbol{\alpha}, \boldsymbol{\alpha}]$$

Definition: 2.5[22]

Let X be a nonempty set and $\hat{\tau}$ be a collection of intervals valued fuzzy sets on X. The collection $\hat{\tau} \mathrel{\hat{\subseteq}}$ IVF(X) is called an interval valued fuzzy topology(Chang) on X iff $\hat{\tau}$ satisfies the following axioms:

(i) $\hat{0}$, $\hat{1}$ $\in$ $\hat{\tau}$

(ii) $\hat{\mu}$, $\hat{\lambda}$ $\in$ $\hat{\tau}$ implies $\hat{\mu} \hat{\cap} \hat{\lambda}$ $\in$ $\hat{\tau}$

(iii) $\hat{\mu}_j$ $\in$ $\hat{\tau}$ for each j $\in J$ implies ($\hat{\cup}_{j \in J} \hat{\mu}_j$) $\in$ $\hat{\tau}$

The pair (X, $\hat{\tau}$) is called an interval valued fuzzy topological space.

The family $\hat{\tau}$ is called an interval valued fuzzy topology (Lowen) on X iff $\hat{\tau}$ satisfies the following conditions:

(i) All constants $\hat{\alpha}$ $\in$ $\hat{\tau}$

(ii) $\hat{\mu}$, $\hat{\lambda}$ $\in$ $\hat{\tau}$ implies $\hat{\mu} \hat{\cap} \hat{\lambda}$ $\in$ $\hat{\tau}$

(iii) $\hat{\mu}_j$ $\in$ $\hat{\tau}$ for each j $\in J$ implies ($\hat{\cup}_{j \in J} \hat{\mu}_j$) $\in$ $\hat{\tau}$

The pair (X, $\hat{\tau}$) is called an interval valued fuzzy topological space.

The pair (X, τ) is called a fuzzy topological space. The elements in τ are called open fuzzy sets of the fuzzy topological space (X, τ). Complements of open sets are called closed fuzzy sets of (X, τ).

Definition: 2.6[18]

Let X be a nonempty set. A second order interval valued fuzzy set in X is a map $\hat{\hat{\mu}}$: X $\rightarrow$ $[I]^I$ where [I] is the set of all closed subintervals of [0, 1] and I is the closed unit interval [0, 1]. The family of second order interval valued fuzzy sets is denoted by SIVF(X).

For every $\hat{\hat{\mu}} \in$ SIVF(X), x $\in X$ and α $\in$ I $\hat{\hat{\mu}}$ (x) (α) = [$\hat{\mu}^-$ (x)(α), $\hat{\mu}^+$(x)(α)] is called the degree of membership of an element x to $\hat{\hat{\mu}}$, where $\hat{\mu}^-$: X $\rightarrow$ I^I and $\hat{\mu}^+$: X $\rightarrow$ I^I are called a lower second order fuzzy set and an upper second order fuzzy set in X respectively. Here $\hat{\hat{\mu}}$ is written as [$\hat{\mu}^-$,$\hat{\mu}^+$].

For any two second order interval valued fuzzy sets $\hat{\hat{\mu}}$, $\hat{\hat{\lambda}}$ in SIVF(X)

(i) $\hat{\hat{\mu}}$ $\hat{\hat{\subseteq}}$ $\hat{\hat{\lambda}}$ iff $\hat{\mu}^-$ (x) $\leq$ $\hat{\lambda}^-$(x)and $\hat{\mu}^+$ (x) $\leq$ $\hat{\lambda}^+$(x), for every x $\in X$

(i.e.) iff $\hat{\mu}^-$ (x) (α) $\leq$ $\hat{\lambda}^-$(x) (α) and $\hat{\mu}^+$ (x) (α) $\leq$ $\hat{\lambda}^+$(x) (α), for every x $\in X$ and for every α $\in$ I

(ii) $\hat{\hat{\mu}}$ = $\hat{\hat{\lambda}}$ iff $\hat{\hat{\mu}}$ $\hat{\hat{\subseteq}}$ $\hat{\hat{\lambda}}$ and $\hat{\hat{\lambda}}$ $\hat{\hat{\subseteq}}$ $\hat{\hat{\mu}}$

(iii)The union $\hat{\hat{\mu}}$ $\hat{\hat{\cup}}$ $\hat{\hat{\lambda}}$ and intersection $\hat{\hat{\mu}}$ $\hat{\hat{\cap}}$ $\hat{\hat{\lambda}}$ are defined respectively, by

$\hat{\hat{\mu}} \hat{\hat{\cup}} \hat{\hat{\lambda}} = [\hat{\mu}^- \vee \hat{\lambda}^-, \hat{\mu}^+ \vee \hat{\lambda}^+]$

$\hat{\hat{\mu}} \hat{\hat{\cap}} \hat{\hat{\lambda}} = [\hat{\mu}^- \wedge \hat{\lambda}^-, \hat{\mu}^+ \wedge \hat{\lambda}^+]$

(iv)For a family $\{\hat{\hat{\mu}}_j / j \in J\}$ of second order interval valued fuzzy sets on a set X, the union $\hat{\bigcup}_{j\in J} \hat{\hat{\mu}}_j$ and the intersection $\hat{\hat{\bigcap}}_{j\in J} \hat{\hat{\mu}}_j$ are defined, respectively, by

$$\hat{\bigcup}_{j\in J} \hat{\hat{\mu}}_j = [\bigvee_{j\in J} (\hat{\mu}_j^-), \bigvee_{j\in J} (\hat{\mu}_j^+)],$$

$$\hat{\hat{\bigcap}}_{j\in J} \hat{\hat{\mu}}_j = [\bigwedge_{j\in J} (\hat{\mu}_j^-), \bigwedge_{j\in J} (\hat{\mu}_j^+)]$$

(v) The constant second order interval valued fuzz sets zero and one are denoted as $\hat{\hat{0}}$ and $\hat{\hat{1}}$ which are defined, respectively, by $\hat{\hat{0}} = [\hat{0}, \hat{0}], \hat{\hat{1}} = [\hat{1}, \hat{1}]$

(vi) A constant second order interval valued fuzzy set alpha is denoted as $\hat{\hat{\alpha}}$ and is defined by $\hat{\hat{\alpha}} = [\hat{\alpha}, \hat{\alpha}]$

3. Second Order Interval Valued Fuzzy Topological Spaces

In this section, the notion of second order interval valued fuzzy topological space is introduced with suitable examples and its basic properties are discussed.

Definition: 3.1

Let X be a nonempty set and $\hat{\hat{\tau}}$ be a family of second order interval valued fuzzy sets on X. The family $\hat{\hat{\tau}}$ is called a second order interval valued fuzzy topology (Chang) on X iff $\hat{\hat{\tau}}$ satisfies the following axioms:

(i) $\hat{\hat{0}}, \hat{\hat{1}} \in \hat{\hat{\tau}}$

(ii) $\hat{\hat{\mu}}, \hat{\hat{\lambda}} \in \hat{\hat{\tau}}$ implies $\hat{\hat{\mu}} \hat{\hat{\cap}} \hat{\hat{\lambda}} \in \hat{\hat{\tau}}$

(iii) $\hat{\hat{\mu}}_j \in \hat{\hat{\tau}}$ for each $j \in J$ implies $(\hat{\bigcup}_{j\in J} \hat{\hat{\mu}}_j) \in \hat{\hat{\tau}}$

The pair (X, $\hat{\hat{\tau}}$) is called a second order interval valued fuzzy topological space (Chang).

A second order interval valued fuzzy topology (Lowen) on X is defined by replacing axiom (i), in the above definition by axiom (i)′

(i)′ All constant second order interval valued fuzzy sets belong to $\hat{\hat{\tau}}$

The pair (X, $\hat{\hat{\tau}}$) is called a second order interval valued fuzzy topological space (Lowen). The element of $\hat{\hat{\tau}}$ are called second order interval valued fuzzy open sets.

Example: 3.2

Let X be a nonempty set and n be a positive integer. Let $\hat{\hat{\tau}} = \{\hat{\hat{1}}\} \hat{\cup} \{\hat{\hat{\mu}} \in \text{SIVF}(X) / \hat{\mu}^-(x)(\alpha) = 0,$ $\hat{\mu}^+(x)(\alpha) = 0,$ for every $x \in X$ and for $\alpha \neq \frac{r}{n}$, r = 0, 1, . . . n}. Then $\hat{\hat{\tau}}$ is a second order interval valued fuzzy topology on X.

Proof:

(i) Since $\hat{0}(x)(\alpha) = 0,\ \hat{0}(x)(\alpha) = 0$, for every $x \in X$ and for every $\alpha \in I$ Therefore $\hat{\hat{0}} \in \hat{\hat{\tau}}$

(ii) $\hat{\hat{\mu}}_j \in \hat{\hat{\tau}}$ for $j \in J$

implies $\hat{\mu}_j^-(x)(\alpha) = 0,\ \hat{\mu}_j^+(x)(\alpha) = 0$, for $\alpha \neq \frac{r}{n}$, r = 0, 1, .. n, for every $x \in X$and for every $j \in J$.

Therefore $\vee_{j\in J}(\hat{\mu}_j^-(x)(\alpha)) = 0,\ \vee_{j\in J}(\hat{\mu}_j^+(x)(\alpha)) = 0$, for $\alpha \neq \frac{r}{n}$, r = 0, 1, . . . n, for every $x \in X$

implies $(\hat{\hat{\cup}}_{j\in J}\ \hat{\hat{\mu}}_j) \in \hat{\hat{\tau}}$

(iii) $\hat{\hat{\mu}}_i \in \hat{\hat{\tau}}$ for i = 1 to m

implies $\hat{\mu}_i^-(x)(\alpha) = 0,\ \hat{\mu}_i^+(x)(\alpha) = 0$, for $\alpha \neq \frac{r}{n}$, r = 0, 1, . . n, for every $x \in X$ and for i = 1 to m

Therefore $\wedge_{i=1}^{m}(\hat{\mu}_i^-(x)(\alpha)) = 0,\ \wedge_{i=1}^{m}(\hat{\mu}_i^+(x)(\alpha)) = 0$, for $\alpha \neq \frac{r}{n}$, r = 0, 1, 2, . . . n, for every $x \in X$

implies $(\hat{\hat{\cap}}_{i=1}^{m}\ \hat{\hat{\mu}}_i) \in \hat{\hat{\tau}}$

Therefore $\hat{\hat{\tau}}$ is a second order interval valued fuzzy topology on X.

Definition:3.3

For a second order interval valued fuzzy set $\hat{\hat{\mu}}$ in X, the complement $\hat{\hat{\mu}}$ is defined in two different ways:

(i) $(\hat{\hat{\mu}})^c(x)(\alpha) = [1 - \hat{\mu}^+(x)(\alpha),\ 1 - \hat{\mu}^-(x)(\alpha)]$

for every $x \in X$, for every $\alpha \in I$

$= [(\hat{\mu}^+)^c(x)(\alpha),\ (\hat{\mu}^-)^c(x)(\alpha)]$

for every $x \in X$, for every $\alpha \in I$

(ii) $(\hat{\hat{\mu}})_c(x)(\alpha) = [(\hat{\mu}^-)_c\ (x)(\alpha),\ (\hat{\mu}^+)_c(x)(\alpha)]$

for every x $\in X$, for every $\alpha \in$ I

$$= [\hat{\mu}^-(x)(1-\alpha),\ \hat{\mu}^+(x)(1-\alpha)]$$

for every x $\in X$, for every $\alpha \in$ I

The following example shows that $\hat{\hat{\mu}}$ and $(\hat{\hat{\mu}})_c$ are different:

Example: 3.4

Consider the closed unit interval I. For a fixed $\alpha \in$ I, define $(\hat{\hat{\mu}})_\alpha : I \to [I]^I$ such that $(\hat{\mu}^-)_\alpha(\beta)(\gamma) = \alpha\beta\gamma$, $(\hat{\mu}^+)_\alpha(\beta)(\gamma) = \alpha\beta\gamma$, for every $\beta, \gamma \in$ I (1)

Now $((\hat{\hat{\mu}})_\alpha)_c(\beta)(\gamma) = [(\hat{\mu}^-)_\alpha(\beta)(1-\gamma), (\hat{\mu}^+)_\alpha(\beta)(1-\gamma)]$, for every $\beta, \gamma \in$ I

$$= [\alpha\beta(1-\gamma), \alpha\beta(1-\gamma)],$$

for every $\beta, \gamma \in$ I (2)

Therefore (1) and (2) implies $((\hat{\hat{\mu}})_\alpha)_c \neq (\hat{\hat{\mu}})_\alpha$

Remark: 3.5

(i) $((\hat{\hat{\mu}})_c)_c = \hat{\hat{\mu}}$

(ii) $((\hat{\hat{\mu}})^c)^c = \hat{\hat{\mu}}$

Proof:

(i) By definition of $((\hat{\hat{\mu}})_c)_c$, for every x $\in X$, for every $\alpha \in$ I

$$((\hat{\hat{\mu}})_c)_c(x)(\alpha) = [(\hat{\mu}^-)_c(x)(1-\alpha), (\hat{\mu}^+)_c(x)(1-\alpha)]$$

$$= [\hat{\mu}^-(x)(1-(1-\alpha)),\ \hat{\mu}^+(x)(1-(1-\alpha))]$$

$$= \hat{\mu}^-(x)(\alpha),\ \hat{\mu}^+(x)(\alpha)]$$

$$= \hat{\hat{\mu}}(x)(\alpha),$$

Therefore $((\hat{\hat{\mu}})_c)_c = \hat{\hat{\mu}}$

(ii) By definition of $(\hat{\hat{\mu}})^c$, for every x $\in X$, for every $\alpha \in$ I

$$(\hat{\hat{\mu}})^c(x)(\alpha) = [1 - \hat{\mu}^+(x)(\alpha), 1 - \hat{\mu}^-(x)(\alpha)],$$

$$((\hat{\hat{\mu}})^c)^c(x)(\alpha) = [1-(1-\hat{\mu}^-(x)(\alpha)), 1-(1-\hat{\mu}^+(x)(\alpha))],$$

$$= [\hat{\mu}^-(x)(\alpha), \hat{\mu}^+(x)(\alpha)],$$

$$= \hat{\hat{\mu}}(x)(\alpha)$$

Therefore $((\hat{\hat{\mu}})^c)^c = \hat{\hat{\mu}}$

Example: 3.6

Let the closed unit interval I be a nonempty set. Let $\hat{\hat{\mu}}$, $\hat{\hat{\lambda}}$ be second order interval valued fuzzy sets in I defined as

$\hat{\mu}^-(\beta)(\alpha) = 0,\ 0 \le \beta \le \frac{1}{2},\quad 0 \le \alpha \le \frac{1}{2}$

$= (2\beta - 1)\alpha,\ \frac{1}{2} \le \beta \le 1,\ \frac{1}{2} \le \alpha \le 1$

$\hat{\mu}^+(\beta)(\alpha) = 0,\ 0 \le \beta \le \frac{1}{2},\quad 0 \le \alpha \le \frac{1}{2}$

$= (2\beta - 1)\alpha,\ \frac{1}{2} \le \beta \le 1,\ \frac{1}{2} \le \alpha \le 1$

$\hat{\lambda}^-(\beta)(\alpha) = \alpha,\quad 0 \le \beta \le \frac{1}{4},\ 0 \le \alpha \le \frac{1}{4}$

$= (-4\beta + 2)\alpha,\ \frac{1}{4} \le \beta \le \frac{1}{2},\ \frac{1}{4} \le \alpha \le \frac{1}{2}$

$= 0,\ \frac{1}{2} \le \beta \le 1, \frac{1}{2} \le \alpha \le 1$

$\hat{\lambda}^+(\beta)(\alpha) = \alpha,\quad 0 \le \beta \le \frac{1}{4},\ 0 \le \alpha \le \frac{1}{4}$

$= (-4\beta + 2)\alpha,\quad \frac{1}{4} \le \beta \le \frac{1}{2},\ \frac{1}{4} \le \alpha \le \frac{1}{2}$

$= 0, \frac{1}{2} \le \beta \le 1,\ \frac{1}{2} \le \alpha \le 1$

Then $\hat{\hat{\tau}} = \{\hat{\hat{0}}, \hat{\hat{1}}, \hat{\hat{\mu}}, \hat{\hat{\lambda}}, \hat{\hat{\mu}} \hat{\cup} \hat{\hat{\lambda}}\}$ is a second order interval valued fuzzy topology on I. The complements of $\hat{\hat{\mu}}$ and $\hat{\hat{\lambda}}$ can be obtained as

$(\hat{\mu}^-)_c(\beta)(\alpha) = (2\beta - 1)(1 - \alpha),\ \frac{1}{2} \le \beta \le 1, 0 \le \alpha \le \frac{1}{2}$

$= 0, 0 \le \beta \le \frac{1}{2},\quad \frac{1}{2} \le \alpha \le 1$

$(\hat{\mu}^+)_c(\beta)(\alpha) = (2\beta - 1)(1 - \alpha), \frac{1}{2} \le \beta \le 1, 0 \le \alpha \le \frac{1}{2}$

$= 0, 0 \le \beta \le \frac{1}{2},\quad \frac{1}{2} \le \alpha \le 1$

$(\hat{\lambda}^-)_c(\beta)(\alpha) = 0,\ \frac{1}{2} \le \beta \le 1, 0 \le \alpha \le \frac{1}{2}$

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$= (-4\beta + 2)(1-\alpha), \quad \frac{1}{4} \le \beta \le \frac{1}{2}, \quad \frac{1}{2} \le \alpha \le \frac{3}{4}$

$= 1-\alpha, \quad 0 \le \beta \le \frac{1}{4}, \quad \frac{3}{4} \le \alpha \le 1$

$(\hat{\lambda}^{+})_c\ (\beta)(\alpha) \ = \ 0, \ \frac{1}{2} \le \beta \le 1, 0 \le \alpha \le \frac{1}{2}$

$= (-4\beta + 2)(1-\alpha), \quad \frac{1}{4} \le \beta \le \frac{1}{2}, \quad \frac{1}{2} \le \alpha \le \frac{3}{4}$

$= 1-\alpha, \quad 0 \le \beta \le \frac{1}{4}, \quad \frac{3}{4} \le \alpha \le 1$

Then $(\hat{\hat{\tau}})_c = \{ (\hat{\hat{0}})_c, (\hat{\hat{1}})_c, (\hat{\hat{\mu}})_c, (\hat{\hat{\lambda}})_c, (\hat{\hat{\mu}} \ \hat{\cup} \ \hat{\hat{\lambda}})_c \}$ forms a second order interval valued fuzzy topology on I.

Here $\hat{\hat{\mu}} \ne (\hat{\hat{\mu}})_c$, $\hat{\hat{\lambda}} \ne (\hat{\hat{\lambda}})_c$ .Therefore $\hat{\hat{\tau}} \ne (\hat{\hat{\tau}})_c$

Definition: 3.7

For any two second order interval valued fuzzy sets $\hat{\hat{\mu}}, \hat{\hat{\lambda}}$ in a set X, where $\hat{\hat{\mu}} = [\hat{\mu}^-, \hat{\mu}^+]$ and $\hat{\hat{\lambda}} = [\hat{\lambda}^-, \hat{\lambda}^+]$

(i) $\hat{\hat{\mu}} \ \hat{\hat{\cap}}_1 \ \hat{\hat{\lambda}} = \hat{\hat{0}}$ means given $x \in X$, either $\hat{\mu}^-(x) = \mathbf{0}$ and $\hat{\mu}^+(x) = \mathbf{0}$ or $\hat{\lambda}^-(x) = \mathbf{0}$ and $\hat{\lambda}^+(x) = \mathbf{0}$

(ii) $\hat{\hat{\mu}} \ \hat{\hat{\cap}}_2 \ \hat{\hat{\lambda}} = \hat{\hat{0}}$ means given $x \in X$ and $\alpha \in I$,

either $\hat{\mu}^-(x)(\alpha) = 0$ and $\hat{\mu}^+(x)(\alpha) = 0$ or $\hat{\lambda}^-(x)(\alpha) = 0$ and $\hat{\lambda}^+(x)(\alpha) = 0$

Remark:3.8

If $\hat{\hat{\mu}} \ \hat{\hat{\cap}}_1 \ \hat{\hat{\lambda}} = \hat{\hat{0}}$ then $\hat{\hat{\mu}} \ \hat{\hat{\cap}}_2 \ \hat{\hat{\lambda}} = \hat{\hat{0}}$

Following example shows that $\hat{\hat{\mu}} \ \hat{\hat{\cap}}_2 \ \hat{\hat{\lambda}} = \hat{\hat{0}}$ but $\hat{\hat{\mu}} \ \hat{\hat{\cap}}_1 \ \hat{\hat{\lambda}} \ne \hat{\hat{0}}$

Example:3.9

Let X = {a, b}

Define $\hat{\mu}^-(a)(\alpha) = 0.5$,

$\hat{\mu}^+(a)(\alpha) = 0.5$, for every $\alpha \in I$

$\hat{\mu}^-(b)(\alpha) = 0.2$,

$\hat{\mu}^+(b)(\alpha) = 0.2$, for every $\alpha \in I$

$\hat{\lambda}^{-}$(a) (α) = 0.6, for α = 1

= 0, for $\alpha \neq 1$

$\hat{\lambda}^{+}$(a) (α) = 0.6, for α = 1

= 0, for $\alpha \neq 1$

$\hat{\lambda}^{-}$(b) (α) = 0.7

$\hat{\lambda}^{+}$(b) (α) = 0.7, for every $\alpha \in$ I

Here for, a $\in$ X and $\alpha \neq 1$

$\hat{\mu}^{-}$(a) (α) = 0.5, $\hat{\mu}^{+}$(a) (α) = 0.5 and

$\hat{\lambda}^{-}$(a) (α) = 0, $\hat{\lambda}^{+}$(a) (α) = 0

Therefore $\hat{\hat{\mu}} \;\hat{\hat{\cap}}_2\; \hat{\hat{\lambda}} = \hat{\hat{0}}$

For all x $\in X$

$\hat{\mu}^{-}$(x) $\neq$ **0**, $\hat{\mu}^{+}$(x) $\neq$ **0** or $\hat{\lambda}^{-}$(x) $\neq$ **0**, $\hat{\lambda}^{+}$(x) $\neq$ **0**

Therefore $\hat{\hat{\mu}} \;\hat{\hat{\cap}}_1\; \hat{\hat{\lambda}} \neq \hat{\hat{0}}$

4. Connections Between First Order Interval Valued Fuzzy and Second Order Interval Valued Fuzzy Topological Spaces

In this section, connections between first order interval valued fuzzy and second order interval valued fuzzy topological spaces are established.

Theorem: 4.1

Let X be a nonempty set. Then every first order interval valued fuzzy topology (Lowen) $\hat{\tau} = \{\hat{\mu}_j / j \in J\}$ on X defines a second order interval valued fuzzy topology (Lowen) $\hat{\hat{\tau}} = \{\hat{\hat{\mu}}_j / \hat{\mu}_j \in \hat{\tau}\}$ on X, where $\hat{\mu}_j^{-}$(x) (α) = μ_j^{-}(x) and $\hat{\mu}_j^{+}$(x) (α) = μ_j^{+}(x), for every x $\in X$ and for every $\alpha \in$ I. C_I denotes the correspondence $\hat{\tau} \rightarrow \hat{\hat{\tau}}$

Proof:

To prove $\hat{\hat{\tau}}$ is a second order interval valued fuzzy topology (Lowen) on X. By the definition of $\hat{\hat{\mu}}_j$, the correspondence $\hat{\mu}_j \rightarrow \hat{\hat{\mu}}_j$ is one – one.

(i) Since $\hat{0}$, $\hat{1}$, $\hat{\alpha} \in \hat{\tau}$, $\hat{\hat{0}}$, $\hat{\hat{1}}$, $\hat{\hat{\alpha}} \in \hat{\hat{\tau}}$

(ii) To prove $\hat{\hat{\tau}}$ is closed with respect to arbitrary union

Given $\hat{\hat{\mu}}_j \in \hat{\hat{\tau}}$ for j $\in J_0 \subseteq$ J

To prove: $\hat{\widehat{\bigcup}}_{j\in J_0} \hat{\hat{\mu}}_j \in \hat{\hat{\tau}}$

$\hat{\hat{\mu}}_j \in \hat{\hat{\tau}}$ for $j \in J_0 \subseteq J$

implies $\widehat{\mu}_j \in \widehat{\tau}$, for $j \in J_0 \subseteq J$

implies $\widehat{\bigcup}_{j\in J_0} \widehat{\mu}_j \in \widehat{\tau}$

implies $\widehat{(\widehat{\bigcup}_{j\in J_0} \widehat{\mu}_j)} \in \hat{\hat{\tau}}$

Let $\widehat{(\widehat{\bigcup}_{j\in J_0} \widehat{\mu}_j)} = [\widehat{\vee_{j\in J_0} \mu_j^-}, \widehat{\vee_{j\in J_0} \mu_j^+}]$

Now

$(\hat{\widehat{\bigcup}}_{j\in J_0} \hat{\hat{\mu}}_j)(x)(\alpha) = [(\vee_{j\in J_0} \hat{\mu}_j^-)(x)(\alpha), (\vee_{j\in J_0} \hat{\mu}_j^+)(x)(\alpha)],$

for every $x \in X$, for every $\alpha \in I$

$= [\vee_{j\in J_0} (\hat{\mu}_j^-(x)(\alpha)), \vee_{j\in J_0} (\hat{\mu}_j^+(x)(\alpha))],$

for every $x \in X$, for every $\alpha \in I$

$= [\vee_{j\in J_0} (\mu_j^-(x)), \vee_{j\in J_0} (\mu_j^+(x))]$, for every $x \in X$

$= [(\vee_{j\in J_0} \mu_j^-)(x), (\vee_{j\in J_0} \mu_j^+)(x)]$, for every $x \in X$

$= [\widehat{(\vee_{j\in J_0} \mu_j^-)}(x)(\alpha), \widehat{(\vee_{j\in J_0} \mu_j^+)}(x)(\alpha)],$

for every $x \in X$, for every $\alpha \in I$

$= \widehat{(\widehat{\bigcup}_{j\in J_0} \widehat{\mu}_j)}(x)(\alpha)$, for every $x \in X$, for every $\alpha \in I$

Therefore $\hat{\widehat{\bigcup}}_{j\in J_0} \hat{\hat{\mu}}_j \ \widehat{(\widehat{\bigcup}_{j\in J_0} \widehat{\mu}_j)} \in \widehat{\tau}$

(i) To prove $\hat{\hat{\tau}}$ is closed with respect to finite intersection.

Given $\hat{\hat{\mu}}_i \in \hat{\hat{\tau}}$ for i = 1 to m

To prove: $\hat{\widehat{\bigcap}}_{i=1}^{m} \hat{\hat{\mu}}_i \in \hat{\hat{\tau}}$

$\hat{\hat{\mu}}_i \in \hat{\hat{\tau}}$, for i = 1 to m

implies $\widehat{\mu}_i \in \widehat{\tau}$, for i = 1 to m implies $\hat{\bigcap}_{i=1}^{m} \hat{\mu}_i \in \hat{\tau}$

implies $\widehat{\hat{\bigcap}_{i=1}^{m} \hat{\mu}_i} \in \hat{\hat{\tau}}$

Let

Now $\widehat{\hat{\bigcap}_{i=1}^{m} \hat{\mu}_i} = [\widehat{\bigwedge_{i=1}^{m} \mu_i}^{-}, \widehat{\bigwedge_{i=1}^{m} \mu_i}^{+}]$

$$(\hat{\hat{\bigcap}}_{i=1}^{m} \hat{\hat{\mu}}_i)(x)(\alpha) = [(\bigwedge_{i=1}^{m} \hat{\mu}_i^{-})(x)(\alpha), (\bigwedge_{i=1}^{m} \hat{\mu}_i^{+})(x)(\alpha)], \text{ for every } x \in X, \text{ for every } \alpha \in I$$

$$= [\bigwedge_{i=1}^{m} (\hat{\mu}_i^{-}(x)(\alpha)), \bigwedge_{i=1}^{m} (\hat{\mu}_i^{+}(x)(\alpha))], \text{ for every } x \in X, \text{ for every } \alpha \in I$$

$$= [\bigwedge_{i=1}^{m} (\mu_i^{-}(x)), \bigwedge_{i=1}^{m} (\mu_i^{+}(x))], \text{ for every } x \in X$$

$$= [(\bigwedge_{i=1}^{m} \mu_i^{-})(x), (\bigwedge_{i=1}^{m} \mu_i^{+})(x)], \text{ for every } x \in X$$

$$= [(\widehat{\bigwedge_{i=1}^{m} \mu_i}^{-})(x)(\alpha), (\widehat{\bigwedge_{i=1}^{m} \mu_i}^{+})(x)(\alpha)], \text{ for every } x \in X, \text{ for every } \alpha \in I$$

$$= \widehat{(\hat{\bigcap}_{i=1}^{m} \hat{\mu}_i)}(x)(\alpha), \text{ for every } x \in X, \text{ for every } \alpha \in I$$

Therefore $\hat{\hat{\bigcap}}_{i=1}^{m} \hat{\hat{\mu}}_i = \widehat{\hat{\bigcap}_{i=1}^{m} \hat{\mu}_i} \in \hat{\hat{\tau}}$

Therefore $\hat{\hat{\tau}}$ is a second order interval valued fuzzy topology (Lowen) on X.

Theorem :4.2

Let X be a nonempty set. Let $\hat{\hat{\tau}} = \{\hat{\hat{\mu}}_j / j \in J\}$ be a second order interval valued fuzzy topology (Lowen) on X. Fix $\alpha \in I$. Then the collection $(\hat{\hat{\tau}})_\alpha$ = distinct elements of the collection $\{(\hat{\hat{\mu}}_j)_\alpha / \hat{\hat{\mu}}_j \in \hat{\hat{\tau}}\}$ defines a first order interval valued fuzzy topology (Lowen) on X, where $(\hat{\mu}_j^{-})_\alpha : X \to I$, $(\hat{\mu}_j^{+})_\alpha : X \to I$ such that $(\hat{\mu}_j^{-})_\alpha(x) = \hat{\mu}_j^{-}(x)(\alpha)$, $(\hat{\mu}_j^{+})_\alpha(x) = \hat{\mu}_j^{+}(x)(\alpha)$, for every $x \in X$, for every $\alpha \in I$. The correspondence $\hat{\hat{\tau}} \to (\hat{\hat{\tau}})_\alpha$ is denoted as C_2.

Proof:

To prove $(\hat{\hat{\tau}})_\alpha$ is a first order interval valued fuzzy topology (Lowen) on X. By the definition of $(\hat{\hat{\tau}})_\alpha$, there exists $J_0 \subseteq J$ such that for $j \neq k$, $j, k \in J_0$, $(\hat{\hat{\mu}}_j)_\alpha \neq (\hat{\hat{\mu}}_k)_\alpha$ and $(\hat{\hat{\tau}})_\alpha$ can be written as $(\hat{\hat{\tau}})_\alpha = \{(\hat{\hat{\mu}}_j)_\alpha / j \in J_0\}$.

(i) Consider $\hat{\hat{0}}, \hat{\hat{1}}, \hat{\hat{\beta}} \in \hat{\hat{\tau}}$

Now $(\hat{\hat{0}})_\alpha (x) = [(\hat{0})_\alpha (x), (\hat{0})_\alpha (x)]$, for every $x \in X$

$= [\hat{0}(x)(\alpha), \hat{0}(x)(\alpha)]$, for every $x \in X$ and for every $\alpha \in I$

$= [0, 0]$

Therefore $(\hat{\hat{0}})_\alpha = \mathbf{[0, 0]} = \hat{0} \in (\hat{\hat{\tau}})_\alpha$

Similarly for $\hat{\hat{1}}, \hat{\hat{\beta}} \in \hat{\hat{\tau}}$, $(\hat{\hat{1}})_\alpha, (\hat{\hat{\beta}})_\alpha \in (\hat{\hat{\tau}})_\alpha$

(ii) To prove $(\hat{\hat{\tau}})_\alpha$ is closed with respect to arbitrary union.

Given $(\hat{\hat{\mu}}_j)_\alpha \in (\hat{\hat{\tau}})_\alpha$, for $j \in J_0 \subseteq J$

To prove $\underset{j \in J_0}{\cup} (\hat{\hat{\mu}}_j)_\alpha \in (\hat{\hat{\tau}})_\alpha$

$(\hat{\hat{\mu}}_j)_\alpha \in (\hat{\hat{\tau}})_\alpha$

implies $\hat{\hat{\mu}}_j \in \hat{\hat{\tau}}$ such that $\hat{\mu}_j^-(x)(\alpha) = (\hat{\mu}_j^-)_\alpha (x)$, $\hat{\mu}_j^+(x)(\alpha) = (\hat{\mu}_j^+)_\alpha (x)$, for every $x \in X$, for every $\alpha \in I$

Now $(\underset{j \in J_0}{\hat{\cup}} \hat{\hat{\mu}}_j) \in \hat{\hat{\tau}}$

implies $(\underset{j \in J_0}{\hat{\cup}} \hat{\hat{\mu}}_j)_\alpha \in (\hat{\hat{\tau}})_\alpha$

Consider

$(\hat{\hat{\mu}}_j)_\alpha = [(\hat{\mu}_j^-)_\alpha, (\hat{\mu}_j^+)_\alpha]$

$(\hat{\hat{\mu}}_j)_\alpha (x) = [(\hat{\mu}_j^-)_\alpha (x), (\hat{\mu}_j^+)_\alpha (x)]$, for every $x \in X$

$\underset{j \in J_0}{\cup} (\hat{\hat{\mu}}_j)_\alpha (x) = [\underset{j \in J_0}{\vee} (\hat{\mu}_j^-)_\alpha (x), \underset{j \in J_0}{\vee} (\hat{\mu}_j^+)_\alpha (x)]$, for every $x \in X$

$= [\underset{j \in J_0}{\vee} \hat{\mu}_j^-(x)(\alpha), \underset{j \in J_0}{\vee} \hat{\mu}_j^+(x)(\alpha)]$, for every $x \in X$, for every $\alpha \in I$

$= [(\underset{j \in J_0}{\vee} \hat{\mu}_j^-)(x)(\alpha), (\underset{j \in J_0}{\vee} \hat{\mu}_j^+)(x)(\alpha)]$, for every $x \in X$, for every $\alpha \in I$

$= (\underset{j \in J_0}{\hat{\cup}} \hat{\hat{\mu}}_j)(x)(\alpha)$, for every $x \in X$, for every $\alpha \in I$

$= (\underset{j \in J_0}{\hat{\cup}} \hat{\hat{\mu}}_j)_\alpha (x)$, for every $x \in X$

Therefore $\underset{j\in J_0}{\hat{\cup}} (\hat{\hat{\mu}}_j)_\alpha \in (\hat{\hat{\tau}})_\alpha$

(iii) To prove $(\hat{\hat{\tau}})_\alpha$ is closed with respect to finite intersection.

Given $(\hat{\hat{\mu}}_i)_\alpha \in (\hat{\hat{\tau}})_\alpha$, for i = 1 to m

To prove $\overset{m}{\underset{i=1}{\hat{\cap}}} (\hat{\hat{\mu}}_i)_\alpha \in (\hat{\hat{\tau}})_\alpha$

$(\hat{\hat{\mu}}_i)_\alpha \in (\hat{\hat{\tau}})_\alpha$

implies $\hat{\hat{\mu}}_i \in \hat{\hat{\tau}}$ such that $(\hat{\mu}_i^-)(x)(\alpha) = (\hat{\mu}_i^-)_\alpha(x)$, $(\hat{\mu}_i^+)(x)(\alpha) = (\hat{\mu}_i^+)_\alpha(x)$, for every $x \in X$, for every $\alpha \in I$.

Now $(\overset{m}{\underset{i=1}{\hat{\hat{\cap}}}} \hat{\hat{\mu}}_i) \in \hat{\hat{\tau}}$

implies $(\overset{m}{\underset{i=1}{\hat{\hat{\cap}}}} \hat{\hat{\mu}}_i)_\alpha \in (\hat{\hat{\tau}})_\alpha$

Consider

$$(\hat{\hat{\mu}}_i)_\alpha = [(\hat{\mu}_i^-)_\alpha, (\hat{\mu}_i^+)_\alpha]$$

$$(\hat{\hat{\mu}}_i)_\alpha(x) = [(\hat{\mu}_i^-)_\alpha(x), (\hat{\mu}_i^+)_\alpha(x)], \text{ for every } x \in X$$

$$\overset{m}{\underset{i=1}{\hat{\cap}}} (\hat{\hat{\mu}}_i)_\alpha(x) = [\overset{m}{\underset{i=1}{\wedge}} (\hat{\mu}_i^-)_\alpha(x), \overset{m}{\underset{i=1}{\wedge}} (\hat{\mu}_i^+)_\alpha(x)], \text{ for every } x \in X$$

$$= [\overset{m}{\underset{i=1}{\wedge}} \hat{\mu}_i^-(x)(\alpha), \overset{m}{\underset{i=1}{\wedge}} \hat{\mu}_i^+(x)(\alpha)], \text{ for every } x \in X, \text{ for every } \alpha \in I$$

$$= [(\overset{m}{\underset{i=1}{\wedge}} \hat{\mu}_i^-)(x)(\alpha), (\overset{m}{\underset{i=1}{\wedge}} \hat{\mu}_i^+)(x)(\alpha)], \text{ for every } x\in X, \text{ for every } \alpha\in I$$

$$= (\overset{m}{\underset{i=1}{\hat{\hat{\cap}}}} \hat{\hat{\mu}}_i)_\alpha(x), \text{ for every } x \in X$$

Therefore $\overset{m}{\underset{i=1}{\hat{\cap}}} (\hat{\hat{\mu}}_i)_\alpha \in (\hat{\hat{\tau}})_\alpha$

Therefore $(\hat{\hat{\tau}})_\alpha$ is a first order interval valued fuzzy topology (Lowen) on X.

Theorem: 4.3

Let X be a nonempty set. Let $\hat{\hat{\tau}} = \{(\hat{\hat{\mu}}_j) / j \in J\}$ be a second order interval valued fuzzy topology (Lowen) on X. Then the collection $(\hat{\hat{\tau}})_c = \{(\hat{\hat{\mu}}_j)_c / \hat{\hat{\mu}}_j \in \hat{\hat{\tau}}\}$ is also a second order interval valued fuzzy topology (Lowen) on X. The correspondence $\hat{\hat{\tau}} \to (\hat{\hat{\tau}})_c$ is denoted as C_3.

Proof:

By the definition of $(\hat{\hat{\tau}})_c$, there exists $J_0 \subseteq J$ such that $j \neq k$, $j, k \in J_0$, $(\hat{\hat{\mu}}_j)_c \neq (\hat{\hat{\mu}}_k)_c$ and $(\hat{\hat{\tau}})_c$ can be written as $(\hat{\hat{\tau}})_c = \{(\hat{\hat{\mu}}_j)_c \,/\, j \in J_0\}$

(i) $\hat{\hat{0}}, \hat{\hat{1}}, \hat{\hat{\alpha}} \in \hat{\hat{\tau}}$

implies $(\hat{\hat{0}})_c, (\hat{\hat{1}})_c, (\hat{\hat{\alpha}})_c \in (\hat{\hat{\tau}})_c$

(ii) To prove $(\hat{\hat{\tau}})_c$ is closed with respect to arbitrary union.

Given $(\hat{\hat{\mu}}_j)_c \in (\hat{\hat{\tau}})_c$ for $j \in J_0 \subseteq J$

To prove $\underset{j \in J_0}{\hat{\cup}} (\hat{\hat{\mu}}_j)_c \in (\hat{\hat{\tau}})_c$

Since $(\hat{\hat{\mu}}_j)_c \in (\hat{\hat{\tau}})_c$ implies $\hat{\hat{\mu}}_j \in \hat{\hat{\tau}}$

implies $\underset{j \in J_0}{\hat{\cup}} \hat{\hat{\mu}}_j \in \hat{\hat{\tau}}$

implies $(\underset{j \in J_0}{\hat{\cup}} \hat{\hat{\mu}}_j)_c \in (\hat{\hat{\tau}})_c$

Consider

$$(\underset{j \in J_0}{\hat{\cup}} (\hat{\hat{\mu}}_j)_c)(x)(\alpha) = [(\underset{j \in J_0}{\vee} (\hat{\mu}_j^-)_c)(x)(\alpha), (\underset{j \in J_0}{\vee} (\hat{\mu}_j^+)_c)(x)(\alpha)],$$

for every $x \in X$, for every $\alpha \in I$

$$= [\underset{j \in J_0}{\vee} ((\hat{\mu}_j^-)_c (x)(\alpha)), \underset{j \in J_0}{\vee} ((\hat{\mu}_j^+)_c (x)(\alpha))],$$

for every $x \in X$, for every $\alpha \in I$

$$= [\underset{j \in J_0}{\vee} (\hat{\mu}_j^- (x)(1-\alpha)), \underset{j \in J_0}{\vee} (\hat{\mu}_j^+ (x)(1-\alpha)],$$

for every $x \in X$, for every $\alpha \in I$

$$= [(\underset{j \in J_0}{\vee} \hat{\mu}_j^-)(x)(1-\alpha), (\underset{j \in J_0}{\vee} \hat{\mu}_j^+)(x)(1-\alpha)],$$

for every $x \in X$, for every $\alpha \in I$

$$= [(\underset{j \in J_0}{\vee} \hat{\mu}_j^-)_c(x)(\alpha), (\underset{j \in J_0}{\vee} \hat{\mu}_j^+)_c(x)(\alpha)],$$

for every $x \in X$, for every $\alpha \in I$

$$= (\underset{j \in J_0}{\hat{\cup}} \hat{\hat{\mu}}_j)_c(x)(\alpha), \text{ for every } x \in X, \text{ for every } \alpha \in I$$

Therefore $\hat{\bigcup}_{j \in J_0} (\hat{\hat{\mu}}_j)_c = (\hat{\bigcup}_{j \in J_0} \hat{\hat{\mu}}_j)_c \in (\hat{\hat{\tau}})_c$

(iii) To prove $(\hat{\hat{\tau}})_c$ is closed with respect to finite intersection.

Consider $(\hat{\hat{\mu}}_i)_c \in (\hat{\hat{\tau}})_c$, for i = 1 to m

To prove $\hat{\hat{\bigcap}}_{i=1}^{m} (\hat{\hat{\mu}}_i)_c \in (\hat{\hat{\tau}})_c$

$(\hat{\hat{\mu}}_i)_c \in (\hat{\hat{\tau}})_c$

implies $(\hat{\hat{\mu}}_i) \in \hat{\hat{\tau}}$

implies $\hat{\hat{\bigcap}}_{i=1}^{m} \hat{\hat{\mu}}_i \in \hat{\hat{\tau}}$

implies $(\hat{\hat{\bigcap}}_{i=1}^{m} \hat{\hat{\mu}}_i)_c \in (\hat{\hat{\tau}})_c$

Consider

$$(\hat{\hat{\bigcap}}_{i=1}^{m} (\hat{\hat{\mu}}_i)_c)(x)(\alpha) = [(\bigwedge_{i=1}^{m} (\hat{\mu}_i^-)_c)(x)(\alpha), (\bigwedge_{i=1}^{m} (\hat{\mu}_i^+)_c)(x)(\alpha)],$$

for every $x \in X$, for every $\alpha \in I$

$$= [\bigwedge_{i=1}^{m} ((\hat{\mu}_i^-)_c(x)(\alpha)), \bigwedge_{i=1}^{m} ((\hat{\mu}_i^+)_c(x)(\alpha))],$$

for every $x \in X$, for every $\alpha \in I$

$$= [\bigwedge_{i=1}^{m} (\hat{\mu}_i^-(x)(1-\alpha)), \bigwedge_{i=1}^{m} (\hat{\mu}_i^+(x)(1-\alpha))],$$

for every $x \in X$, for every $\alpha \in I$

$$= [(\bigwedge_{i=1}^{m} \hat{\mu}_i^-)(x)(1-\alpha), (\bigwedge_{i=1}^{m} \hat{\mu}_i^+)(x)(1-\alpha)],$$

for every $x \in X$, for every $\alpha \in I$

$$= [(\bigwedge_{i=1}^{m} \hat{\mu}_i^-)_c(x)(\alpha), (\bigwedge_{i=1}^{m} \hat{\mu}_i^+)_c(x)(\alpha)],$$

for every $x \in X$, for every $\alpha \in I$

$$= (\hat{\hat{\bigcap}}_{i=1}^{m} \hat{\hat{\mu}}_i)_c(x)(\alpha), \text{ for every } x \in X, \text{ for every } \alpha \in I$$

Therefore $\mathop{\hat{\hat{\cap}}}_{i=1}^{m} (\hat{\hat{\mu}}_i)_c = (\mathop{\hat{\hat{\cap}}}_{i=1}^{m} \hat{\hat{\mu}}_i)_c \in (\hat{\hat{\tau}})_c$

Therefore $(\hat{\hat{\tau}})_c$ is a second order interval valued fuzzy topology (Lowen) on X.

Theorem: 4.4

If the second order interval valued fuzzy topology $\hat{\hat{\tau}}$ on X is got from the first order interval valued fuzzy topology $\hat{\tau}$ on X through the association C_I, then $\hat{\hat{\tau}} = (\hat{\hat{\tau}})_c$

Proof:

$$(\hat{\hat{\mu}}_j)_c\ (x)\ (\alpha) = [(\hat{\mu}_j^-)_c\ (x)(\alpha),\ (\hat{\mu}_j^+)_c (x)\ (\alpha)], \text{ for every } x \in X, \text{ for every } \alpha \in I$$

$$= [\hat{\mu}_j^-\ (x)(1-\alpha),\ \hat{\mu}_j^+\ (x)(1-\alpha)], \text{ for every } x \in X, \text{ for every } \alpha \in I$$

$$= [\mu_j^-\ (x),\ \mu_j^+\ (x)], \text{ for every } x \in X, \text{ (since by } C_I)$$

$$= [\hat{\mu}_j^-\ (x)\ (\alpha),\ \hat{\mu}_j^+\ (x)\ (\alpha)], \text{ for every } x \in X, \text{ for every } \alpha \in I$$

(since by C_I)

$$= \hat{\hat{\mu}}_j (x)\ (\alpha)$$

Therefore $(\hat{\hat{\mu}}_j)_c = (\hat{\hat{\mu}}_j)$

This is true for every $\hat{\mu}_j \in \hat{\tau}$

Therefore $\hat{\hat{\tau}} = (\hat{\hat{\tau}})_c$

5.Connections Between K-Hausdorffness in First Order Interval Valued Fuzzy and K-Hausdorffness in Second Order Interval Valued Fuzzy Topological Spaces.

Definition:5.1

An interval valued fuzzy topological space (X, $\hat{\tau}$) is said to be an interval valued fuzzy K-Hausdorff, denoted by (IVFK-H) or interval valued fuzzy K-T_2, if for all pair of disjoint points x, y $\in$ X, there exists two interval valued fuzzy open sets $\hat{\mu} = [\mu^-, \mu^+] \in \hat{\tau}$ and $\hat{\lambda} = [\lambda^-, \lambda^+] \in \hat{\tau}$ such that $\mu^-(x) > 0$, $\mu^+(x) > 0$, $\lambda^-(y) > 0$, $\lambda^+(y) > 0$ and $\hat{\mu} \mathbin{\hat{\cap}} \hat{\lambda} = \hat{0}$.

Definition:5.2

A second order interval valued fuzzy topological space (X, $\hat{\hat{\tau}}$) is said to be second order interval valued fuzzy K-Hausdorff of type 1, denoted by $(\text{SIVFK-H})_1$, if for every x, y $\in$ X, x $\neq$ y, there exists two interval valued fuzzy open sets $\hat{\hat{\mu}} = [\hat{\mu}^-, \hat{\mu}^+]$, $\hat{\hat{\lambda}} = [\hat{\lambda}^-, \hat{\lambda}^+] \in \hat{\hat{\tau}}$, such that $\hat{\mu}^-(x) > \mathbf{0}$, $\hat{\mu}^+(x) > \mathbf{0}$, $\hat{\lambda}^-(y) > \mathbf{0}$, $\hat{\lambda}^+(y) > \mathbf{0}$ and $\hat{\hat{\mu}} \hat{\hat{\cap}}_1 \hat{\hat{\lambda}} = \hat{\hat{0}}$.

Definition:5.3

A second order interval valued fuzzy topological space (X, $\hat{\hat{\tau}}$) is said to be second order interval valued fuzzy K-Hausdorff space of type 2 is denoted by $(SIVFK\text{-}H)_2$, is defined by replacing the condition $\hat{\hat{\mu}} \hat{\hat{\cap}}_1 \hat{\hat{\lambda}} = \hat{\hat{0}}$ in the above definition by $\hat{\hat{\mu}} \hat{\hat{\cap}}_2 \hat{\hat{\lambda}} = \hat{\hat{0}}$.

Theorem: 5.4

$(X, \hat{\tau})$ is (IVFK-H) iff $(X, \hat{\hat{\tau}})$ is $(SIVFK\text{-}H)_1$, where $(X, \hat{\hat{\tau}})$ got from $(X, \hat{\tau})$ through the association C_1.

Proof:

Assume $(X, \hat{\tau})$ is (IVFK-H).

Given $\hat{\tau}$, $\hat{\hat{\tau}} = \{\hat{\hat{\mu}} / \hat{\mu} \in \hat{\tau}\}$, where $\hat{\mu}^-(x)(\alpha) = \mu^-(x)$, $\hat{\mu}^+(x)(\alpha) = \mu^+(x)$, for every $x \in X$ and for every $\alpha \in I$.

Consider $x, y \in X$ such that $x \neq y$. There exist $\hat{\mu} = [\mu^-, \mu^+]$, $\hat{\lambda} = [\lambda^-, \lambda^+] \in \hat{\tau}$ such that $\mu^-(x) > 0$, $\mu^+(x) > 0$, $\lambda^-(y) > 0$, $\lambda^+(y) > 0$and $\hat{\mu} \hat{\cap} \hat{\lambda} = \hat{0}$.

Therefore $\hat{\hat{\mu}}, \hat{\hat{\lambda}} \in \hat{\hat{\tau}}$.

$\mu^-(x) > 0$implies $\hat{\mu}^-(x)(\alpha) > 0$, for every $x \in X$ and for every $\alpha \in I$

implies $\hat{\mu}^-(x) > \mathbf{0}$, for every $x \in X$

$\mu^+(x) > 0$ implies $\hat{\mu}^+(x)(\alpha) > 0$, for every $x \in X$ and for every $\alpha \in I$

implies $\hat{\mu}^+(x) > \mathbf{0}$, for every $x \in X$

Similarly, $\hat{\lambda}^-(x) > \mathbf{0}$, $\hat{\lambda}^+(x) > \mathbf{0}$

Claim : $\hat{\mu} \hat{\cap} \hat{\lambda} = \hat{0}$ implies $\hat{\hat{\mu}} \hat{\hat{\cap}}_1 \hat{\hat{\lambda}} = \hat{\hat{0}}$

Let $\hat{\mu}^-(x) \neq \mathbf{0}$, $\hat{\mu}^+(x) \neq \mathbf{0}$, for every $x \in X$

implies $\hat{\mu}^-(x)(\alpha) \neq 0$, $\hat{\mu}^+(x)(\alpha) \neq 0$, for every $x \in X$ and for some $\alpha \in I$

implies $\mu^-(x) \neq 0$, $\mu^+(x) \neq 0$, for every $x \in X$

implies $\lambda^-(x) = 0$, $\lambda^+(x) = 0$, for every $x \in X$ (since $\hat{\mu} \hat{\cap} \hat{\lambda} = \hat{0}$)

implies $\hat{\lambda}^-(x)(\alpha) = 0$, $\hat{\lambda}^+(x)(\alpha) = 0$, for every $x \in X$ and for every $\alpha \in I$

implies $\hat{\lambda}^{-}(x) = \mathbf{0}$, $\hat{\lambda}^{+}(x) = \mathbf{0}$, for every $x \in X$

implies $\hat{\hat{\mu}} \; \hat{\hat{\cap}}_1 \; \hat{\hat{\lambda}} = \hat{\hat{0}}$

Hence $(X, \hat{\hat{\tau}})$ is $(SIVFK\text{-}H)_1$.

To prove the converse, it is enough to observe that $\hat{\hat{\mu}} \; \hat{\hat{\cap}}_1 \; \hat{\hat{\lambda}} = \hat{\hat{0}}$ implies $\hat{\mu} \; \hat{\cap} \; \hat{\lambda} = \hat{0}$

$\hat{\hat{\mu}} \; \hat{\hat{\cap}}_1 \; \hat{\hat{\lambda}} = \hat{\hat{0}}$

implies $\hat{\mu}^{-}(x) = \mathbf{0}$, $\hat{\mu}^{+}(x) = \mathbf{0}$ (or) $\hat{\lambda}^{-}(x) = \mathbf{0}$, $\hat{\lambda}^{+}(x) = \mathbf{0}$, for every $x \in X$

implies $\hat{\mu}^{-}(x)(\alpha) = 0, \hat{\mu}^{+}(x)(\alpha) = 0$ (or) $\hat{\lambda}^{-}(x)(\alpha) = 0, \hat{\lambda}^{+}(x)(\alpha) = 0$,

for every $x \in X$, for every $\alpha \in I$

implies $\mu^{-}(x) = 0$, $\mu^{+}(x) = 0$ (or) $\lambda^{-}(x) = 0$, $\lambda^{+}(x) = 0$, for every $x \in X$

implies $\hat{\mu} \; \hat{\cap} \; \hat{\lambda} = \hat{0}$

Hence $(X, \hat{\tau})$ is (IVFK-H).

Theorem:5.5

If $(X, \hat{\hat{\tau}})$ is $(SIVFK\text{-}H)_1$ space, then for every $\alpha \in I$, $(X, (\hat{\hat{\tau}})_\alpha)$ is (IVFK-H) where $(X, (\hat{\hat{\tau}})_\alpha)$ is got from $(X, \hat{\hat{\tau}})$ through the association C_2.

Proof:

Assume $(X, \hat{\hat{\tau}})$ is $(SIVFK\text{-}H)_1$

Given $\hat{\hat{\tau}}$, for $\alpha \in I$. $(\hat{\hat{\tau}})_\alpha$ = Distinct elements of the collection $\{(\hat{\hat{\mu}})_\alpha \,/\, \hat{\hat{\mu}} \in \hat{\hat{\tau}}\}$ defines a first order interval valued fuzzy topology where, $(\hat{\mu}^{-})_\alpha(x) = \hat{\mu}^{-}(x)(\alpha)$, $(\hat{\mu}^{+})_\alpha(x) = \hat{\mu}^{+}(x)(\alpha)$, for every $x \in X$.

Consider $x, y \in X$, such that $x \neq y$

Since $(X, \hat{\hat{\tau}})$ is $(SIVFK\text{-}H)_1$, there exist $\hat{\hat{\mu}}, \hat{\hat{\lambda}} \in \hat{\hat{\tau}}$ such that $\hat{\mu}^{-}(x) > \mathbf{0}$, $\hat{\mu}^{+}(x) > \mathbf{0}$, $\hat{\lambda}^{-}(y) > \mathbf{0}$, $\hat{\lambda}^{+}(y) > \mathbf{0}$ and $\hat{\hat{\mu}} \; \hat{\hat{\cap}}_1 \; \hat{\hat{\lambda}} = \hat{\hat{0}}$

Therefore $(\hat{\hat{\mu}})_\alpha, (\hat{\hat{\lambda}})_\alpha \in (\hat{\hat{\tau}})_\alpha$

$(\hat{\mu}^{-})_\alpha(x) = \hat{\mu}^{-}(x)(\alpha)$, for every $x \in X$ and for every $\alpha \in I$

$> \mathbf{0}(\alpha)$, for every $\alpha \in I$

> 0

$(\hat{\mu}^{+})_{\alpha}(x) = \hat{\mu}^{+}(x)(\alpha)$, for every $x \in X$ and for every $\alpha \in I$

$> \mathbf{0}(\alpha)$, for every $\alpha \in I$

> 0

Similarly, $(\hat{\lambda}^{-})_{\alpha}(y) > 0$, $(\hat{\lambda}^{+})_{\alpha}(y) > 0$

Let $\hat{\hat{\mu}} \,\hat{\hat{\cap}}_1\, \hat{\hat{\lambda}} = \hat{\hat{0}}$

implies $\hat{\mu}^{-}(x) = \mathbf{0}$, $\hat{\mu}^{+}(x) = \mathbf{0}$ (or) $\hat{\lambda}^{-}(x) = \mathbf{0}$, $\hat{\lambda}^{+}(x) = \mathbf{0}$, for every $x \in X$

implies $\hat{\mu}^{-}(x)(\alpha) = 0$, $\hat{\mu}^{+}(x)(\alpha) = 0$ (or) $\hat{\lambda}^{-}(x)(\alpha) = 0$, $\hat{\lambda}^{+}(x)(\alpha) = 0$,

for every $x \in X$ and for every $\alpha \in I$

implies $(\hat{\mu}^{-})_{\alpha}(x) = 0$, $(\hat{\mu}^{+})_{\alpha}(x) = 0$ (or) $(\hat{\lambda}^{-})_{\alpha}(x) = 0$, $(\hat{\lambda}^{+})_{\alpha}(x) = 0$, for every $x \in X$

implies $(\hat{\hat{\mu}})_{\alpha} \,\hat{\cap}\, (\hat{\hat{\lambda}})_{\alpha} = \hat{0}$.

Hence $(X, (\hat{\hat{\tau}})_{\alpha})$ is (IVFK-H)

Theorem:5.6

$(X, \hat{\hat{\tau}})$ is (SIVFK-H)$_1$, iff $(X, (\hat{\hat{\tau}})_c)$ is (SIVFK-H)$_1$, where $(X, (\hat{\hat{\tau}})_c)$ is got from $(X, \hat{\hat{\tau}})$ through the association C_3.

Proof:

Assume $(X, \hat{\hat{\tau}})$ is (SIVFK-H)$_1$

Given $\hat{\hat{\tau}}$, $(\hat{\hat{\tau}})_c = \{(\hat{\hat{\mu}})_c \,/\, \hat{\hat{\mu}} \in \hat{\hat{\tau}}\}$ where $(\hat{\mu}^{-})_c(x)(\alpha) = \hat{\mu}^{-}(x)(1-\alpha)$, $(\hat{\mu}^{+})_c(x)(\alpha) = \hat{\mu}^{+}(x)(1-\alpha)$, for every $x \in X$ and for every $\alpha \in I$.

Consider $x, y \in X$ such that $x \neq y$

Since $(X, \hat{\hat{\tau}})$ is (SIVFK-H)$_1$, there exist $\hat{\hat{\mu}}, \hat{\hat{\lambda}} \in \hat{\hat{\tau}}$ such that $\hat{\mu}^{-}(x) > \mathbf{0}$, $\hat{\mu}^{+}(x) => \mathbf{0}$, $\hat{\lambda}^{-}(y) > \mathbf{0}$, $\hat{\lambda}^{+}(y) > \mathbf{0}$ and $\hat{\hat{\mu}} \,\hat{\hat{\cap}}_1\, \hat{\hat{\lambda}} = \hat{\hat{0}}$.

Therefore $(\hat{\hat{\mu}})_c$, $(\hat{\hat{\lambda}})_c \in (\hat{\hat{\tau}})_c$

Consider

$(\hat{\mu}^{-})_c(x)(\alpha) = \hat{\mu}^{-}(x)(1-\alpha)$, for every $x \in X$ and for every $\alpha \in I$

$> \mathbf{0}(1-\alpha)$, for every $\alpha \in I$

> 0

$(\hat{\mu}^-)_c(x) > \mathbf{0}$, for every $x \in X$

$(\hat{\mu}^+)_c(x)(\alpha) = \hat{\mu}^+(x)(1-\alpha)$, for every $x \in X$ and for every $\alpha \in I$

$> \mathbf{0}(1-\alpha)$, for every $\alpha \in I$

> 0

$(\hat{\mu}^+)_c(x) > \mathbf{0}$, for every $x \in X$

Similarly, $(\hat{\lambda}^-)_c(x) > \mathbf{0}$, $(\hat{\lambda}^+)_c(x) > \mathbf{0}$

Let $\hat{\mu} \;\hat{\hat{\cap}}_1\; \hat{\hat{\lambda}} = \hat{\hat{0}}$

implies $\hat{\mu}^-(x) = \mathbf{0}$, $\hat{\mu}^+(x) = \mathbf{0}$ (or) $\hat{\lambda}^-(x) = \mathbf{0}$, $\hat{\lambda}^+(x) = \mathbf{0}$, for every $x \in X$

implies $\hat{\mu}^-(x)(\alpha) = 0$, $\hat{\mu}^+(x)(\alpha) = 0$ (or) $\hat{\lambda}^-(x)(\alpha) = 0$, $\hat{\lambda}^+(x)(\alpha) = 0$,

for every $x \in X$ and for every $\alpha \in I$

implies $\hat{\mu}^-(x)(1-\alpha) = 0$, $\hat{\mu}^+(x)(1-\alpha) = 0$ (or) $\hat{\lambda}^-(x)(1-\alpha) = 0$, $\hat{\lambda}^+(x)(1-\alpha) = 0$,

for every $x \in X$ and for every $\alpha \in I$

implies $(\hat{\mu}^-)_c(x)(\alpha) = 0$, $(\hat{\mu}^+)_c(x)(\alpha) = 0$ (or) $(\hat{\lambda}^-)_c(x)(\alpha) = 0$, $(\hat{\lambda}^+)_c(x)(\alpha) = 0$,

for every $x \in X$ and for every $\alpha \in I$

implies $(\hat{\mu}^-)_c(x) = \mathbf{0}$, $(\hat{\mu}^+)_c(x) = \mathbf{0}$ (or) $(\hat{\lambda}^-)_c(x) = \mathbf{0}$, $(\hat{\lambda}^+)_c(x) = \mathbf{0}$,

for every $x \in X$

implies $(\hat{\mu})_c \;\hat{\hat{\cap}}_1\; (\hat{\hat{\lambda}})_c = \hat{\hat{0}}$

Hence $(X, (\hat{\tau})_c)$ is $(SIVFK\text{-}H)_1$.

To prove the converse part, it is enough to observe that

$(\hat{\mu})_c \;\hat{\hat{\cap}}_1\; (\hat{\hat{\lambda}})_c = \hat{\hat{0}}$ implies $\hat{\mu} \;\hat{\hat{\cap}}_1\; \hat{\hat{\lambda}} = \hat{\hat{0}}$

$(\hat{\mu})_c \;\hat{\hat{\cap}}_1\; (\hat{\hat{\lambda}})_c = \hat{\hat{0}}$

implies $(\hat{\mu}^-)_c(x) = \mathbf{0}$, $(\hat{\mu}^+)_c(x) = \mathbf{0}$ (or) $(\hat{\lambda}^-)_c(x) = \mathbf{0}$, $(\hat{\lambda}^+)_c(x) = \mathbf{0}$, for every $x \in X$

implies $(\hat{\mu}^-)_c(x)(\alpha) = 0$, $(\hat{\mu}^+)_c(x)(\alpha) = 0$ (or) $(\hat{\lambda}^-)_c(x)(\alpha) = 0$, $(\hat{\lambda}^+)_c(x)(\alpha) = 0$,

for every $x \in X$ and for every $\alpha \in I$

implies $(\hat{\mu}^-)_c(x)(1-\alpha) = 0$, $(\hat{\mu}^+)_c(x)(1-\alpha) = 0$ (or)

$(\hat{\lambda}^{-})_c(x)\,(1-\alpha) = 0, (\hat{\lambda}^{+})_c(x)\,(1-\alpha) = 0$, for every $x \in X$ and for every $\alpha \in I$

implies $\hat{\mu}^{-}(x) = \mathbf{0}$, $\hat{\mu}^{+}(x) = \mathbf{0}$ (or) $\hat{\lambda}^{-}(x) = \mathbf{0}$, $\hat{\lambda}^{+}(x) = \mathbf{0}$, for every $x \in X$

implies $\hat{\hat{\mu}}\ \hat{\hat{\cap}}_1\ \hat{\hat{\lambda}} = \hat{\hat{0}}$

Hence $(X, \hat{\hat{\tau}})$ is $(SIVFK\text{-}H)_1$

Note:5.7

Theorems 5.4 to 5.6 proved for the Hausdorff separation axiom $(SIVFK\text{-}H)_1$ have exact parallels for the Hausdorff separation axiom $(SIVFK\text{-}H)_2$.

6.Conclusion:

Thus, in this paper, we introduced a new definition of second order interval valued fuzzy topological spaces and their properties were studied with suitable examples. Connections between first order interval valued fuzzy and second order interval valued fuzzy topological spaces were studied. K-Hausdorff separation axiom in first order interval valued fuzzy and second order interval valued fuzzy topological spaces has been presented, and the connections between first order interval valued fuzzy K- Hausdorff spaces and second order interval valued fuzzy K- Hausdorff spaces were studied.

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