Advances in Nonlinear Variational Inequalities
ISSN: 1092-910X
Vol 28 No. 1s (2025)

Weakly Quasi-Primary K- Modules

Muntaha Abdul- Razaq Hasan
Department of Mathematics ,College of Basic Education , Mustansiriyah University ,Baghdad-,Iraq.

Article History:

Received: *13-08-2024*

Revised: *01-10-2024*

Accepted: *17-10-2024*

Abstract:

Important areas for study in the field of modules include prime modules .There has already been an introduction to the idea of quasi-prime modules .Our focus recently has been on weakly quasi-prime modules as a means for investigating one of these generalizations .The main objective of this work is to introduce a new type of module , namely weakly quasi-primary K-modules . A module D whenever $ann_K D = \sqrt{ann_K\, r\, D}$

For each r $\notin ann_K$ D, is called weakly quasi-primary K-module , which is a generalization of weakly quasi-prime K-module ,whenever ann_K D=ann_K r D for every r$\notin ann_K$ D , then D is weakly quasi-prime K-module . Here the author obtain some characterize of weakly quasi-primary K-modules and investigated some properties of weakly quasi-primary K-modules .We prove relationships between weakly quasi-primary K-module and other modules , like prime module, quasi-prime module , weakly quasi-prime module and primary module are study in this paper .

Keywords: Prime modules , Quasi-Prime modules ,Primary modules ,Weakly quasi-primary modules.

1. Introduction

In this work , K be commutative ring with every ideal is prime and D be a unitary K-module . In [1] the concept of prime module was introduce ,where an K-module D is called prime if for each B sub-module of D, ann_K B =ann_K D . It is proved in [2], that a K-module is primary if and only if $\sqrt{\text{ann}_K D}$=$\sqrt{\text{ann}_K B}$ for each sub-module B of D . In [3] Roman give another definition of primary K-modules ,where an K-module is primary if ann_K B is primary ideal for every non-zero submodule B of an K-module D .In [4],Fuchs define proper ideal J is primary if for x y ∈ J,then either x∈ J or y^n ∈ J for every x, y in K . Hasan , in [5] give a definition of weakly quasi-prime modules , where D is weakly quasi-prime modules if $ann_K D = ann_K rD$ for each r$\notin ann_K D$. the concept of weakly quasi-prime K- module motivated us to introduce and study weakly quasi-primary K-module and we give several characterized by this module . A part 2 of this paper devoted to study the definition of weakly quasi-primary K-module and some description for this module . When D be multiplication K-module and L$\nsubseteq ann_K$r D , where L be any proper ideal of the ring K , r∈ K , then every sub-module of weakly quasi-primary K-module is weakly quasi-primary K-module, we investigate the condition make weakly quasi-primary K-module equivalent with prime K-module and weakly quasi-prime K-module and we end the section by the results of direct sum and direct summand of weakly quasi-primary K-module . In section 3 we study the relation between weakly quasi-primary K-module , quasi prime and primary K-module.

2. Weakly quasi-primary K-modules

The main goal for this section has been to establish a new concept known as weakly quasi- primary K-module . The study also explores the hereditary characteristics of weakly quasi-primary K-module.

Definition (2-1):D is said to be weakly quasi-primary K- module if $ann_K D$ =$\sqrt{ann_K rD}$,for every r∉ $ann_K D$.

Examples and Remarks (2.2):

1-Z as Z-module is weakly quasi-primary Z-module , since

ann_Z Z=0 = $\sqrt{ann_Z\, rZ}$.

2- Z_6 as Z-module is not weakly quasi-primary module ,since

ann_Z Z_6 =6Z and $\sqrt{ann_Z 2Z_6}$ = $\sqrt{3Z}$ =3Z, so 6Z≠ 3Z.

3-Z_n as Z-module is weakly quasi-primary module if and only if n is prime number.

Proof :to prove n is prime number ,suppose n is not prime number ,so there is k ,w ∈ N;k,w < n ,n=kw ,so ann_Z Z_n = $\sqrt{ann_Z rZ_n}$; r ∉ ann_Z Z_n ,so $ann_Z Z_{KW}$=kw ,but $\sqrt{ann_{rZ_{kw}}}$ = $\sqrt{kw}$,while kw ≠ $\sqrt{kw}$ implies that n must be prime number .The conversely ,it is clear.

4−Z_{p^∞} is weakly quasi-primary K-module , since $ann_Z Z_{p^\infty}$ =$\sqrt{ann_K\, rZ_{P^\infty}}$ =0, for r∉ $ann_K Z_{P^\infty}$.

5-It is clear that not every divisible module over integral domain is weakly quasi-primary module.

6-Every weakly quasi-primary K-module is weakly quasi-prime K-module.

Proof :we must prove that ann_K D=ann_K rD for every r∉ $ann_K D$ Since r D ⊆ D ,so ann_K D ⊆ ann_K r D .Let x∈ ann_K r D ⊆ $\sqrt{ann_K rD}$,by[6].Since D is weakly quasi-primary K-module so x ∈ ann_K D leads to ann_K rD ⊆ ann_KD,which mean D is weakly quasi-prime K-module.

But the converse is not true for example : every divisible module over integral domain is weakly quasi-prime module ,see [5],but not weakly quasi-primary module (5 in Examples and Remarks 2.2).

We know that a ring K is called cosemiprime ring if and only if every proper ideal in K is semi-prime ,[6].

Now, we can give the following remark.

Remark (2.3) :let K be cosemiprime ring ,then D is weakly quasi-prime K-module if and only if D is weakly quasi –primary K-module.

Theorem (2.4): Let L be an ideal of ring K; r ∉ ann_K LD,L⊈ $ann_K rD$,D is weakly quasi-primary K-module if and only if $\sqrt{ann_K LD}$=ann_K D.

Proof :since LD ⊆ D leads us to ann_K D ⊆ ann_K LD and by [6],$ann_K LD$ ⊆ $\sqrt{ann_K LD}$,so we have $ann_K D$ ⊆ $\sqrt{ann_K LD}$.Let x ∈ $\sqrt{ann_K LD}$,so x^n ∈ $ann_K LD$ implies x^nLD=0 equivalently to rx^nLD=0,leads us x^nLrD=0,so x^nL⊆ ann_KrD,but L⊈ $ann_K rD$ so x^n ∈ ann_KrD,implies x ∈ $\sqrt{ann_K rD}$,but D is weakly quasi-primary K-module so x ∈ $ann_K D$,so we have the result.

Contrariwise ,if ann_K D=$\sqrt{ann_K LD}$,to prove D is weakly quasi-primary K-module ,since r D⊆ D ,so ann_K D ⊆ ann_K rD⊆ $\sqrt{ann_K rD}$;r∉ $ann_K D$,let x ∈ $\sqrt{ann_K rD}$,so x^n ∈ ann_KrD,implies x^nrD=0 equivalently to Lx^nrD=0 ,but the ring is commutative ,so x^nrLD=0 ,so x^nr∈ ann_KLD so by our hypothesis lead us to x^n ∈ ann_KLD,so x∈ $\sqrt{ann_K}$LD=ann_KD,then we have the result.

Proposition (2.5): every prime K-module is weakly quasi-primary K-module.

Proof: because r D⊆D, so $ann_K D \subseteq ann_K$rD,by [6]we have ann_KD⊆ $\sqrt{ann_K rD}$. Let x∈ $\sqrt{ann_K rD}$; r∉ ann_KD ,then $x^n \in ann_K$r D, thus x^n rD=0,implies x^nr∈ ann_KD, since D is prime module implies by[7] ann_KD is prime ideal,thus $x^n \in ann_K$D,leads us to $x.x^{n-1} \in ann_K$D ,so x.x^{n-1}D=0,suppose that x^{n-1}=h∈ K,so x∈ ann_KhD,but D is prime module so by [7] implies that x∈ ann_KD,thus D is weakly quasi-primary K-module.

But the contrariwise is not true for example :let D=Z⊕ Z_n,so ann_KD=$\sqrt{ann_K r(Z \oplus Z_n})$=0 such that r∉ ann_K(Z⊕ Z_n),so D is weakly quasi-primary module,but D is not prime module ,see[7].

Proposition (2-6):let D be multiplication K-module and L⊈ann_Kr D ,where L be any proper ideal of the ring K ,r∈ K ,then every sub-module of weakly quasi-primary K-module is weakly quasi-primary K-module.

Proof : let D be weakly quasi-primary K-module and B be non-zero submodule of D ,to prove that B is weakly quasi-primary module ;r∉ ann_KB.Since rB⊆B,so ann_KB⊆ ann_KrB,thus ann_KB⊆ $\sqrt{ann_K rB}$.

Let x∈ $\sqrt{ann_K rB}$,so$x^n \in ann_K$rB,implies x^n r B =0,since M is multiplication ,so there exist L of K ,such that B=LD ,leads us x^nrLD=0 equivalently to x^nLrD=0,so x^nL∈ ann_KrD,since L⊈ ann_KrD ,so $x^n \in ann_K$rD,leads us to x∈ $\sqrt{ann_K}$rD,but D is weakly quasi-primary ,so x∈ ann_KD⊆ ann_KB,which mean x∈ $ann_K B$,so we have the result.

Theorem (2-7): the direct sum of two weakly quasi-primary K-module is also weakly quasi-primary K-module.

Proof : let D =$D_1 \oplus D_2$,where $D_1$and D_2 are two weakly quasi-primary K-module to prove that D is weakly quasi-primary K-module. We must achieve ann_KD=$\sqrt{ann_K rD}$, for every r ∉ ann_KD.

$\sqrt{ann_K rD}$ =$\sqrt{ann_K r(D_1 \oplus D_2)}$ =$\sqrt{ann_K (rD_1 \oplus rD_2)}$, see[8].

$= \sqrt{ann_K rD_1 \cap ann_K rD_2}$, by[8]

$=\sqrt{ann_K rD_1} \cap \sqrt{ann_K rD_2}$

$=ann_K D_1 \cap ann_K D_2 = ann_K(D_1 \oplus D_2) = ann_K D.$

By Mathematical induction we have the following:

Corollary (2-8):let D be an K-module ,if D is weakly quasi-primary K-module ,then for any positive integer n,D^n is weakly quasi-primary K-module,where $D^n$is the direct sum of n copies of D.

Recall that a sub module B of an K-module D is called a direct summand of D if and only if there exist a sub-module C of D such that D=B⊕C,[9]

Remark (2-9): a direct summand of weakly quasi- primary K-module is not weakly quasi-primary K-module , for example :let D=Z⊕ Z_8 ,it is clear that ann_KD=$\sqrt{ann_K rD}$=0,but Z_8 is not weakly quasi-primary K-module ,see (3 in examples and Remarks 2-2).

Theorem (2-10): let L be an ideal of K ,which is not contained in ann_KD ,D be multiplication K-module ,then the following statement are equivalent

1-D is prime K-module.

2-D is weakly quasi-primary K-module.

3- D is weakly quasi-prime K-module.

Proof: 1→ 2 , by proposition (2-5)

2 → 3, by (6 in examples and remarks (2-2)

3 → 1, to prove ann_KD=ann_KB, for every B be non-trivial submodule of D. Since B⊆ D, so ann_KD ⊆ ann_KB.Let x ∈ ann_KB,so by [7] we have x∈ ann_Kr B , for every r ∉ ann_KB, implies that x r B=0,but D is multiplication K-module ,so B=LD, where L be an ideal of a ring K .Thus x r LD=0,which mean x L ⊆ ann_KrD,but D is weakly quasi-prime K-module and L⊈ ann_KD ,implies x ∈ ann_KD

3-Relation of weakly quasi-primary K-module with quasi-prime and primary K-module.

This section we delve into the relationships between weakly quasi-primary module and different types of module , like quasi-prime and primary K-modules . Recall that in [7] an K-module D is said to be quasi-prime K-module if and only if ann_KB is a prime ideal for every non-zero submodule B of D and the author obtain some characterization of quasi-prime K-module , where the definition of quasi-prime K-module equivalent with $ann_K B$ =$ann_K rB$,for each sub-module B of D ,r∈ K ,r B≠ 0in general not every weakly quasi-primary module is quasi-prime module for example:Z_{P^∞} is weakly quasi-primary K-module, see (4 in examples and remarks(2-2)),but $Z_{P^\infty}$is not quasi-prime K-module [7]. But if we put the condition in proposition (3-1) we have the result.

Proposition (3-1): let L be an ideal of commutative ring ,where L ⊈ ann_KD,D be multiplication K-module ,then every weakly quasi-primary K-module is quasi-prime K-module.

Proof : by [7] we must prove that ann_KB=ann_KrB, for each submodule B of D such that rB≠0;r∈ K.Since rB⊆ B,so ann_KB⊆ ann_KrB.Let x ∈ ann_Kr B, then x r B=0 , but B is multiplication K-module ,so B=LD ,for L be ideal of K ,thus x r LD =0,implies x L⊆ $ann_K rD$ ⊆ $\sqrt{ann_K} rD$, but D is weakly quasi-primary K-module, so x L⊆ $ann_K D$, where L⊈ $ann_K D$, implies x∈ $ann_K D$,so x∈ $ann_K B$, which mean B is quasi-prime K-module.

Proposition (3-2): let D be multiplication K-module , then every quasi-prime K-module is weakly quasi-primary K-module.

Proof : by[7] we have every multiplication quasi-prime K-module is prime K-module and by proposition(2-5)we have the result.

Now , we give the following theorem from proposition(3-1)and proposition(3-2)

Theorem (3-3): let L be an ideal of commutative ring ,where L ⊈ann_KD,D be multiplication K-module,then D is weakly quasi-primary K-module if and only if D is quasi-prime K-module.

Proposition (3-4): every cyclic quasi-prime K-module is weakly quasi-primary K-module.

Proof : from [7] and proposition (2-5) we have the result.

Roman in[3] give a definition of primary K-module ,where an K-module is primary if $ann_K B$ is primary ideal, for every non-zero sub-module B of K-module D ,where an ideal L is primary ideal if for x y∈L,implies eather x∈L or y^n ∈L;n>0,[8].

Theorem (3-5): let D be multiplication, Let L be an ideal of ring K; r ∉$ann_K LD$, L⊈$ann_K rD$, then D is primary K-module if and only if D is weakly quasi-primary K-module.

Proof : to prove D is weakly quasi-primary K-module, since r D⊆D,so ann_KD⊆ $ann_K rD$ ⊆ $\sqrt{ann_K rD}$. Let x∈ $\sqrt{ann_K rD}$,implies x^n ∈ $ann_K rD$,which mean x.x^{n-1} ∈ $ann_K rD$,but D is primary,so either x∈ $ann_K rD$ or x^{n-1} ∈ $ann_K rD$.if x∈ $ann_K rD$,then xrD=0,which mean xr∈ $ann_K D$,where r∉ $ann_K D$,implies x∈ $ann_K D$,so we have the result.If x^{n-1} ∈ $ann_K rD$,so

x^{n-1}rD=0,which mean x.x^{n-2}rD=0 ,by n –copies we have x∈ $ann_K D$.Thus D is weakly quasi-primary K-module.

Contrariwise , by [2],an K-module D is primary if and only if $\sqrt{ann_K D}$=$\sqrt{ann_K B}$ .Since B⊆ D,so $ann_K D \subseteq ann_K B \subseteq \sqrt{ann_K B}$ $*$. Let x∈ $\sqrt{ann_K D}$, so $x^n \in ann_K D$, implies x^nD=0.Thus $x^n rD = 0$,so $x^n \in ann_K rD$,which mean x∈ $\sqrt{ann_K rD}$,but D is weakly quasi-primary K-module ,implies x∈ $ann_K D$,therefore by *we obtain x∈ $\sqrt{ann_K B}$. conversely ,let x∈ $\sqrt{ann_K} B$,so $x^n \in ann_K B$,which mean $x^n B$=0,but D is multiplication ,so there exist an ideal L of a ring K ,such that B=L D. Thus $x^n LD = 0$,which mean $x^n LD$=0,lead us $x^n \in ann_K LD$,so x∈ $\sqrt{ann_K} LD$ and by theorem (2-4),we obtain x∈ $ann_K D$,so x∈ $\sqrt{ann_K} D$,which mean that D is primary K-module.

Note that ,the condition M is multiplication in theorem(3-5)is necessary as it is show in the following example:$Z_{P^\infty}$is weakly quasi-primary K-module, see examples and remarks(4 in 2-2),but Z_{P^∞} is not primary ,since $\sqrt{ann_K Z_{P^\infty}}$=0,but $\sqrt{ann_K(\frac{1}{P^2} + Z)}$=$\sqrt{P^2}Z$=P$Z \neq 0$.

Conclusion

In this article ,we introduced a novel concept called weakly quasi-primary K-module and achieved several intriguing findings , including various new characterizations for that notion .The link between weakly quasi-primary K-module with other modules is adopted and we found the conditions for equivalence of weakly quasi-primary K-modules with other modules.

Acknowledgement

The author thanks the editor and anonymous reviewers for their useful comments and suggestions . Additionally , the author would like to thank Mustansiriyah University

(www.uomustansiriyah.edu.iq)Baghdad-Iraq .for its support in the present research.

Refrences

[1] J.Dauns.1980."Prime Modules and One- Sided ideals in Ring Theory and Algeba III",Proceeding of the Third Oklahoma Conference,,Dekker,New yourk(1980):.301-344.
[2] L.H.ALomairy."On Coprimary Modules.".Journal of AL-Nahrain university13,no.3 (2010):.175-177.
[3] J.Roman and F.,Roard.2005."Advanced Linear Algebra",Irvine,Usa.
[4] L.Fuch.."On Quasi-Primary Ideals,Acta Sci.Math. Szeged 11 (1947):174-183.
[5] Hasan,M.,Abdul-Razaq." On Weakly Quasi-prime Modules .", Ibn-AL-Haitham Journal for Pure and Applied Sciences .25,no.1 (2011):421-426.
[6] Hasan,M.Abdul-Razaq."Cosemiprime Ring ."Journal of Global Scientific Research 8,no.1 (2023) :2847-2850.
[7] Hasan,M.Abdul-Razaq.1999."Quasi-Prime Module and Quasi-Prime Submodules", M.Sc.thesis,university of Baghdad,Iraq.
[8] Andersone.F..W.and Fulle,K.R.1973."Ring and Categories of Modules ." University of Oregon.
[9] Kash,F.1982." Modues and Rings ."Academic Press,London,New York.