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Fermatean Picture Fuzzy Sets

Dr. V. Chitra 1, Ms. B. Maheswari 2*
1.Department of Mathematics, Nallamuthu Gounder Mahalingam College, Pollachi, Tamil Nadu 642001.
E-mail: chitrasakthi1811@gmail.com
2.Department of Mathematics, Dr. Mahalingam College of Engineering and Technology, Pollachi,Tamil Nadu 642003.
E-mail: mahes.sint@gmail.com

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Abstract:

This paper introduces the concept of the Fermatean Picture Fuzzy Set (FPFS). We define several operations on these set. Additionally, we establish the concepts of images and preimages within the context of FPFS. Finally, we extend our discussion to the Fermatean Picture fuzzy topological space and define the properties using the theorems.

Keywords: Fermatean Picture Fuzzy Sets (FPFS); Fermatean Picture Fuzzy (FPF) topological spaces; FPF closure; FPF interior; Fermatean Picture Fuzzy continuous.

1. Introduction

Fuzzy set theory introduced by Zadeh[1] in 1965, and it has greatly improved our ability to deal with ambiguity and uncertainty in a variety of contexts. Over the years, several extensions of fuzzy sets has been developed to better model complex real-world scenarios. An example of such an extension is the Intuitionistic Fuzzy Set (IFS), which provide a more intricate representation of doubt by creating a second level of hesitation were introduced and developed by Atanassov [2],[3],[4]. By establishing a second level of hesitation, IFS allows a more complex representation of uncertainty. Building on this, Picture Fuzzy Sets (PFS) were introduced by Cuong [5], to further account for situations where opinions can be divided into degrees of acceptance, rejection and hesitancy, who also examined several operations on PFS and some attributes that were derived from these operations. Cuong and Hai [6] built primary operations for fuzzy inference processes in image fuzzy systems and examined the primary fuzzy logic operators—negations, conjunctions, disjunctions, and implications—on picture fuzzy sets. A few arithmetic aggregation operators, such as type-2 picture fuzzy weighted, ordered weighted and hybrid weighted aggregation operators, were created by Yang and Li [10]. Their associated features are also covered.

For uncertainty, Fermatean Fuzzy Sets (FFS) offer a more comprehensive framework. FFS are classified according to degrees of hesitation, membership and non-membership, which are constrained by the Fermatean ratio. Fermatean fuzzy sets (FFS) were proposed by Senapati and Yager [7], who also established the accuracy and score functions for FFS ranking. Zhe Liu [11] proposed a novel triangular divergence-based distance metric for FFSs. By addressing the shortcomings in the existing measure, the new measure aims to provide a more efficient means of studying FFSs. First, we define the fundamental operations on Fermatean Picture Fuzzy Sets, which include the definition of the Fermatean Picture Fuzzy Set. Next, we investigate the concepts of images and preimages within the framework of FPFS, laying the fundamental basis for their utilization in many domains. Additionally, we provide Interval-Valued Fermatean Picture Fuzzy Sets, expanding the use of FPFS to scenarios in which interval values are a better fit for encapsulating uncertainty.

In this paper sections are organized as follows: A brief definition of Fermatean Picture Fuzzy Sets is given in Section 2. In Section 3, images and preimages within FPFS are examined, along with thorough descriptions of Fermatean Picture Fuzzy Sets. Fermatean Picture fuzzy topological spaces and the characteristics of FPF topological space are described in section 4.

2. Preliminaries

Definition 2.1. [3]

Let X be a universe of discourse, then a fuzzy set A is an object having the following formulation: $A = \{\langle x, \mu_A(x)\rangle | x \in X\}$ where $\mu_A: X \to [0,1]$ and $\mu_A(x)$ is called the membership degree of x in X.

Definition 2.2. [3]

Let U be a universe. An Intuitionistic fuzzy set A on U can be defined as follows:

$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ where $\mu_A: U \to [0,1]$ and $\nu_A: X \to [0,1]$ such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for any $x \in U$, where $\mu_A(x)$ and $\nu_A(x)$ are the degree of membership and degree of nonmembership of the element x respectively.

Definition 2.3. [9]

Let X be a non-empty set and I the unit interval [0,1]. A Pythagorean fuzzy set A is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ where $\mu_A: U \to [0,1]$ and $\nu_A: X \to [0,1]$ denotes respectively the degree of membership and degree of non-membership of each element $x \in X$ to the set A, and $(\mu_A(x))^2 + (\nu_A(x))^2 \leq 1$ for each $x \in X$.

Definition 2.4. [2]

Let X be a non-empty set. A cubic set in X is a structure of the form $C = \{\langle x, A(x), \lambda(x)\rangle | x \in X\}$ where A is an interval valued fuzzy set in X and λ is a fuzzy set in X.

3. Fermatean Picture Fuzzy Set

Definition 3.1.

A Fermatean Picture Fuzzy Set $\mathcal{A}$ in a universe U is an object of the form $\mathcal{A} = \{\langle x, \langle x, \alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \gamma_{\mathcal{A}}(x)\rangle | x \in U\rangle\}$ where $\alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \gamma_{\mathcal{A}}(x)$ are respectively called the degree of positive membership, the degree of neutral membership, the degree of negative membership of x in $\mathcal{A}$ and the following conditions are satisfied

$0 \leq \alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \gamma_{\mathcal{A}}(x) \leq 1,$

$0 \leq \alpha_{\mathcal{A}}(x) + \beta_{\mathcal{A}}(x) + \gamma_{\mathcal{A}}(x) \leq 1$ and

$0 \leq \alpha_{\mathcal{A}}{}^{3}(x) + \beta_{\mathcal{A}}{}^{3}(x) + \gamma_{\mathcal{A}}{}^{3}(x) \leq 1, \forall x \in U.$

Then $\forall x \in U$, $\pi_{\mathcal{A}}(x) = 1 - \alpha_{\mathcal{A}}{}^{3}(x) + \beta_{\mathcal{A}}{}^{3}(x) + \gamma_{\mathcal{A}}{}^{3}(x)$ is called the degree of refusal membership of x in Ã.

When dealing with human opinions that involve multiple types of responses such as "yes," "abstain," "no," and "refusal," Fermatean Picture Fuzzy Sets (FPFS) offer a suitable mathematical framework to handle the complexity and uncertainty inherent in such scenarios. For an example, in feedback mechanisms for products or services, users might express satisfaction (yes), dissatisfaction (no), neutrality (abstain), or refuse to provide feedback.

Numerical Example 3.2.

Let $X = \{a, 0.3, 0.4, 0.1\}$

Here $\alpha_{\mathcal{A}}(x) + \beta_{\mathcal{A}}(x) + \gamma_{\mathcal{A}}(x) = 0.3 + 0.4 + 0.1 = 0.8 < 1$ and

$0 \leq 0.027 + 0.064 + 0.001 \leq 1$

$\Longrightarrow 0.092 < 1$

Then $\pi_{\mathcal{A}}(a) = 1 - 0.092 = 0.908$ is called the degree of refusal membership of a in X.

Definition 3.3.

Let X be a non-empty set and the Fermatean Picture Fuzzy sets A and B be in the form

$A = \{\langle x, \langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle | x \in X\rangle\}$ and $B = \{\langle x, \langle x, \alpha_B(x), \beta_B(x), \gamma_B(x)\rangle | x \in X\rangle\}$

1. $(A) \subseteq (B)$ iff $\alpha_A(x) \leq \alpha_B(x), \beta_A(x) \leq \beta_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$
2. $(A) = (B)$ iff $\alpha_A(x) = \alpha_B(x), \beta_A(x) = \beta_B(x)$ and $\gamma_A(x) = \gamma_B(x)$ $\forall x \in X$
3. $A \cap B = \{\langle x, \alpha_{AB}(x), \beta_{AB}(x), \gamma_{AB}(x) \rangle | x \in X\}$ where
   I. $\alpha_{A\cap B}(x) = min\{\alpha_A(x), \alpha_B(x)\}$
   II. $\beta_{A\cap B}(x) = min\{\beta_A(x), \beta_B(x)\}$
   III. $\gamma_{A\cap B}(x) = max\{\gamma_A(x), \gamma_B(x)\}$
4. $A \cup B = \{\langle x, \alpha_{AB}(x), \beta_{AB}(x), \gamma_{AB}(x) \rangle | x \in X\}$ where
   I. $\alpha_{A\cup B}(x) = max\{\alpha_A(x), \alpha_B(x)\}$
   II. $\beta_{A\cup B}(x) = min\{\beta_A(x), \beta_B(x)\}$
   III. $\gamma_{A\cup B}(x) = min\{\gamma_A(x), \gamma_B(x)\}$

Definition 3.4.

Let $A = \{\langle x, \langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle | x \in X\rangle\}$ is a fermatean picture fuzzy set on X, then the complement of the set A ($C(A)$ in short) may be defined as

$$C(A) = \left\{\langle x, \gamma_A(x), \sqrt[3]{1 - \left(\alpha_A^3(x) + \beta_A^3(x) + \gamma_A^3(x)\right)}, \alpha_A(x)\rangle\right\} | x \in X$$

Definition 3.5.

If $A = \{\langle x, \langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle | x \in X\rangle\}$ and $B = \{\langle x, \langle x, \alpha_B(x), \beta_B(x), \gamma_B(x)\rangle | x \in X\rangle\}$ are any two fermatean picture fuzzy sets of X then the following are true

I. $C(A \cap B) = C(A) \cup C(B)$

II. $C(A \cup B) = C(A) \cap C(B)$

Definition 3.6.

Let $\{A_i : i \in J\}$ be an arbitrary family of fermatean picture fuzzy sets in X then

a) $\cap A_i = \left\{\langle x, \wedge T_{A_i}(x), \wedge N_{A_i}(x), \vee F_{A_i}(x)\rangle : x \in X\right\}$

b) $\cup A_i = \left\{\langle x, \vee T_{A_i}(x), \wedge_{A_i}(x), \wedge F_{A_i}(x)\rangle : x \in X\right\}$

Definition 3.7.

Let A and B are Fermatean Picture Fuzzy sets in X, then A|B may be defined as

$A|B = \{\langle x, T_A(x) \wedge F_B(x), N_A(x) \wedge N_B(x), F_A(x) \vee T_B(x)\rangle : x \in X\}$

Definition 3.8.

The Fermatean Picture Fuzzy sets $0_{\mathcal{F}}$ and $1_{\mathcal{F}}$ in X are defined as follows:

$0_{\mathcal{F}}$ may be defined as:

(0_1) $0_{\mathcal{F}} = \{\langle x, 0,0,1\rangle : x \in X\}$

(0_2) $0_{\mathcal{F}} = \{\langle x, 0,0,0\rangle : x \in X\}$

$1_{\mathcal{F}}$ may be defined as:

(1_1) $1_{\mathcal{F}} = \{\langle x, 1,0,0\rangle : x \in X\}$

(1_2) $1_{\mathcal{F}} = \{\langle x, 0,1,0\rangle : x \in X\}$

where 1 and 0 represent the constant maps sending every element of X to 1 and 0, respectively.

Proposition 3.9.

For any Fermatean Picture Fuzzy sets A the following are holds

a) $0_{\mathcal{F}} \subseteq A, 0_{\mathcal{F}} \subseteq 0_{\mathcal{F}}$

b) $A \subseteq 1_{\mathcal{F}}, 1_{\mathcal{F}} \subseteq 1_{\mathcal{F}}$

Proposition 3.10.

Let A and B are Fermatean Picture Fuzzy sets in X, then the following holds.

a. $A \subseteq B$ and $C \subseteq D \Longrightarrow A \cup C \subseteq B \cup D$ and $C \subseteq D \Longrightarrow A \cap C \subseteq B \cap D$

b. $A \subseteq B, A \subseteq C \Longrightarrow A \subseteq B \cap C$

c. $A \subseteq C, B \subseteq C \Longrightarrow A \cup B \subseteq C$

d. $A \subseteq B, B \subseteq C \Longrightarrow A \subseteq C$

e. $\overline{A \cup B} = \bar{A} \cap \bar{B}$

f. $\overline{A \cap B} = \bar{A} \cup \bar{B}$

g. $A \subseteq B \Longrightarrow \bar{B} \subseteq \bar{A}$

h. $\overline{(\bar{A})} = A$

i. $\bar{1}_{\mathcal{F}} = 0_{\mathcal{F}}$ and $\bar{0}_{\mathcal{F}} = \sqrt{2}_{\mathcal{F}}$

Example:

Let us consider the Fermatean Picture Fuzzy sets

$$A = \{< x, \alpha_A(x) = 0.2, \beta_A(x) = 0.3, \gamma_A(x) = 0.5 >\}$$

$$B = \{< x, \alpha_B(x) = 0.3, \beta_B(x) = 0.4, \gamma_B(x) = 0.4 >\}$$

$$C = \{< x, \alpha_C(x) = 0.4, \beta_C(x) = 0.5, \gamma_C(x) = 0.3 >\}$$

$$D = \{< x, \alpha_D(x) = 0.45, \beta_D(x) = 0.55, \gamma_D(x) = 0.1 >\}$$

We can observe that the above sets satisfy the Proposition 3.10.

Definition 3.11. Images of Fermatean Picture Fuzzy Set

Let X and Y be two non-empty set and $f: X \longrightarrow Y$ be a function. If $A = \{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle | x \in X\}$ is a FPF set in X, then the image of A under f denoted by $f(A)$ is the FPFS in Y defined by

$$f(A) = \left\{\langle y, f(\alpha_A)(y), f(\beta_A)(y), 1 - f\big(1 - (\gamma_A)(y)\big)\rangle | y \in Y\right\}$$

$$f(\alpha_A)(y) = \begin{cases} sup_{x \epsilon f^{-1}(y)} \alpha_A(x) & if\ f^{-1}(y) \neq \phi \\ 0, & otherwise \end{cases}$$

$$f(\beta_A)(y) = \begin{cases} inf_{x \epsilon f^{-1}(y)} \beta_A(x) & if\ f^{-1}(y) \neq \phi \\ 0, & otherwise \end{cases}$$

$$1 - f\big(1 - (\gamma_A)(y)\big) = \begin{cases} inf_{x \epsilon f^{-1}(y)} \gamma_A(x) & if\ f^{-1}(y) \neq \phi \\ 0, & otherwise \end{cases}$$

Definition 3.12. Preimages of Fermatean Picture Fuzzy Set

Let X and Y be two non-empty set and $f: X \longrightarrow Y$ be a function. If

$B = \{\langle y, \alpha_B(y), \beta_B(y), \gamma_B(y)\rangle | y \in Y\}$ is a FPF set in Y, then the preimage of B under f denoted by $f^{-1}(B)$ is the FPF set in X defined by

$$f^{-1}(B) = \{\langle x, f^{-1}(\alpha_B)(x), f^{-1}(\beta_B)(x), f^{-1}(\gamma_A)(x)\rangle | x \in X\}$$

Where $f^{-1}(\alpha_B)(x) = \alpha_B(f(x)), f^{-1}(\beta_B)(x) = \beta_B(f(x)), f^{-1}(\gamma_B)(x) = \gamma_B(f(x))$

Proposition 3.13.

Let $A, A_i((i \epsilon J)$ be FPF set in X, $B, B_j((j \epsilon K)$ be FPF set in Y and $f: X \longrightarrow Y$ be a function. Then

a) $A_1 \subseteq A_2 \Longrightarrow f(A_1) \subseteq f(A_2)$

b) $B_1 \subseteq B_2 \Longrightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$

c) $A \subseteq f^{-1}(f(A))$ and if f is injective, then $A = f(f^{-1}(A))$

d) $f(f^{-1}(B)) \subseteq B$ and if f is surjective, then $f\left(f^{-1}(B)\right) = B$

e) $f^{-1}\left(\cup B_j\right) = \cup f^{-1}\left(B_j\right)$

f) $f^{-1}\left(\cap B_j\right) = \cap f^{-1}\left(B_j\right)$

g) $f(\cup A_I) = \cup f(A_I)$

h) $f(\cap A_I) = \cap f(A_I)$ [if f is injective then $f(\cap A_I) = \cap f(A_I)$]

i) $f^{-1}(1_{FPF}) = 1_{FPF}$

j) $f^{-1}(0_{FPF}) = 0_{FPF}$

k) $\overline{f(A)} \subseteq f(\bar{A})$ [if f is surjective]

l) $f^{-1}(\bar{B}) \subseteq \overline{f^{-1}(B)}$

Definition 3.14. Interval-valued Fermatean Picture Fuzzy sets

Let X be a non-empty set. An interval-valued Fermatean Picture Fuzzy Set $\mathcal{A}_P$ in X is of the form

$$\mathcal{A}_P = \left\{x: \left[\alpha^-_{\mathcal{A}_P(x)}, \alpha^+_{\mathcal{A}_P(x)}\right]\left[\beta^-_{\mathcal{A}_P(x)}, \beta^+_{\mathcal{A}_P(x)}\right]\left[\gamma^-_{\mathcal{A}_P(x)}, \gamma^+_{\mathcal{A}_P(x)}\right]/x\epsilon X\right\}$$

Where $\alpha^-_{\mathcal{A}_P(x)}, \beta^-_{\mathcal{A}_P(x)}, \gamma^-_{\mathcal{A}_P(x)}\ /x\epsilon X \rightarrow [0,1]$,

$0 \leq \alpha^-_{\mathcal{A}_P(x)} + \beta^-_{\mathcal{A}_P(x)} + \gamma^-_{\mathcal{A}(x)} \leq 1$,

$0 \leq \left[\alpha^-_{\mathcal{A}_P(x)}\right]^3 + \left[\beta^-_{\mathcal{A}_P(x)}\right]^3 + \left[\gamma^-_{\mathcal{A}(x)}\right]^3 \leq 1$ then

$\pi^-_{\mathcal{A}_P(x)} = 1 - \left(\left[\alpha^-_{\mathcal{A}_P(x)}\right]^3 + \left[\beta^-_{\mathcal{A}_P(x)}\right]^3 + \left[\gamma^-_{\mathcal{A}_P(x)}\right]^3\right)$

And $\alpha^+_{\mathcal{A}_P(x)}, \beta^+_{\mathcal{A}_P(x)}, \gamma^+_{\mathcal{A}_P(x)}\ /x\epsilon X \rightarrow [0,1]$,

$0 \leq \alpha^+_{\mathcal{A}_P(x)} + \beta^+_{\mathcal{A}_P(x)} + \gamma^+_{\mathcal{A}_P(x)} \leq 1$,

$0 \leq \left[\alpha^+_{\mathcal{A}_P(x)}\right]^3 + \left[\beta^+_{\mathcal{A}_P(x)}\right]^3 + \left[\gamma^+_{\mathcal{A}_P(x)}\right]^3 \leq 1$ then

$\pi^-_{\mathcal{A}_P(x)} = 1 - \left(\left[\alpha^+_{\mathcal{A}(x)}\right]^3 + \left[\beta^+_{\mathcal{A}(x)}\right]^3 + \left[\gamma^+_{\mathcal{A}_P(x)}\right]^3\right)$

Where $\beta^-_{\mathcal{A}_P(x)} = min\left\{1, 1 - \alpha^-_{\mathcal{A}_P(x)} - \gamma^-_{\mathcal{A}_P(x)}\right\}$

$\beta^+_{\mathcal{A}_P(x)} = max\left\{1, 1 - \alpha^+_{\mathcal{A}_P(x)} - \gamma^+_{\mathcal{A}(x)}\right\}$

Definition 3.15. Fermatean Picture Fuzzy Cubic set

Let X be a non-empty set. A Fermatean Picture Fuzzy Cubic set ς_P is a structure of the form

$\varsigma_P = \{\langle x, A_P(x), \lambda_P(x)\rangle / x\epsilon X\}$ where $A_P(x)$ is an interval valued Fermatean Picture Fuzzy set in X and $\lambda_P(x)$ is an Fermatean Picture Fuzzy Set in X.

Fermatean Picture Fuzzy topological space

Definition 4.1.

Let X be a non-empty set and τ be a family of Fermatean Picture Fuzzy ($\mathscr{F}$) subsets of X. If

1. $1_X, 0_X \in \tau$
2. for any $\mathscr{F}_1, \mathscr{F}_2 \in \tau$, we have $\mathscr{F}_1 \cap \mathscr{F}_2 \in \tau$,
3. for any $\{\mathscr{F}_i\}_{i \in I} \subset \tau$, we have $\bigcup_{i \in I} \mathscr{F}_i \in \tau$ where I is an arbitrary index set then τ is called a Fermatean Picture Fuzzy topology on X.

Fermatean Picture Fuzzy topological space is defined as the pair (X, τ). Every element in τ is referred to as an open Fermatean Picture Fuzzy subset. A closed Fermatean Picture Fuzzy set is the complement of an open Fermatean Picture Fuzzy subset. An illustration of a Fermatean Picture Fuzzy topological space is shown below.

Example 4.2.

Let $X = \{x_1, x_2\}$, consider the family of Fermatean Picture Fuzzy subsets $\tau = \{1_X, 0_X, \mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3\}$, where

$$\mathscr{F}_1 = \{\langle x_1, \alpha_{\mathscr{F}_1}(x_1) = 0.4, \beta_{\mathscr{F}_1}(x_1) = 0.3, \gamma_{\mathscr{F}_1}(x_1) = 0.1\rangle,$$
$$\langle x_2, \alpha_{\mathscr{F}_1}(x_2) = 0.2, \beta_{\mathscr{F}_1}(x_2) = 0.1, \gamma_{\mathscr{F}_1}(x_2) = 0.01\rangle\}$$
$$\mathscr{F}_2 = \{\langle x_1, \alpha_{\mathscr{F}_2}(x_1) = 0.45, \beta_{\mathscr{F}_2}(x_1) = 0.32, \gamma_{\mathscr{F}_2}(x_1) = 0.11\rangle,$$
$$\langle x_2, \alpha_{\mathscr{F}_2}(x_2) = 0.21, \beta_{\mathscr{F}_2}(x_2) = 0.12, \gamma_{\mathscr{F}_2}(x_2) = 0.15\rangle\}$$
$$\mathscr{F}_3 = \{\langle x_1, \alpha_{\mathscr{F}_3}(x_1) = 0.5, \beta_{\mathscr{F}_3}(x_1) = 0.15, \gamma_{\mathscr{F}_3}(x_1) = 0.26\rangle,$$
$$\langle x_2, \alpha_{\mathscr{F}_3}(x_2) = 0.26, \beta_{\mathscr{F}_3}(x_2) = 0.43, \gamma_{\mathscr{F}_3}(x_2) = 0.3\rangle\}$$

Observe that (X, τ) is a Fermatean Picture Fuzzy topological space.

Remark 4.3.

The topology containing all Fermatean Picture fuzzy subsets is referred to as the discrete Fermatean Picture fuzzy topological space, while the family $\{1_X, 0_X\}$ is known as the indiscrete Fermatean Picture fuzzy topological space. Further, if $\tau_1 \subset \tau_2$, then a Fermatean Picture fuzzy topology τ_1 on a set is coarser than a Fermatean Picture fuzzy topology τ_2 on the same set.

Definition 4.4.

$\mathcal{A} = \{x, \alpha_{\mathscr{F}}(x), \beta_{\mathscr{F}}(x), (\gamma_{\mathscr{F}}(x)) : x \in X\}$ is a Fermatean picture fuzzy set in X. Let (X, τ) be a Fermatean Picture fuzzy topological space. Fermatean picture fuzzy closure and interior are defined by

1. $FPFcl(\mathcal{A}) = \cap \{H : H \textit{ is closed Fermatean Picture fuzzy set in } X \textit{ and } \mathscr{F} \subset H\}$.
2. $FPFint(\mathcal{A}) = \cup \{G : G \textit{ is open Fermatean Picture fuzzy set in } X \textit{ and } G \subset \mathscr{F}\}$.

Remark 4.5.

Let $\mathscr{F}$ be any Fermatean picture fuzzy set in X, and let (X,τ) be a Fermatean picture fuzzy topological space. Following that,

1. An open Fermatean picture fuzzy set is $FPFint(\mathscr{F})$.
2. A closed Fermatean picture fuzzy set is $FPFcl(\mathscr{F})$.
3. $FPFint(1_X) = 1_X$, $FPFint(0_X) = 0_X$.
4. $FPFcl(1_X) = 1_X$, $FPFcl(0_X) = 0_X$.

Theorem 4.6.

Let (X,τ) be a Fermatean picture fuzzy topological space and $\mathscr{F}_1, \mathscr{F}_2$ be Fermatean picture fuzzy sets in X. Then, the following properties hold:

1. $FPFint\ (\mathscr{F}_1) \subset \mathscr{F}_1$ and $\mathscr{F}_1 \subset FPFcl(\mathscr{F}_1)$.
2. If $\mathscr{F}_1 \subset \mathscr{F}_2$, then $FPFint(\mathscr{F}_1) \subset FPFint(\mathscr{F}_2)$ and $FPFcl(\mathscr{F}_1) \subset FPFcl(\mathscr{F}_2)$.
3. $\mathscr{F}_1$ is open Fermatean picture fuzzy if and only if $\mathscr{F}_1 = FPFint(\mathscr{F}_1)$.
4. $\mathscr{F}_1$ is closed Fermatean picture fuzzy if and only if $\mathscr{F}_1 = FPFcl(\mathscr{F}_1)$.
5. $FPFint(\mathscr{F}_1) \cup FPFint(\mathscr{F}_2) \subset FPFint(\mathscr{F}_1 \cup \mathscr{F}_2)$.
6. $FPFcl(\mathscr{F}_1 \cap \mathscr{F}_2) \subset FPFcl(\mathscr{F}_1) \cap FPFcl(\mathscr{F}_2)$.
7. $FPFint(\mathscr{F}_1 \cap \mathscr{F}_2) = FPFint(\mathscr{F}_1) \cap FPFint(\mathscr{F}_2)$.
8. $FPFcl(\mathscr{F}_1) \cup FPFcl(\mathscr{F}_2) = FPFcl(\mathscr{F}_1 \cup \mathscr{F}_2)$.

Proof.

We know that $FPFint\ (\mathscr{F}_1) \subset \mathscr{F}_1$ and $\mathscr{F}_1 \subset FPFcl(\mathscr{F}_1)$ is true and also if $\mathscr{F}_1 \subset \mathscr{F}_2$, then $FPFint(\mathscr{F}_1) \subset FPFint(\mathscr{F}_2)$ and $FPFcl(\mathscr{F}_1) \subset FPFcl(\mathscr{F}_2)$ is true.

(3) and (4) follow from (1) and Definition 4.4.

From $FPFint(\mathscr{F}_1 \cap \mathscr{F}_2) \subset FPFint(\mathscr{F}_1)$ and $FPFint(\mathscr{F}_1 \cap \mathscr{F}_2) \subset FPFint(\mathscr{F}_2)$, we obtain $FPFint(\mathscr{F}_1 \cap \mathscr{F}_2) \subset FPFint(\mathscr{F}_1) \cap FPFint(\mathscr{F}_2)$. On the other hand, from the facts, $FPFint(\mathscr{F}_1) \subset \mathscr{F}_1$ and $FPFint(\mathscr{F}_2) \subset \mathscr{F}_2$, we have $FPFint(\mathscr{F}_1) \cap FPFint(\mathscr{F}_2) \subset \mathscr{F}_1 \cap \mathscr{F}_2$ and $FPFint(\mathscr{F}_1) \cap FPFint(\mathscr{F}_2) \in \tau$ we see that $FPFint(\mathscr{F}_1) \cap FPFint(\mathscr{F}_2) \subset FPFint(\mathscr{F}_1 \cap \mathscr{F}_2)$ and hence $FPFint(\mathscr{F}_1 \cap \mathscr{F}_2) = FPFint(\mathscr{F}_1) \cap FPFint(\mathscr{F}_2)$. From (7), (8) is readily inferred.

Remark 4.7.

Let (X,τ) be a Fermatean picture fuzzy topological space and $\mathscr{F}$ be a Fermatean picture fuzzy set in . Then, the ensuing characteristics are true:

1. $FPFcl(\mathscr{F}^c) = FPFint(\mathscr{F})^c$.
2. $FPFint(\mathscr{F}^c) = FPFcl(\mathscr{F})^c$.
3. $FPFcl(\mathscr{F}^c)^c = FPFint(\mathscr{F})$.
4. $FPFint(\mathscr{F}^c)^c = FPFcl(\mathscr{F})$.

Definition 4.8.

Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two Fermatean picture fuzzy subsets in a Fermatean picture fuzzy topological space. Then, if there is an open Fermatean picture fuzzy subset A, such as $\mathscr{F}_1 \subset A \subset \mathscr{F}_2$, then $\mathscr{F}_2$ is said to be a neighbourhood of $\mathscr{F}_1$.

Definition 4.9.

Let $g : X \to Y$ be a function and let (X, τ_1) and (Y, τ_2) be two Fermatean picture fuzzy topological spaces. If there exists a neighbourhood U of $\mathscr{F}_1$ such that $g[U] \subset V$ for every neighbourhood V of $g[\mathscr{F}_1]$ and for any Fermatean picture fuzzy subset $\mathscr{F}_1$ of X, then g is said to be Fermatean picture fuzzy continuous.

Theorem 4.10.

Let $g : X \to Y$ be a function, (X, τ_1) and (Y, τ_2) be two Fermatean picture fuzzy topological spaces. Then, the following facts are equivalent:

(1)g is Fermatean picture fuzzy continuous.

(2) There exists a neighbourhood U of $\mathscr{F}_1$ such that for any $\mathscr{F}_2 \subset U$ we have $g[\mathscr{F}_2] \subset V$ for any Fermatean picture fuzzy subset $\mathscr{F}_1$ of X and for any neighbourhood V of $g[\mathscr{F}_1]$.

(3) There exists a neighbourhood U of A such that $U \subset g^{-1}[V]$ for any Fermatean picture fuzzy subset $\mathscr{F}_1$ of X and for any neighbourhood V of $g[\mathscr{F}_1]$.

(4) $g^{-1}[V]$ is a neighbourhood of $\mathscr{F}_1$ for any Fermatean picture fuzzy subset $\mathscr{F}_1$ of X and for any neighbourhood V of $g[\mathscr{F}_1]$.

Proof.

$(1) \Rightarrow (2)$: Suppose that g is Fermatean picture fuzzy continuous. Given a Fermatean picture fuzzy subset of X called $\mathscr{F}_1$, and a neighbourhood of $g[\mathscr{F}_1]$ is V. Then, there exists a neighbourhood U of $\mathscr{F}_1$ such that $g[U] \subset V$. Now, we get $g[B] \subset g[U] \subset V$ if $B \subset U$.

$(2) \Rightarrow (3)$: Given a Fermatean picture fuzzy subset of X called $\mathscr{F}_1$, and a neighbourhood of $g[\mathscr{F}_1]$ is V. According to (2), $\mathscr{F}_1$ has a neighborhood U such that $[\mathscr{F}_2] \subset V$ for every $\mathscr{F}_2 \subset U$. $\mathscr{F}_2 \subset g^{-1}[g[\mathscr{F}_2]] \subset g^{-1}[V]$ can then be written. $U \subset g^{-1}[V]$ since $\mathscr{F}_2$ is an arbitrary subset of U.

$(3) \Rightarrow (4)$: Let $\mathscr{F}_1$ be a Fermatean picture fuzzy set of X and V be a neighbourhood of $g[\mathscr{F}_1]$. According to (3), there exists a neighbourhood U of $\mathscr{F}_1$ such that $U \subset g^{-1}[V]$. There exists an open Fermatean picture fuzzy subset K of X such that $\mathscr{F}_1 \subset K \subset U$, Since U is a neighbourhood of $\mathscr{F}_1$. Conversely, if $U \subset g^{-1}[V]$, then $\mathscr{F}_1 \subset K \subset g^{-1}[V]$ indicating that $g^{-1}[V]$ is a neighbourhood of $\mathscr{F}_1$.

$(4) \Rightarrow (1)$: Let $\mathscr{F}_1$ be a Fermatean picture fuzzy set of X and V be a neighbourhood of $g[\mathscr{F}_1]$. According to the theory, $\mathscr{F}_1$'s neighborhood is $g^{-1}[V]$. Consequently, $g[K] \subset g[g^{-1}[V]] \subset V$ since there is an open Fermatean Picture fuzzy subset K of X such that $\mathscr{F}_1 \subset K \subset g^{-1}[V]$. Furthermore, K is in the neighborhood of $\mathscr{F}_1$ because it is open. Consequently, g is a Fermatean picture fuzzy continuous.

Remark 4.11.

We note that any intuitionistic picture fuzzy subset or Pythagorean picture fuzzy subset of a set can be regarded as a Fermatean picture fuzzy subset, but the converse need not be true.

Example:

Let $X = \{x_1, x_2\}$.Consider the following family of Fermatean Picture Fuzzy subsets

$\tau = \{1_X, 0_X, \mathscr{F}_1, \mathscr{F}_2\}$, where

$$\mathscr{F}_1 = \{\langle x_1, \alpha_{\mathscr{F}_1}(x_1) = 0.4, \beta_{\mathscr{F}_1}(x_1) = 0.5, \gamma_{\mathscr{F}_1}(x_1) = 0.1\rangle, \langle x_2, \alpha_{\mathscr{F}_1}(x_2) = 0.2, \beta_{\mathscr{F}_1}(x_2) = 0.1, \gamma_{\mathscr{F}_1}(x_2) = 0.3\rangle\}$$

$$\mathscr{F}_2 = \{\langle x_1, \alpha_{\mathscr{F}_2}(x_1) = 0.4, \beta_{\mathscr{F}_2}(x_1) = 0.3, \gamma_{\mathscr{F}_2}(x_1) = 0.1\rangle, \langle x_2, \alpha_{\mathscr{F}_2}(x_2) = 0.2, \beta_{\mathscr{F}_2}(x_2) = 0.1, \gamma_{\mathscr{F}_2}(x_2) = 0.1\rangle\}$$

Note that although (X, τ) is not an intuitionistic picture fuzzy set or a Pythagorean picture fuzzy set, it is a Fermatean picture fuzzy set.

Conclusion

In this paper, Fermatean Picture fuzzy set is defined and some properties were presented. Further We defined the concept of Fermatean Picture fuzzy topology, Fermatean Picture fuzzy continuous mapping, neighborhood and their properties. In future, We will try to explain the properties of Fermatean Picture fuzzy continuous, compactness and connectedness in Fermatean Picture fuzzy topological space.

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