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Fuzzy Baer Subrings: A Fuzzified Extension of Baer Rings

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Abstract:

In classical ring theory, a ring $\mathcal{K}$ is classified as a Baer ring if, for any subset $\mathcal{B} \subseteq \mathcal{K}$ the left (or right) annihilator $L(\mathcal{B})$ is generated by an idempotent element in $\mathcal{R}$ This paper introduces the concept of fuzzy Baer subrings, extending the principles of Baer rings to the fuzzy setting by defining a fuzzy subset generated by an element and utilizing fuzzy left and right annihilators. Additionally, we develop the theory of fuzzy Rickart subrings by introducing the concept of fuzzy points, establishing that each fuzzy Baer subring inherently qualifies as a fuzzy Rickart subring. Further contributing to fuzzy algebra, we define fuzzy idempotent subrings, capturing broader generalizations in the fuzzy context. This research bridges classical and fuzzy set theories by adapting Baer rings, Rickart rings, and idempotent rings within the framework of fuzzy algebra.

Keywords: Fuzzy subring, Fuzzy Baer Subring, Fuzzy Rickart Subring, Fuzzy Idempotent Subring

1. Introduction

Fuzzy set theory, introduced by Zadeh in 1965 [23], has become a crucial mathematical framework with extensive applications in fields such as medical diagnostics, computer networks, artificial intelligence, and several branches of mathematics, including fuzzy algebra, fuzzy topology, optimization, graph theory, and measure theory. It extends classical set theory by associating each element with a membership degree within the interval $[0, 1]$, thus offering a structured approach to manage uncertainty and imprecision. The practical applications of this theory are well-documented across various domains [8, 18].

The development of fuzzy algebraic structures began with Rosenfeld's introduction of fuzzy subgroups in 1971 [20], which paved the way for extensive research on fuzzy analogs of algebraic structures like rings, fields, modules, and topologies. Liu's work on fuzzy subrings in 1982 [11] marked a key advancement, opening the door to the study of fuzzy ideals and algebraic extensions. Since then, various types of fuzzy subrings have been explored, each adding new layers of flexibility and capturing distinct aspects of uncertainty within algebraic frameworks.

Notable advancements include Banerjee's work on intuitionistic fuzzy subrings [3], which incorporate both membership and non-membership degrees for a more comprehensive representation of uncertainty, and Rasuli's introduction of $Q-$ and anti- $Q-$fuzzy subrings [1], utilizing $t-$norms and $t-$conorms to integrate logical operators into fuzzy theory. Further contributions include Maheswari's exploration of bipolar fuzzy sets [13], which account for both positive and negative membership degrees, and Dogra's introduction of picture fuzzy subrings [7], which introduce a dimension of neutrality to address ambivalence in uncertain scenarios. Massa'deh [16] also extended the theory with bipolar $Q-$fuzzy soft $\Gamma-$semirings and their homomorphisms.

In this paper, we extend these foundational concepts by introducing the theory of fuzzy Baer subrings, where we define fuzzy subsets generated by individual elements and utilize fuzzy left and right annihilators to extend the classical properties of Baer rings to the fuzzy domain. We also introduce fuzzy Rickart subrings, developing the concept of fuzzy points, and prove that every fuzzy Baer subring is naturally a fuzzy Rickart subring. The paper further contributes to fuzzy algebra by defining fuzzy idempotent subrings, thereby generalizing classical results in the fuzzy setting.

Baer rings, initially introduced by Kaplansky in 1955 [9], are defined as rings in which, for any subset $\mathcal{B} \subseteq \mathcal{R}$, the left or right annihilator of $\mathcal{B}$ is an ideal generated by an idempotent element. Baer rings have strong connections to functional analysis and operator algebras. Clark's quasi-Baer rings [4] later emerged as a relaxation of these conditions, requiring only that the right annihilator of any right ideal is generated by an idempotent. Every Baer ring is thus also a quasi-Baer ring, reflecting their robust annihilator structure.

Throughout this paper, $\mathcal{K}$ refers to an associative ring with an identity element. For any non-empty subset $\mathcal{S} \subseteq \mathcal{K}$, the left annihilator of $\mathcal{S}$ in $\mathcal{K}$ is represented by $L(\mathcal{S})$, while the right annihilator is denoted by $R(\mathcal{S})$. An element $e \in \mathcal{K}$ is said to be idempotent if $e^2 = e$. Additionally, I denotes an indexing set wherever needed.

2. Preliminaries

This section outlines the fundamental terms necessary for a comprehensive understanding of the paper.

2.1 Definition [23]

Let ϕ and ψ be fuzzy subsets of $\mathcal{K}$. The product of two fuzzy subsets is defined as follows:

$$(\phi\psi)(p) = \begin{cases} \sup\limits_{p=rs} \inf\{\phi(r), \psi(s)\}, & where\ r, s \in \mathcal{K}\ , \\ 0\ if\ p\ is\ not\ expressible\ as\ p = rs\ for\ all\ r, s \in \mathcal{K}. \end{cases}$$

2.2 Definition [23]

let Γ be fuzzy subset of $\mathcal{K}$. The level subset $\mathcal{K}$. The level subset Γ_t for $t \in [0, 1]$ is defined as the set $\Gamma_t = \{r \in \mathcal{K} \mid \Gamma(r) \geq t\}$.

2.3 Definition [20]

Let Γ represent a fuzzy subset of a group G. The fuzzy subset Γ is called a fuzzy subgroup of G if, for any $g, h \in G$, the following conditions hold:

i. $\Gamma(gh) \geq \min\{\Gamma(g), \Gamma(h)\}$,
ii. $\Gamma(g^{-1}) \geq \Gamma(g)$.

If G is a group with identity e, then $\Gamma(g^{-1}) = \Gamma(g) \leq \Gamma(e)$.

2.4 Definition [11]

Let Ω represent a fuzzy subset of a ring $\mathcal{K}$. The fuzzy subset Ω is called a fuzzy subring of $\mathcal{K}$ if, for any $r, s \in \mathcal{K}$, the following conditions hold:

i. $\Omega(r - s) \geq \min\{\Omega(r), \Omega(s)\}$,
ii. $\Omega(rs) \geq \min\{\Omega(r), \Omega(s)\}$.

2.5 Definition [11]

Let Ω represent a fuzzy subset of a ring $\mathcal{K}$. The fuzzy subset Ω is called a fuzzy left (or right) ideal of $\mathcal{K}$ if, for any $r, s \in \mathcal{K}$, the following conditions hold:

i. $\Omega(r - s) \geq \min\{\Omega(r), \Omega(s)\}$,
ii. $\Omega(rs) \geq \Omega(s) \left(or\ \Omega(r)\right)$,

A fuzzy subset Ω is called a fuzzy ideal of the ring $\mathcal{K}$ if it satisfies the conditions of being both a fuzzy left ideal and a fuzzy right ideal.

2.6 Definition [21]

For any fuzzy subsets Ω and Γ of a ring $\mathcal{K}$, the sum $(\Omega + \Gamma)$ is given by,

$(\Omega + \Gamma)(z) = \sup_{z=r+s} \left(\inf\left(\Omega(r), \Gamma(s)\right)\right)$, where $z, s, r \in \mathcal{K}$.

2.7 Theorem [22]

A fuzzy subset Ω of a ring $\mathcal{K}$ is considered a fuzzy subring (or fuzzy ideal) of $\mathcal{K}$ if and only if, for every $p \in Im\, \Omega$, the corresponding level subset Ω_p is a level subring (or level ideal) of $\mathcal{K}$.

2.8 Definition [17]

Let Ω represent a fuzzy subset of $\mathcal{K}$. The left fuzzy annihilator $L(\Omega)$ of the fuzzy subset Ω is defined as,

$$L(\Omega)(z) = \begin{cases} \max\{p \mid z \in L(\Omega_p)\}, \\ 0\ if\ z \notin L(\Omega_p)\ for\ any\ p \in Im\, \Omega. \end{cases}$$

2.9 Definition [17]

Let Ω represent a fuzzy subset of $\mathcal{K}$. The right fuzzy annihilator $R(\Omega)$ of the fuzzy subset Ω is defined as,

$$R(\Omega)(z) = \begin{cases} \max\{p \mid z \in R(\Omega_p)\}, \\ 0 \; if \; z \notin L(\Omega_p) \; for \; any \; p \in Im\,\Omega. \end{cases}$$

If $\mathcal{R}$ is commutative ring, then $L(\Omega) = R(\Omega)$.

2.10Theorem [17]

If Ω and Γ are fuzzy subsets of $\mathcal{K}$, then,

i.$\Omega \subset L(R\,(\Omega))$,
ii.if $\Omega \subset \Gamma$ then $R(\Gamma) \subset R(\Omega)$ and $L(\Gamma) \subset L(\Omega)$,
iii.$R(\Omega) = R\left(L\left(R\,(\Omega)\right)\right)$.

2.11Definition [14]

Let $\{\Omega_\alpha \mid \alpha \in I\}$ be a family of fuzzy subsets of ring $\mathcal{K}$. The direct sum of these fuzzy subsets is denoted by $\sum_{\alpha\in I}\Omega_\alpha$ and for all $r \in \mathcal{K}$ defined as,

$$\left(\sum_{\alpha\in I}\Omega_\alpha\right)(r) = \sup\left\{\inf_{\alpha\in I}\;\{\Omega_\alpha(r_\alpha)\} \mid r = \sum_{\alpha\in I} r_\alpha\right\}$$

2.12Theorem [14]

If $\{\Omega_\alpha \mid \alpha \in I\}$ is a collection of fuzzy subrings (or fuzzy ideals) of $\mathcal{K}$, then $\sum_{\alpha\in I}\Omega_\alpha$ is a fuzzy subring (or fuzzy ideal) of ring $\mathcal{K}$ and $\Omega_\alpha \subseteq \sum_{\alpha\in I}\Omega_\alpha$ for all $\alpha \in I$.

2.13Theorem [14]

If $\{\Omega_i \mid i \in I\}$ is a collection of subrings of $\mathcal{K}$, then $\bigcap_{i\in I}\Omega_i$ forms a fuzzy subring of $\mathcal{K}$.

2.14Theorem [19]

Let Ω represent a fuzzy subset of $\mathcal{K}$. For an element $r \in \mathcal{K}$ and $0 < \beta \leq 1$, the fuzzy point r_β associated with Ω is defined by the following rule:

$$r_\beta(s) = \begin{cases} \alpha & if\; s = r, \\ 0 & if\; s \neq r. \end{cases}$$

3. Fuzzy Baer Subring

This section outlines a fuzzy framework for introducing the concepts of fuzzy Baer subrings and fuzzy Rickart subrings. Within this framework, we introduce the idea of a fuzzy subring generated by an element. Additionally, we utilize the concept of fuzzy left (or right) annihilators, as introduced by Medhi [17]. Furthermore, the fundamental properties of fuzzy Baer subrings and fuzzy Rickart subrings are examined in detail in this section.

3.1 Definition

A fuzzy subring Ω of $\mathcal{K}$ is said to be generated by an element a in $\mathcal{K}$ if and only if,

$$\Omega_a = \{r \in \mathcal{K} \mid \Omega(r) = \Omega(0)\} = a\mathcal{K}$$

3.2 Example

Consider a fuzzy subring $\Omega: \mathbb{Z} \to [0, 1]$ defined by

$$\Omega(r) = \begin{cases} 0.8 & if\ r \in 2\mathbb{Z}, \\ 0 & otherwise. \end{cases}$$

The set $\{r \in \mathcal{R} \mid \Omega(r) = \Omega(0)\} = 2\mathbb{Z}$. Therefore, Ω represents a fuzzy subring of $\mathbb{Z}$ generated by the element $2 \in \mathbb{Z}$.

3.3 Theorem

If Ω is a left fuzzy ideal of ring $\mathcal{K}$, then $L\ (\Omega)$ is a left fuzzy ideal of $\mathcal{K}$.

Proof:

We have, Ω is a left fuzzy ideal of ring $\mathcal{K}$. Let $r, s \in \mathcal{K}$. It suffices to prove that $L\ (\Omega)(r - s) \geq \min\{L\ (\Omega)(r), L\ (\Omega)(s)\}$ and $L\ (\Omega)(rs) \geq L\ (\Omega)(s)$. Consider $L\ (\Omega)(r) = p$ and $L\ (\Omega)(y) = q$. Since Ω is a fuzzy ideal of ring $\mathcal{K}$, by Theorem 2.7, $\Omega_{\min\{p,q\}}$ is an ideal of ring $\mathcal{K}$. Hence, $r - s \in L\left(\Omega_{\min\{p,q\}}\right)$. Therefore, $L\ (\Omega)(r - s) \geq \min\{p, q\} = \min\{L\ (\Omega\ )(r), L\ (\Omega)(s)\}$. Thus, $L\ (\Omega)(r - s) \geq \min\{L\ (\Omega)(r), L\ (\Omega)(s)\}$.

Additionally, $\Omega_{\max\{p,q\}}$ is an ideal of ring $\mathcal{K}$. Hence, $rs \in L\left(\Omega_{\max\{p,q\}}\right)$. Thus, $L\ (\Omega)(rs) \geq \max\{p, q\} = \max\{L\ (\Omega)(r), L\ (\Omega)(s)\} \geq L\ (\Omega)(s)$. Hence, $L\ (\Omega)$ is a left fuzzy ideal of $\mathcal{K}$.

3.4 Example

Consider a fuzzy ideal $\Omega: \mathbb{Z}_6 \to [0, 1]$ defined by

$$\Omega(r) = \begin{cases} 0.9 & if\ r \in \{0, 2, 4\}, \\ 0.1 & otherwise. \end{cases}$$

The left fuzzy annihilator $L\ (\Omega): \mathbb{Z}_6 \to [0, 1]$ given by

$$L\ (\Omega) = \begin{cases} 0.9 & if\ r \in \{0, 3\}, \\ 0 & otherwise. \end{cases}$$

It can be easily observed that $L\ (\Omega)$ is a left fuzzy ideal of $\mathbb{Z}_6$.

3.5 Definition

A fuzzy subring Ω of $\mathcal{K}$ is said to be generated by an idempotent e in $\mathcal{K}$ if the set

$\{r \in \mathcal{K} \mid \Omega\ (r) = \Omega\ (0)\} = e\mathcal{K}$.

3.6 Theorem

Let e be an idempotent in a ring $\mathcal{K}$. A fuzzy subring Ω is generated by an idempotent e if and only if $L\ (\Omega)$ is the fuzzy subring generated by an idempotent $1 - e$.

Proof:

Let Ω be fuzzy subring of $\mathcal{K}$ generated by an idempotent e. Hence, we have $\{r \in \mathcal{K} \mid \Omega(r) = \Omega(0)\} = e\mathcal{K}$. By Definition 2.8, the left fuzzy annihilator $L\ (\Omega)$, is defined as:

$$L\ (\Omega)(z) = \begin{cases} \max\{t \mid z \in L\ (\Omega_t)], \\ 0\ if\ z \notin L\ (\Omega_t)\ for\ any\ t \in Im\ \Omega. \end{cases}$$

Consider $r \in\ \mathcal{K}$ such that $L\ (\Omega)(r) = L\ (\Omega)(0)$. Hence $r \in L\ (\Omega_t)$ for $t = \sup_{z \in \mathcal{K}}\{\Omega(z)\}$ and $\Omega(e\mathcal{K}) = \Omega(0)$. Therefore $e\mathcal{K} \subseteq \Omega_t$ and $e\mathcal{K}r = 0$. Then $er = 0$ and $r = (1-e)r \in (1-e)\mathcal{K}$.

Conversely, suppose $L\ (\Omega)$ is fuzzy subring generated by an idempotent $1-e$. Consider $r \in \mathcal{K}$ such that $\Omega(r) = \Omega(0)$. Therefore $r \in \Omega_k$ for some $k = \sup_{z \in \mathcal{K}}\{\Omega(z)\}$. Hence there exist an element $(1-e)s$ for some non zero $s \in \mathcal{K}$ such that $r(1-e)s = (r-re)s = 0$. Thus $r = re \in e\mathcal{K}$.

3.7 Example

Let $e = 3$ be idempotent of $\mathbb{Z}_6$ and Ω be fuzzy subring of $\mathbb{Z}_6$ generated by idempotent $e = 3$ defined by,

$$\Omega(r) = \begin{cases} 0.9 & if\ r \in \{0,3\}, \\ 0.1 & otherwise. \end{cases}$$

Then

$$L\ (\Omega)(r) = \begin{cases} if & r \in \{0,2,4\}, \\ 0 & otherwise. \end{cases}$$

It is clearly observed that $L\ (\Omega)$ is the fuzzy subring generated by an idempotent $1-e = 1-3 = -2 = 4$ in $\mathbb{Z}_6$.

Kaplansky [9] introduced the concept of a Baer ring. A ring $\mathcal{K}$ is defined as a Baer ring if the left (or right) annihilator of every nonempty subset of $\mathcal{K}$ is a left (or right) ideal generated by an idempotent in $\mathcal{K}$. The following is a fuzzy framework for fuzzy Baer subrings, utilizing the notion of a fuzzy subring generated by an idempotent and the fuzzy left (or right) annihilator of a ring.

3.8 Definition

Let $\mathcal{B}$ be a fuzzy subring of ring $\mathcal{K}$. If for any subset $\Omega \subseteq\ \mathcal{B}$ the left annihilator $L\ (\Omega)$ is generated by an idempotent in $\mathcal{K}$, then $\mathcal{B}$ is called the fuzzy Bear subring of ring $\mathcal{K}$.

3.9 Example

Let $\mathcal{B}$ be a fuzzy subring of $\mathbb{Z}_4$ defined by

$$\mathcal{B}(r) = \begin{cases} 0.8 & if\ r \in \{0,2\}, \\ 0.1 & otherwise. \end{cases}$$

For any subset of $\Omega \subseteq\ \mathcal{B}$ the left annihilator $L\ (\Omega)$ is given by

$$L\ (\Omega)(r) = \begin{cases} k & if\ r \in \{0,2\}, \\ 0 & otherwise, \end{cases} \quad 0 \le k \le 0.8$$

Since the set $\{r \in \mathbb{Z}_4 \mid L(\Omega)(r) = L(0)\} = \{0, 2\}$ is not generated by an idempotent in $\mathbb{Z}_4$. Hence $\mathcal{B}$ is not the fuzzy Baer subring of $\mathbb{Z}_4$

3.10 Example

Let $\mathcal{B}$ be a fuzzy subring of $\mathbb{Z}_5$ defined by

$$\mathcal{B}(r) = \begin{cases} 0.9 & if\ r = 0, \\ 0.2 & otherwise. \end{cases}$$

For any subset of $\Omega \subseteq \mathcal{B}$ the left annihilator $L(\Omega)$ is given by

$$L(\Omega)(r) = \begin{cases} k & if\ r = 0, \\ 0 & otherwise, \end{cases} \quad 0 \le k \le 0.9$$

Since the set $\{r \in \mathbb{Z}_5 \mid L(\Omega)(r) = L(0)\} = \{0\}$ is generated by an idempotent in $\mathbb{Z}_5$. Hence $\mathcal{B}$ is the fuzzy Baer subring of $\mathbb{Z}_5$.

3.11 Theorem

Let Ω be a fuzzy Baer subring of $\mathcal{K}$. If $\psi \subseteq \mathcal{K}$ is any fuzzy subset of Ω, then ψ is also a fuzzy Baer subring of $\mathcal{K}$.

Proof:

Straightforward.

3.12 Theorem [23]

If $\{\Omega_i \mid i \in I\}$ is a family of left fuzzy ideals of $\mathcal{K}$, then $\bigcap_{i \in I} \Omega_i$ is a fuzzy left ideal of $\mathcal{K}$.

Proof:

Let $\Omega = \bigcap_{i \in I} \Omega_i$ and $r, s \in \mathcal{K}$. Since intersection of any family of fuzzy subrings is fuzzy subrings, so it is enough to prove that $\Omega(rs) \ge \Omega(s)$. Consider $\Omega(rs) = \bigcap_{i \in I} \Omega_i(rs) = \inf_{i \in I}\{\Omega_i(rs)\} \ge \inf_{i \in I}\{\Omega_i(s)\} = \bigcap_{i \in I} \Omega_i(s) = \Omega(s)$

3.13 Theorem

If $\{\Omega_i \mid i \in I\}$ is a family of right fuzzy ideals of $\mathcal{K}$, then $\bigcap_{i \in I} \Omega_i$ is a fuzzy right ideal of $\mathcal{K}$.

Proof:

Straightforward.

3.14 Theorem

If $\{\Omega_i \mid i \in I\}$ is a family of fuzzy ideals of $\mathcal{K}$, then $\bigcap_{i \in I} \Omega_i$ is a fuzzy ideal of $\mathcal{K}$.

Proof:

It follows directly from Theorem 3.12 and Theorem 3.13.

3.15 Theorem

If $\mathcal{B}$ is fuzzy Baer subring of $\mathcal{K}$ and $\Omega_i \subseteq \mathcal{B}$ is any family of subsets of $\mathcal{B}$, then $\bigcap_{i \in I} \Omega_i$ is also fuzzy Baer subring of $\mathcal{K}$.

Proof:

It follows directly from Theorem 3.11 and Theorem 3.14.

3.16 Theorem

If $\{\Omega_i \mid i \in I\}$ is collection of fuzzy Baer subrings of $\mathcal{K}$, then $\sum_{\alpha\in I} \Omega_\alpha$ is a fuzzy Baer subring of $\mathcal{K}$.

Proof:

It is enough to prove that $L\left(\sum_{\alpha\in I} \Omega_\alpha\right) = \bigcap_{i \in I} L(\Omega_\alpha)$. Since $\Omega_\alpha \subseteq \sum_{\alpha\in I} \Omega_\alpha$, we have $L\left(\sum_{\alpha\in I} \Omega_\alpha\right) \subseteq L(\Omega_\alpha)$. Hence, $L\left(\sum_{\alpha\in I} \Omega_\alpha\right) \subseteq \bigcap_{i \in I} L(\Omega_\alpha)$.

Let $\bigcap_{i \in I} L(\Omega_\alpha)(r) = p$, which implies $L(\Omega_\alpha)(r) \geq p$. Let $L(\Omega_\alpha)(r) = p_\alpha$. Therefore $r \in L\big((\Omega_\alpha)_{p_\alpha}\big)$. If $q = \sup_{\alpha \in I} \{p_\alpha\}$ and $(\Omega_\alpha)_q \subseteq (\Omega_\alpha)_{p_\alpha}$. Thus $L\big((\Omega_\alpha)_{p_\alpha}\big) \subseteq L\big((\Omega_\alpha)_q\big)$, and therefore $r \in L\big((\Omega_\alpha)_q\big)$. Hence $r \in \bigcap_{\alpha \in I} L\big((\Omega_\alpha)_q\big) \subseteq L\left(\sum_{\alpha\in I}(\Omega_\alpha)_q\right) \subseteq L\left(\sum_{\alpha\in I} \Omega_\alpha\right)_q \subseteq \left(L\left(\sum_{\alpha\in I} \Omega_\alpha\right)\right)_q$, which implies $L\left(\sum_{\alpha \in I} \Omega_\alpha\right)(r) \geq q \geq p_\alpha$. It follows that $L\left(\sum_{\alpha \in I} \Omega_\alpha\right)(r) \geq L(\Omega_\alpha)(r)$ and $L\left(\sum_{\alpha \in I} \Omega_\alpha\right)(r) \geq \bigcap_{\alpha \in I} L(\Omega_\alpha)(r)$. Thus, $\bigcap_{\alpha \in I} L(\Omega_\alpha) \subseteq L\left(\sum_{\alpha \in I} \Omega_\alpha\right)$. Hence $L\left(\sum_{\alpha \in I} \Omega_\alpha\right) = \bigcap_{\alpha \in I} L(\Omega_\alpha)$. By Theorem 3.15, $\sum_{\alpha \in I} \Omega_\alpha$ is a fuzzy Baer subring of $\mathcal{K}$.

3.17 Definition

Let Ω be a fuzzy subring of ring $\mathcal{K}$. If for any fuzzy point x_α of Ω, where $x \in \mathcal{K}$ and $0 < \alpha \leq \Omega(x)$, the left annihilator $L(x_\alpha)$ is generated by an idempotent in $\mathcal{K}$, then Ω is fuzzy Rickart subring of $\mathcal{K}$.

3.18 Example

Let Ω be a fuzzy subring of $\mathbb{Z}$ defined as,

$$\Omega(r) = \begin{cases} 0.9 & if\ r \in 2\mathbb{Z}, \\ 0.2 & otherwise. \end{cases}$$

For any $x \in \mathbb{Z}$ and $0 < \alpha \leq \Omega(r)$, the fuzzy point x_α is defined as,

$$x_\alpha(y) = \begin{cases} \alpha & y = x, \\ 0 & y \neq x. \end{cases}$$

Then,

$$L(x_\alpha)(y) = \begin{cases} \alpha & y = 0, \\ 0 & otherwise, \end{cases}$$

And $\{y \in \mathbb{Z} \mid L(x_\alpha)(y) = L(x_\alpha)(0)\} = \{0\}$ is generated by an idempotent in $\mathbb{Z}$. Hence, Ω is fuzzy Rickart subring of $\mathbb{Z}$.

3.19 Example

Let Ω be a fuzzy subring of $\mathbb{Z}_8$ defined as,

$$\Omega(r) = \begin{cases} 0.8 & if\ r \in \{0, 2, 4, 6\}, \\ 0.2 & otherwise. \end{cases}$$

For $2 \in \mathbb{Z}_8$ and $0 < 0.1 \leq \Omega(2)$, the fuzzy point $2_{0.1}$ is defined as,

$$2_{0.1}(y) = \begin{cases} 0.1 & y = 2, \\ 0 & y \neq 2. \end{cases}$$

Then,

$$L\,(2_{0.1})(y) = \begin{cases} 0.1 & y \in \{0, 4\}, \\ 0 & otherwise. \end{cases}$$

and $\{y \in \mathbb{Z}_8 \mid L\,(2_{0.1})(y) = L(2_{0.1})(0)\} = \{0, 4\}$ is not generated by an idempotent in $\mathbb{Z}_8$. Hence, Ω is not fuzzy Rickart subring of $\mathbb{Z}_8$.

3.20 Theorem

Every fuzzy Baer subring is fuzzy Rickart subring.

Proof: Straightforward.

4. Fuzzy Idempotent Subring

In this section, we present a fuzzy framework for identifying a fuzzy subset as a fuzzy idempotent subset and explore some of its fundamental properties. This framework provides a systematic approach to analyze and characterize idempotent behaviors within fuzzy subrings, enhancing our understanding of their structural and functional attributes.

4.1 Definition

A fuzzy subset ψ of a ring $\mathcal{K}$ is said to be fuzzy idempotent subset if and only if $\psi^2 = \psi$.

4.2 Theorem

Let e be an idempotent in a ring $\mathcal{K}$. Consider a fuzzy subset ψ of $\mathcal{K}$ defined as,

$$\psi_e(x) = \begin{cases} t_0 & if\ x \in \langle e \rangle, \\ t_1 & otherwise, \end{cases}$$

where $t_0 > t_1$. Then, ψ_e is a fuzzy idempotent subring of $\mathcal{K}$.

Proof:

Since $\langle e \rangle$ is the principal ideal generated by the idempotent element $e \in \mathcal{K}$, it follows that $\langle e \rangle$ is a subring of $\mathcal{K}$. Consequently, ψ_e is a fuzzy subring of $\mathcal{K}$. To establish that ψ_e is a fuzzy idempotent subring, we need to prove that $\psi_e^2(x) = \psi_e(x)$ for all $x \in \mathcal{K}$.

i. If $x \in \langle e \rangle$, there exist some $r \in \mathcal{K}$ such that $x = er = e.er$. Thus,
$\psi_e^2(x) = \min\{\psi_e(e), \psi_e(er)\} = t_0 = \psi_e(x)$.

ii. If $x \notin \langle e \rangle$ then,

$\psi_e^2(x) = \min\{\psi_e(u), \psi_e(v)\} = t_1 = \psi_e(x)$, for some $u, v \notin \langle e \rangle$.

Hence, ψ_e satisfies the condition of being a fuzzy idempotent subring of $\mathcal{K}$.

4.3 Example

Consider the fuzzy subset ψ_3 of the ring $\mathbb{Z}_6$ defined for the idempotent element $3 \in \mathbb{Z}_6$ as follows:

$$\psi_3(x) = \begin{cases} 0.7 & if\ x \in \{0, 3\}, \\ 0.3 & otherwise. \end{cases}$$

Then, ψ_3 is a fuzzy idempotent subring of the ring $\mathbb{Z}_6$.

4.4 Theorem

A fuzzy subring of the form

$$\psi_0(x) = \begin{cases} t_0 & if\ x = 0, \\ t_1 & x \in \mathcal{K} - \{0\}, \end{cases}$$

with $t_0, t_1 \in Im\,\psi$ and $t_0 > t_1$ is always a fuzzy idempotent subring of the ring $\mathcal{K}$.

Note that $1 - \psi_e$ is fuzzy subset of $\mathcal{K}$ defined as, $(\psi_e)(x) = 1 - \psi_e(x)$

4.5 Theorem

Let e be an idempotent and ψ_e be the fuzzy idempotent subset of the ring $\mathcal{K}$, then $(1 - \psi_e)$ is the fuzzy idempotent subset of the ring $\mathcal{K}$.

Proof:

Let $x \in \mathcal{K}$ and ψ_e be the fuzzy idempotent subset of the ring $\mathcal{K}$. Thus, we have $\psi_e^2(x) = \psi_e(x)$ for all x. Now consider,

$$\begin{aligned} (1 - \psi_e)^2(x) &= (1 - \psi_e) \circ (1 - \psi_e)(x) = \sup_{x=yz}\{\min\{(1 - \psi_e)(y), (1 - \psi_e)(z)\}\} \\ &= 1 - \sup_{x=yz}\{\min\{\psi_e(y), \psi_e(z)\}\} = 1 - (\psi_e \circ \psi_e)(x) = 1 - \psi_e^2(x) = 1 - \psi_e(x) \\ &= (1 - \psi_e)(x) \end{aligned}$$

Hence, $(1 - \psi_e)$ is a fuzzy idempotent subset of the ring $\mathcal{K}$.

5. Conclusion

Classical concepts from ring theory have been extended into fuzzy subrings in this paper by introducing the concepts of fuzzy Baer subrings, fuzzy Rickart subrings, and fuzzy idempotent subrings. The gap between traditional algebraic structures and fuzzy set theory has been bridged by utilizing the concepts of fuzzy subsets generated by an element, fuzzy left and right annihilators, and the notion of fuzzy points. The understanding of fuzzy analogs of Baer and Rickart rings has been deepened, and the broader potential of fuzzy algebra in capturing uncertainty within algebraic frameworks has been highlighted. The introduction of fuzzy idempotent subrings has further generalized classical results, opening new avenues for research in fuzzy ring theory and its applications across various fields. Significant contributions have been made to the expanding body of knowledge in fuzzy mathematics and its relationship with classical ring theory, encouraging further exploration of fuzzy structures in algebraic contexts.

Ethics declarations

Conflicts of interest: The authors confirm that there are no conflicts of interest.

Ethical approval: This article does not include any studies involving human participants or animals conducted by the authors.

Informed consent: As the article does not involve human participants, informed consent is not applicable.

Refrences

[1] Rasuli R (2018). Characterization of Q-fuzzy subrings (Anti Q-fuzzy subrings) with respect to a T-norm (T-conorm). Journal of Information and Optimization Sciences 39(4): 827-837.
[2] Banerjee B (2003). Intuitionistic fuzzy subrings and ideals. J. Fuzzy Math. 11(1): 139-155.
[3] Banerjee, B. (2003). Intuitionistic fuzzy subrings and ideals. J. Fuzzy Math., 11(1), 139-155.
[4] Clark, W. E. (1967). Twisted matrix units semigroup algebras.
[5] Dixit VN, Kumar & Ajmal N (1992) On fuzzy rings. Fuzzy Sets and Systems 49(2):205–213. https://doi.org/10.1016/0165-0114(92)90325-X
[6] Dogra S & Pal M (2021). Picture fuzzy subring of a crisp ring. Proceedings of the National Academy of Sciences, India Section A: Physical Sciences 91: 429-434.
[7] Dogra, S., & Pal, M. (2021). Picture fuzzy subring of a crisp ring. Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, 91, 429-434.
[8] Engino˘glu, S., & C¸ a˘gman, N. (2020). Fuzzy parameterized fuzzy soft matrices and their application in decision-making. TWMS Journal of Applied and Engineering Mathematics, 10(4), 1105.
[9] Kaplansky I (1968) Rings of operators. W. A. Benjamin, Inc. New York-Amsterdam.
[10] Liu WJ (1982) Fuzzy invariant subgroups and fuzzy ideals. Fuzzy Sets and Systems 8(2):133-139. https://doi.org/10.1016/0165-0114(82)90003-3
[11] Liu, W. J. (1982). Fuzzy invariant subgroups and fuzzy ideals. Fuzzy sets and Systems, 8(2), 133-139.
[12] Maheswari P U, Arjunan K, & Mangayarkarasi R (2016). Notes on bipolar valued fuzzy subrings of a rings. Int. J. Appl. Math. Sci. 9: 89-97.
[13] Maheswari, P. U., Arjunan, K., & Mangayarkarasi, R. (2016). Notes on bipolar valued fuzzy subrings of a rings. Int. J. Appl. Math. Sci, 9, 89-97.
[14] Malik DS, & Mordeson JN. (1992). Fuzzy direct sums of fuzzy rings. Fuzzy sets and systems 45(1): 83-91.
[15] Mandal, D. (2014). Fuzzy ideals and fuzzy interior ideals in ordered semirings. Fuzzy Information and Engineering, 6(1), 101-114.
[16] MASSA'DEH, M., Fallatah, A., & Al-Refai, O. (2021). HOMOMORPHISM IN BIPOLAR Q-FUZZY SOFT Γ-SEMIRING. TWMS Journal of Applied & Engineering Mathematics, 11(1).
[17] Medhi U & Saikia HK (2012) Fuzzy Aspects of Rings with Chain Conditions. International Journal of Fuzzy Mathematics and Systems, 2 (1), 11-20.
[18] Mondal, U., Pal, M., & Das, K. (2024). Interval-valued intuitionistic fuzzy soft graphs with application. TWMS Journal Of Applied And Engineering Mathematics.
[19] Pao-Ming, P., & Ying-Ming, L. (1980). Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore-Smith convergence. Journal of mathematical analysis and applications, 76(2), 571-599.
[20] Rosenfeld, A. (1971). Fuzzy groups. Journal of mathematical analysis and applications, 35(3), 512-517.
[21] Wang-jin L (1983) Operations on fuzzy ideals. Fuzzy Sets and Systems 11(1):31-39. https://doi.org/10.1016/S0165-0114(83)80067-0
[22] Yue, Z. (1988). Prime L-fuzzy ideals and primary L-fuzzy ideals. Fuzzy sets and systems, 27(3), 345- 350.
[23] Zadeh, L. A., (1965), Fuzzy sets, Information and control, 8(3), pp. 338-353.