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# Hypersurfaces of a Finsler Manifold with the Square Root Metric

**Robin Kumar[1], Mohammad Rafee[2]**
[1,2]Department of Mathematics, School of Sciences, RIMT University, Punjab, India.
Corresponding Author: Mohammad Rafee, Email Id: mohd_rafee60@yahoo.com

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**Abstract:**

The study of Finsler manifold with (α,β)-metric is very imporatant aspect of Finsler geometry in terms of its applications. Finsler metrics are extensively used in special theory of relativelty, information science, thermodynamics and geometric optics of anisotropic media. The study of hypersurfaces of a Finsler manifold with some important and special (α,β)-metric has already been studied. The emergence of square root metrics in Finsler geometry can be attributed to various classification concerning (α,β) metric. They have also excellent geometric properties in Finsler geometry. In this paper, we have charecterized the hypersurfaces of a Finsler space with square root (α,β)-metric denoted by F=√(α(α+β)), where α=√(a_ij (x)y^i y^j ) is a Riemannian manifold and β=b_i (x)y^i is a differential 1-form on the Finsler manifold (M,F). Furthermore, it has been found the criterion of hypersurface to be a hyperplane of first kind, hyperplane of second kind and not the hyperplane of the third kind.

**Keywords**: Finsler manifold, Finsler metric, Square root metric, Hypersurface of Finsler manifold, Hyperplane.

**AMS Subject Classification**: 53B40, 53C60

## 1 Introduction

Let $M$ be an $n$-dimensional differential manifold. Define a Finsler metric on the differentiable manifold $M$. This Finsler metric is known as Finsler fundamental fuction on the manifold $M$. Let us first define what is exactly mean by Finsler fundamental function.

### Definition 1.1 (Finsler metric)

We say a function $F: TM \to R$ is a Finsler metric or Finsler fundamental function on the manifold $M$ if $F$ satisfies the following conditions:

1. Regularity of $F$: $F \in C^\infty$ on the slit tangent bundle $TM\setminus\{0\}$ away from zero vectors of the tangent spaces:

That is, $F$ is smooth on $TM\setminus\{0\} = \{(x,y)|x \in M, y \in T_xM, y \neq 0\}$. The smoothness property is desired so that we can apply differential calculus on the Finsler metric $F$.

2. Non-negativity of function $F$:

$F(x,y) \geq 0$ for $x \in M$ and all $y \in T_xM$.

This property ensures that the length of a tangent vector $y \in T_xM$ is either positive or zero. In terms of arc length this property also ensures that arc length defined by the integral 1 is either positive or zero.

3. Positive homogeneity of function $F$:

$F(x, \lambda y) = \lambda F(x, y)$, $\forall\ \lambda > 0$; for $x \in M$ and all $y \in T_x M$.

That is, $F$ is positive 1-homogeneous of first degree in the directional argument $y$. This property ensures that, length of a tangent vector $\lambda y \in T_x M$ is nothing but $\lambda F(x, y) \in R$.

4. Strict convexity of the function $F$:

$F: TM \to R$ is strictly convex over the tangent bundle $TM$.

It is important to note that strictly convex condition is equivalent to the hessian matrix $[g_{ij}]$, where $i, j \in \{1,2,3, \ldots\ldots dim(M)\}$, defined by $\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y) = g_{ij}(x, y)$ is positive definite for any $(x, y) \in TM$. It is the convexity condition on the Finsler metric $F$ that guaranties for the arc length minimization, given by the following formula, of the admissible curves $\gamma: [a, b] \to M$ belonging to the set $C^\infty[a, b]$

$$s[\gamma(t)] = \int_a^b F(\gamma(t), \dot{\gamma}(t))dt \tag{1}$$

can be achieved.

In simpler terms, the convexity condition on a Finsler metric is a geometric requirement that ensures the length of a curve can be properly defined and that the arc length functional $s[\gamma(t)]$ behaves well. This is crucial for tasks such as minimizing or finding geodesics according to the given Finsler metric $F$. Without this convexity, the notion of length and related optimization problems might be undefined or lack unique solutions.

In the definition above, $F(x, y)$ represents the magnitude of the vector $y$ in the tangent space $T_x M$ at a point $x$ on the manifold $M$. This quantity is often called the "$F$-length" or simply the "Finslerian length" of the tangent vector $y$ in $T_x M$. For a fixed point $x \in M$ the function $F(x, .)$ takes any tangent vector $y \in T_x M$ and produces a real number. In physical contexts involving position and direction, you can think of the Finsler metric $F(x, y)$ as having two arguments: $x$ for location and $y$ for velocity at any point $p \in M$.

**Definition 1.2** *(Finsler Manifold)*

A differentiable manifold $M$ that is equipped with a Finsler metric $F(x, y)$ is called a Finsler manifold or Finsler space. This is usually represented as $(M, F)$.

**Definition 1.3** *(Finsler Geometry)*

Consider a Finsler manifold d $(M, F)$, where $F(x, y)$ represents the Finsler metric. The area of geometry that uses the Finsler metric $F(x, y)$ defined on the manifold $M$ is known as Finsler geometry.

In Finsler geometry, an important and unique type of Finsler metric is the $(\alpha, \beta)$-metric. This metric is defined as follows:

**Definition 1.4** ($(\alpha, \beta)$-metric) *Consider a Finsler space denoted as $(M, F(x, y))$, where $F(x, y)$ represents the Finsler fundamental function. This space is said to possess an $(\alpha, \beta)$-metric if the fundamental function $F(x, y)$ can be expressed in the following manner:*

$$F(x, y) = F(\alpha(x, y), \beta(x, y)).$$

In this expression, $F(\alpha(x, y), \beta(x, y))$ is a differentiable function of two variables $\alpha(x, y)$ and

$\beta(x,y)$. Here, $\alpha(x,y)$ is the Riemannian fundamental function defined as $\sqrt{a_{ij}(x)y^i y^j}$, where $a_{ij}(x)$ represents a Riemannian metric tensor, and $\beta(x,y)$ is a differential 1-form defined on the tangent bundle $TM$, with $b_i(x)$ representing a covariant vector field.

The class of $(\alpha,\beta)$-metrics was originally introduced by the renowned geometer M. Matsumoto [10].

Some important examples of $(\alpha,\beta)$-metric are:

**Example 1.5** *The metric defined by*

$$F(x,y) = \alpha + \beta$$

is called Randers metric and the space $(M, F = \alpha + \beta)$ constructed with Randers metric is called Randers space. This metric was first introduced by Physicist G. Randers in 1941 [16], in his study of general relativity.

**Example 1.6** *The metric defined by*

$$F = \frac{\alpha^2}{\beta}, \beta > 0$$

is called Kropina metric. A space $\left(M, F = \frac{\alpha^2}{\beta}\right)$ constructed with Kropina metric is called Kropina space. This metric was introduced by the Russian physicist V.K. Kropina [8]. It has many important and interesting applications in physics, electron optics with a magnetic field, dissipative mechanics and irreversible thermodynamics, relativistic field theory, control theory, evolution and developmental biology [1].

**Example 1.7** *The metric defined by*

$$F = \frac{\alpha^{n+1}}{\beta^n}, \beta > 0$$

is called generalized Kropina metric. A space $\left(M, F = \frac{\alpha^{n+1}}{\beta^n}\right)$ constructed with Kropina metric is called generalized Kropina space.

**Example 1.8** *The metric defined by*

$$F = \frac{\alpha^2}{\alpha - \beta}, \alpha - \beta > 0$$

is called Matsumoto metric. A space $\left(M, F = \frac{\alpha^2}{\alpha-\beta}\right)$ constructed with Matsumoto metric is called Matsumoto space. This metric was first introduced by M. Matsumoto [12] while investigating the model of a Finsler space. This metric is also named as slope metric.

**Example 1.9** *The metric defined by*

$$F = \sqrt{\alpha(\alpha + \beta)}$$

is called square root metric. We say the space $\left(M, F = \sqrt{\alpha(\alpha+\beta)}\right)$ constructed with square root metric Finsler space with square root metric.

**Example 1.10** *The metric defined by*

$$F = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}$$

is called generalized square metric. We say the space $\left(M, F = \frac{(\alpha+\beta)^{n+1}}{\alpha^n}\right)$ constructed with generalized square metric the generalized square space or the Shen's generalized square space.

Among these Finsler metrics this was the square root metric that draws our attention to work with hypersurface of a Finsler space $(M, F)$.

Let us first define the meaning of a hypersurface:

**Definition 1.11** *A hypersurface is an embedded submanifold of co-dimension 1. That is, a submanifold of dimension less that 1 of a given manifold is called hypersuraface of the underlying manifold.*

This means, if we have a manifold $M$ such that dimension of $M$ is $n$, then the submanifold having dimension $n-1$ will be understood as hypersurface of the given manifold.

A submanifold of dimension $n-1$ is generally denoted by the symbol $M^{n-1}$. In this paper, we will also denote a hupersurafce of a manifold $M$ by the symbol $M^{n-1}$.

**Example 1.12** *Let $M = R^2$ be a manifold. Then a straight line denoted by $M^{n-1}$ and defined by linear equation in two variables $x$ and $y$, i.e., $ax + by + c = 0$ is a hypersurface of the underlying manifold $M = R^2$, because dim(M)=dim($R^2$)=2 while the dim($M^{n-1}$)=$n - 1 = 2 - 1 = 1$.*

**Example 1.13** *Let $M = R^3$ be a manifold. Then the unit sphere denoted by $M^{n-1}$ and defined by $x^2 + y^2 + z^2 = 1$ is a hypersuarface of the underlying manifold $M = R^3$, because dim(M)=dim($R^3$)=3 while the dim($M^{n-1}$)=$n - 1 = 3 - 1 = 2$.*

Matsumoto [11], a prominent Finslerian, was the first person who studies the hypersurfaces and charecterised the special hypersurfaces $M^{n-1}$ of a Finsler manifold. He, specifically, characterised the properties of hypersurface $M^{n-1}$ of Randers space [16]. After this, the number of Finslerians dramatically increased to show their interest in Finsler hypersurface $M^{n-1}$. Many authors around the world ([7], [6], [4], [17], [18], [21], [3], [14], [2], [5]) did study the properties of special hypersurface $M^{n-1}$ and derived the conditions under which a Finsler hypersurface $M^{n-1}$ of a Finsler manifold $(M, F(x, y))$ becomes a hyperplane of first kind, second kind but not of the third kind. Aim of the present paper is to investigate the hypersurface $M^{n-1}$ of Finsler space using square root metric $F = \sqrt{\alpha(\alpha+\beta)}$.

## 2 Preliminaries

We consider the Finsler space $(M, F)$, where $F$ is the square root metric, that is given by

$$F = \sqrt{\alpha(\alpha+\beta)}. \tag{2}$$

Using (2), we get

$$F_\alpha = \frac{(2\alpha+\beta)}{2\sqrt{\alpha(\alpha+\beta)}} \tag{3}$$

$$F_\beta = \frac{\alpha}{2\sqrt{\alpha(\alpha+\beta)}} \tag{4}$$

$$F_{\alpha\alpha} = \frac{-\alpha^2}{4[\alpha(\alpha+\beta)]^{\frac{3}{2}}} \tag{5}$$

$$F_{\beta\beta} = \frac{-\alpha^2}{4[\alpha(\alpha+\beta)]^{\frac{3}{2}}} \tag{6}$$

$$F_{\alpha\beta} = -\frac{3\alpha\beta}{4[\alpha(\alpha+\beta)]^{\frac{3}{2}}}. \tag{7}$$

We already know that, in a general Finsler manifold $(M, F)$, the normalized element of support $l_i = \frac{\partial F}{\partial y_i}$ and the angular metric tensor $h_{ij}$ [16] are evaluated by the following formula:

$$l_i = \frac{F_\alpha y_i}{\alpha} + F_\beta b_i \tag{8}$$

$$h_{ij} = pa_{ij} + q_0 b_i b_j + q_1(b_i y_j + b_j y_i) + q_2 y_i y_j, \tag{9}$$

and the coefficients are defined and calculated as follows:

$$y_i = a_{ij} y^j$$

$$p = \frac{FF_\alpha}{\alpha} = \frac{2\alpha+\beta}{2\alpha} \tag{10}$$

$$q_0 = FF_{\beta\beta} = -\frac{\alpha}{4(\alpha+\beta)} \tag{11}$$

$$q_1 = \frac{FF_{\alpha\beta}}{\alpha} = -\frac{3\beta}{4\alpha(\alpha+\beta)} \tag{12}$$

$$q_2 = \frac{F(F_{\alpha\alpha} - \frac{F_\alpha}{\alpha})}{\alpha^2}$$

$$= \frac{3\beta^2 - 4\alpha^2 - 6\alpha\beta}{4\alpha^3(\alpha+\beta)}. \tag{13}$$

We also know that, in a general Finsler manifold $(M, F)$, the fundamental metric tensor $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is evaluated by [16] the following formula:

$$g_{ij} = pa_{ij} + p_0 b_i b_j + p_1(b_i y_j + b_j y_i) + p_2 y_i y_j \tag{14}$$

whereas its coefficients $p, p_0, p_1$ and $p_2$ are defined and calculated as follows:

$$p = \frac{FF_\alpha}{\alpha}$$

$$= \frac{2\alpha+\beta}{2\alpha} \tag{15}$$

$$p_0 = q_0 + F_\beta^2$$

$$= 0 \tag{16}$$

$$p_1 = q_1 + \frac{pF_\beta}{F}$$

$$= \frac{\alpha-\beta}{2\alpha(\alpha+\beta)} \tag{17}$$

$$p_2 = q_2 + \frac{p^2}{F^2}$$

$$= \frac{4\alpha^2 + 2\beta^2 - 5\alpha\beta}{2\alpha^3(\alpha+\beta)}. \tag{18}$$

We know that, in a Finsler manifold $(M,F)$, reciprocal metric tensor of a fundamental metric tensor $g_{ij}=\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is denoted by $g^{ij}$ and is evaluated by the formula [16]

$$g^{ij}=\frac{a^{ij}}{p}-S_0 b^i b^j - S_1(b^i y^j + b^j y^i) - S_2 y^i y^j, \tag{19}$$

whereas its coefficients $b^i$, $S_0$, $S_1$ and $S_2$ are evaluated by the following formulae:

$$b^i = a^{ij} b_j$$

$$S_0=\frac{pp_0+(p_0p_2-p_1^2)\alpha^2}{p\zeta} \tag{20}$$

$$S_1=\frac{pp_1+(p_0p_2-p_1^2)\beta}{p\zeta} \tag{21}$$

$$S_2=\frac{pp_2+(p_0p_2-p_1^2)b^2}{p\zeta} \tag{22}$$

$$\zeta = p(p+p_0b^2+p_1\beta)+(p_0p_2-p_1^2)(\alpha^2b^2-\beta^2), \tag{23}$$

where $b^2=a_{ij}b^ib^j$.

Let us define the $hv$-torsion tensor $C_{ijk}=\frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ as follows [19]:

$$C_{ijk}=\frac{p_1(h_{ij}m_k+h_{jk}m_i+h_{ki}m_j)+\gamma_1 m_i m_j m_k}{2p} \tag{24}$$

and its coefficients $\gamma_1$ and $m_i$ are evaluated by the formulae

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta}-3p_1q_0, m_i = b_i - \frac{y_i\beta}{\alpha^2}. \tag{25}$$

Here $m_i$ is known as non-zero covariant vector orthogonal to element of support $y^i$. Let $\Gamma^i_{jk}$be the components of Christoffel symbol of the associated Riemannian space $R^n$ and $\nabla_k$ be the covariant differentiation with respect to $x^k$ relative to Christoffel symbol. Now we put

$$2E_{ij}=b_{ij}+b_{ji} \tag{26}$$

$$2F_{ij}=b_{ij}-b_{ji}, \tag{27}$$

where $b_{ij}=\nabla_j b_i$. Let $C\Gamma=(\Gamma^{*i}_{jk},\Gamma^{*i}_{0k},C^i_{jk})$ be Cartan connection of $(M,F)$. The difference tensor $D^i_{jk}=\Gamma^{*i}_{jk}-\Gamma^i_{jk}$ of the special Finsler manifold $(M,F)$ is given by [19]

$$\begin{aligned} D^i_{jk} &= B^iE_{jk}+F^i_kB_j+F^i_jB_k+B^i_jb_{0k}+B^i_kb_{0j}- \\ & b_{0m}g^{im}B_{jk}-C^i_{jm}A^m_k-C^i_{km}A^m_j+ \\ & C_{jkm}A^m_s g^{is}+ \\ & \lambda^s(C^i_{jm}Csk^m+C^i_{km}C^m_{sj}-C^m_{jk}C^i_{ms}), \end{aligned} \tag{28}$$

where

$$B_k=p_0b_k+p_1y_k \tag{29}$$

$$B^i=g^{ij}B_j \tag{30}$$

$$B_{ij} = \frac{p_1(a_{ij} - \frac{y_i y_j}{\alpha^2}) + \frac{\partial p_0}{\partial \beta} m_i m_j}{2} \tag{31}$$

$$B_i^k = g^{kj} B_{ji} \tag{32}$$

$$A_k^m = B_k^m E_{00} + B^m E_{k0} + B_k F_0^m + B_0 F_k^m \tag{33}$$

$$\lambda^m = B^m E_{00} + 2B_0 F_0^m \tag{34}$$

$$F_i^k = g^{kj} F_{ji} \tag{35}$$

$$B_0 = B_i Y^i. \tag{36}$$

Here as well as henceforward '0' denotes tensorial contraction with $y^i$ besides $p_0$,$q_0$ and $S_0$.

## 3 Induced Cartan Connection

Let $(M, F)$ be a Finsler manifold, where $F = \sqrt{\alpha(\alpha + \beta)}$ is square root metric. Also, let $M^{n-1}$ be a hypersurface of the Finsler manifold $(M, F)$.

We will describe this hypersurface $M^{n-1}$ by following parametric equations:

$$x^i = x^i(u^\alpha), (i = 1,2,3, \ldots\ldots\ldots n; \alpha = 1,2,3, \ldots\ldots\ldots n-1), \tag{37}$$

where $u^\alpha$ is a parameter that represents coordinates on the hypersurface $M^{n-1}$. We considered parametric equation only because of the fact that parametrization always makes our life simpler. Moreover, using parametrizations we were able to calculate speed of the curves as well as tangent planes and normal lines of the sphere. In elementary differential geometry we also have learnt that parametrization helps us to determine the shape, that is, curvature and torsion of a curves and surfaces. Thus a parametrization has more information than the set of points constituting the hypersurface $M^{n-1}$.

Now differentiating the equation (37) of the hypersurface with respect to parameters $u^\alpha$, we get $B_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}$. Here each $B_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}$ for ($\alpha$=1,2,3,........n-1) represents components of tangent vectors and these tangent vectors $B_\alpha^i$ represent a tangent space at a point $p$ of the hypersurface $M^{n-1}$. Let the matrix corresponding to first derivative $B_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}$ be $[B_\alpha^i] = [\frac{\partial x^i}{\partial u^\alpha}]$, and it has maximal rank , namely, (n-1). The maximal rank required here is to ensure that tangent vectors forms linearly independent set so that any generic vector tangent to $M^{n-1}$ is linearly expressible in terms of these linearly independent tangent vectors. To introduce a Finsler structure in the hypersurface $M^{n-1}$ , the supporting element $y^i$ at a point $u^\alpha$ of $M^{n-1}$ is assumed to be tangential to $M^{n-1}$, so that we may write

$$y^i = B_\alpha^i(u) v^\alpha. \tag{38}$$

Therefore $v^\alpha$ is the element of support of hypersurface $M^{n-1}$ at the point $u^\alpha$. The metric tensor $g_{\alpha\beta}$ and hv-torsion tensor $C_{\alpha\beta\gamma}$ of hypersurface $M^{n-1}$ are defined by

$$g_{\alpha\beta} = g_{ij} B_\alpha^i B_\beta^j, C_{\alpha\beta\gamma} = C_{ijk} B_\alpha^i B_\beta^j B_\gamma^k. \tag{39}$$

Now the unit normal vector $N^i(u, v)$ at an arbitrary point $u^\alpha$ of the hypersurface $M^{n-1}$ is defined as follows:

**Definition 3.1** *A vector $N^i(u, v)$ at a point $u^\alpha$ of the hypersurface $M^{n-1}$ is said to be unit normal vector if*

$$g_{ij}(x(u, v), y(u, v)) B_\alpha^i N^j = 0, g_{ij}(x(u, v), y(u, v)) N^i N^j = 1. \tag{40}$$

Let us define angular metric tensor $h_{ij}$ as follows:

**Definition 3.2** *We say the tensor $h_{ij}$ an angular metric tensor, if $h_{ij}$ satisfies the following conditions:*

$$h_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta, h_{ij}B^i_\alpha N^j = 0, h_{ij}N^i N^j = 1. \quad (41)$$

Let $(B^\alpha_i, N_i)$ be the inverse of $(B^i_\alpha, N^i)$, then we have

$$B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta, B^i_\alpha B^\beta_i = \delta^\beta_\alpha, B^\alpha_i N^i = 0,$$

$$B^i_\alpha N_i = 0, N_i = g_{ij}N^i, B^k_i = g^{kj}B_{ji},$$

$$B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j.$$

We shall denote by $\mathscr{J}C\Gamma$ the connection of a hypersurface $M^{n-1}$ induced from the Cartan connection $C\Gamma$. The induced Cartan connection $\mathscr{J}C\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ on hypersurface $M^{n-1}$ induced from the Cartan's connection $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$ is given by [11]

$$\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M^\alpha_\beta H_\gamma$$

$$G^\alpha_\beta = B^\alpha_i(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta)$$

$$C^\alpha_{\beta\gamma} = B^\alpha_i C^i_{jk}B^j_\beta B^k_\gamma, \quad (42)$$

where second fundamental $v$-tensor $M_{\beta\gamma}$ is defined by

$$M_{\beta\gamma} = N_i C^i_{jk}B^j_\beta B^k_\gamma$$

$$= N_i g^{li}C_{ljk}B^j_\beta B^k_\gamma \quad (\text{using } C^i_{jk} = g^{li}C_{ljk})$$

$$= N^l C_{ljk}B^j_\beta B^k_\gamma \quad (\text{using } N^i g^{li} = N^l)$$

$$= C_{ijk}B^i_\beta B^j_\gamma N^k \quad (\text{adjusting the indices j, k and l}) \quad (43)$$

$$M^\alpha_\beta = g^{\alpha\gamma}M_{\beta\gamma}$$

and normal curvature vector $H_\beta$ is defined by

$$H_\beta = N_i(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta),$$

where

$$B^i_{\beta\gamma} = \frac{\partial B^i_\beta}{\partial U^\gamma}$$

$$B^i_{0\beta} = B^i_{\alpha\beta}v^\alpha.$$

The quantities $M_{\beta\gamma}$ and $H_\beta$ appeared in above equations are called the second fundamental $v$-tensor and normal curvature vector respectively [11]. The second fundamental $h$-tensor $H_{\beta\gamma}$ is defined as [11]

$$H_{\beta\gamma} = N_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M_\beta H_\gamma, \quad (44)$$

where

$$M_\beta = C^i_{jk} B^j_\beta N_i N^k = C_{ijk} B^i_\beta N^j N^k. \tag{45}$$

The relative $h$-covariant derivative and $v$-covariant derivative of projection factor $B^i_\alpha$ with respect to induced Cartan connection $\mathscr{J}C\Gamma$ are respectively given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta} N^i \tag{46}$$

$$B^i_{\alpha|\beta} = M_{\alpha\beta} N^i. \tag{47}$$

The equation (44) shows that $H_{\beta\gamma}$ is not always symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta. \tag{48}$$

Thus the above equation simplifies to

$$H_{0\gamma} = H_\gamma, H_{\gamma 0} = H_\gamma + M_\gamma H_0. \tag{49}$$

**Definition 3.3 [(11)]** *A hypersurface $M^{n-1}$ of a Finsler manifold $(M, F)$ is said to be a hyperplane of first kind if each path connecting two different points of the hypersurface $M^{n-1}$ with respect to the induced Cartan connection $\mathscr{J}C\Gamma$ is also becomes the path of the ambient Finsler manifold $(M, F)$ with respect to Cartan connection $C\Gamma$.*

**Definition 3.4 [(11)]** *A hypersurface $M^{n-1}$ of a Finsler manifold $(M, F)$ is said to be a hyperplane of second kind if each h-path of the hypersurface $M^{n-1}$ with respect to the induced Cartan connection $\mathscr{J}C\Gamma$ is also the h-path of the ambient Finsler manifold $(M, F)$ with respect to Cartan connection $C\Gamma$.*

**Definition 3.5 [(11)]** *A hypersurface $M^{n-1}$ of a Finsler manifold $(M, F)$ is said to be a hyperplane of third kind if unit normal vector $B^i$ of the hypersurface $M^{n-1}$ with respect to the metric $F$ is parallel along each curve $(u^\alpha, v^\alpha)$.*

If one wants to prove a hypersurface a hyperplane of first kind, hyperplane of second kind and hyperplane of third kind, it is very difficult to prove it only with the help of definitions mentioned above, in that situation one should incorporate below sufficient conditions given by M. Matsumoto to prove the same:

**Lemma 3.6 [(11)]** *The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if normal curvature vector $H_\beta$ vanishes.*

**Lemma 3.7 [(11)]** *A hypersurface $M^{n-1}$is a hyperplane of first kind if and only if $H_\alpha = 0$.*

**Lemma 3.8 [(11)]** *A hypersurface $M^{n-1}$ is a hyperplane of second kind with respect to Cartan connection $C\Gamma$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.*

**Lemma 3.9 [(11)]** *A hypersurface $M^{n-1}$ is a hyperplane of third kind with respect to Cartan connection $C\Gamma$ if and only if $H_\alpha = 0$, $H_{\alpha\beta} = 0$ and $M_{\alpha\beta} = 0$.*

## 4 Hypersurface $M^{n-1}$ of the special Finsler space

A hypersurface $M^{n-1}$ is an embedded subspace with codimension 1, meaning it has a dimension that is one less than that of the surrounding manifold. In our study, we focus specifically on hypersurfaces, rather than subspaces of arbitrary dimensions. This paper is concerned with Finslerian hypersurfaces $M^{n-1}$. We will now prove the following propositions in the context of Finslerian hypersurfaces.

**Theorem 4.1** *Let $(M,F)$ be a Finsler manifold, where $F(\alpha,\beta)=\sqrt{\alpha(\alpha+\beta)}$, $n\in N$, is a square root metric and $M^{n-1}$ be its hypersurface. Then fundamental function of the hypersurface $M^{n-1}$ induced from the Finsler manifold $(M,F)$ is a Riemannian metric.*

*Proof.* It is given that $(M,F)$ be a Finsler manifold, where $F(\alpha,\beta)=\sqrt{\alpha(\alpha+\beta)}$, $n\in N$, is a square root metric. Let level equation of the hypersurface $M^{n-1}$ be given by

$$b(x)=c,$$

where c is a real number.

Take the gradient of the above level equation representing hypersurface $M^{n-1}$, we get

$$b_i(x)=\partial_i b.$$

Again consider the parametric equation of the same hypersurface $M^{n-1}$ as

$$x^i=x^i(u^\alpha).$$

Differentiating the equation of hypersurface $b(x(u))=c$ with respect to parameter $u^\alpha$, we get

$$\frac{\partial b(x(u))}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha}=0$$

$$b_i(x)B_\alpha^i=0,$$

where $b_i(x)=\frac{\partial b(x(u))}{\partial x^i}$ and $B_\alpha^i=\frac{\partial x^i}{\partial u^\alpha}$.

This implies that $b_i(x)$ are normal vector field (covariant component) of hypersurface $M^{n-1}$.

Thus at any point of the hypersurface $M^{n-1}$ we now have

$$b_iB_\alpha^i=0 \tag{50}$$

$$b_iy^i=0, i.e., \beta=0. \tag{51}$$

Now, we will see how square root metric $F=\sqrt{\alpha(\alpha+\beta)}$, $n\in N$, induces a metric on the hypersurface $M^{n-1}$. In this case we will denote induced metric by $\bar{F}$. First consider the square root metric

$$F=\sqrt{\alpha(\alpha+\beta)}$$

$$=\sqrt{\sqrt{a_{ij}y^iy^j}\left(\sqrt{a_{ij}y^iy^j}+b_iy^i\right)}$$

$$=\sqrt{a_{ij}B_\alpha^i(u)v^\alpha B_\beta^j(u)v^\beta+\sqrt{a_{ij}B_\alpha^i(u)v^\alpha B_\beta^j(u)v^\beta}\,b_iB_\alpha^i(u)v^\alpha},$$

which is the general induced metric on the corresponding hypersurface $M^{n-1}$. Using (50), general induced metric of the hypersurface becomes

$$F(u,v)=\sqrt{a_{ij}B_\alpha^i(u)B_\beta^j(u)v^\alpha v^\beta}$$

$$F(u,v)=\sqrt{a_{\alpha\beta}v^\alpha v^\beta}, \tag{52}$$

where $a_{\alpha\beta} = a_{ij}B^i_\alpha(u)B^j_\beta(u)$.

Thus function represented by (52) is fundamental function or the metric of the hypersurface $M^{n-1}$ induced from the ambient Finsler manifold $(M, F)$.

The fundamental function of the hypersurface $M^{n-1}$ represented by (52) do not have $\beta$ component as $\beta = b_i y^i = 0$ over the hypersurface $M^{n-1}$ therefore fundamental function of the hypersurface $M^{n-1}$ induced from the Finsler manifold $(M, F)$ is a Riemannian metric.

**Theorem 4.2** *Let $(M, F)$ be a Finsler manifold, where $F = \sqrt{\alpha(\alpha+\beta)}$, $n \in N$, is a square root metric and $M^{n-1}$ be its associated hypersurface. Then the covariant and contravariant components of normal vector field on the hypersurface $M^{n-1}$ are given by*

1. $b_i = \sqrt{\frac{4b^2}{4-b^2}}N_i$
2. $b^i = \sqrt{4b^2(4-b^2)}N^i + \frac{4b^2}{\alpha}y^i$.

*Proof.* It is given that $M^{n-1}$ is a hypersurface of the manifold $(M, F)$, where $F = \sqrt{\alpha(\alpha+\beta)}$, $n \in N$, is a square root metric.

Moreover, we know from (51) that $\beta = 0$ over the hypersurface $M^{n-1}$. Let us calculate the value of $p$, $p_0$, $p_1$ and $p_2$. For that, substitute the value of $\beta = 0$ into (15), (16), (17), and (18), we get

$$p = 1, p_0 = 0, p_1 = \frac{1}{2\alpha}, p_2 = \frac{2}{\alpha^2}. \tag{53}$$

Using the values of $p$, $p_0$, $p_1$, $p_2$ into (20), (21), (22) and (23), we get

$$S_0 = \frac{1}{b^2-4} \tag{54}$$

$$S_1 = \frac{2}{\alpha(4-^2)} \tag{55}$$

$$S_2 = -\frac{8-b^2}{\alpha^2(4-b^2)} \tag{56}$$

$$\zeta = \frac{4-b^2}{4}. \tag{57}$$

Substituting the values of $p, S_0, S_1, S_2$ from (53), (54), (55) and (56) into (19), we have

$$g^{ij} = \frac{a^{ij}}{1} - \frac{b^i b^j}{b^2-4} - \frac{2(b^i y^j + b^j y^i)}{\alpha(4-b^2)} + \frac{8-b^2(y^i y^j)}{\alpha^2(4-b^2)}. \tag{58}$$

Multiplying equation (58) by $b_i b_j$ and using $\beta = b_i y^i = 0$, over the hypersurface $M^{n-1}$, it becomes

$$g^{ij}b_ib_j = a^{ij}b_ib_j - \frac{b^ib^jb_ib_j}{b^2-4} - \frac{2(b^iy^j + b^jy^i)b_ib_j}{\alpha(4-b^2)} + \frac{(8-b^2)(y^iy^jb_ib_j)}{\alpha^2(4-b^2)}$$

$$= (a^{ij}b_i)b_j - \frac{b^ib_ib^jb_j}{b^2-4} - \frac{2\{(b^ib_i)(b_jy^j) + (b_ib^j)(b_iy^i)\}}{\alpha(4-b^2)} + \frac{(8-b^2)(b_iy^ib_jy^j)}{\alpha^2(4-b^2)}$$

$$= b^2 - \frac{1}{b^2-4} \times b^4$$

$$g^{ij}b_ib_j = \frac{4b^2}{4-b^2}.$$

Thus at any generic point of the hypersurface $M^{n-1}$, we have

$$g^{ij}b_ib_j = \frac{4b^2}{4-b^2}.$$

Now from the above equation and using (40), we get

$$b_i = \sqrt{\frac{4b^2}{4-b^2}}N_i, \tag{59}$$

which is the covariant component of the normal vector field on the hypersurface $M^{n-1}$. Now from (58) and (59) we get

$$b^i = a^{ij}b_j$$
$$= \sqrt{4b^2(4-b^2)}N^i + \frac{4b^2}{\alpha}y^i, \tag{60}$$

which is the contravariant component of the normal vector field on the hypersurface $M^{n-1}$.

**Theorem 4.3** *Let $(M,F)$ be a Finsler manifold, where $F = \sqrt{\alpha(\alpha+\beta)}$ is a square root metric and $M^{n-1}$ be its associated hypersurface. Then second fundamental v-tensor of hypersurface $M^{n-1}$ is given by*

$$M_{\alpha\beta} = \frac{1}{4\alpha}\left(\sqrt{\frac{4b^2}{4-b^2}}\right)h_{\alpha\beta}$$

and second fundamental h-tensor $H_{\alpha\beta}$ is symmetric, i.e., $H_{\alpha\beta} = H_{\beta\alpha}$.

*Proof.* It is given that $M^{n-1}$ is a hypersurface of the manifold $(M,F)$, where $F = \sqrt{\alpha(\alpha+\beta)}$, $n \in N$, is a square root metric.

Moreover, we know from equation (51) that $\beta = 0$ over the hypersurface $M^{n-1}$. Put the value of $\beta = 0$ into equations (15), (16), (17), and (18), we get

$$p = 1, p_0 = 0, p_1 = \frac{1}{2\alpha}, p_2 = \frac{2}{\alpha^2}.$$

Now, put the values of $p, p_0, p_1$ and $p_2$ obtained above into (14), we get fundamental metric tensor of the hypersurace $M^{n-1}$

$$g_{ij} = a_{ij} + \frac{2}{\alpha^2}y_iy_j + \frac{1}{2\alpha}(b_iy_j + b_jy_i). \tag{61}$$

Let us calculate the value of $q_0$, $q_1$ and $q_2$. For that, substitute the value of $\beta = 0$ into (11), (12), and (13), we get

$$q_0 = -\frac{1}{4}, q_1 = 0, q_2 = -\frac{1}{\alpha^2}.$$

Substituting the values of $p, q_0, q_1$ and $q_2$ into 9, we get angular metric tensor of the hypersurface $M^{n-1}$

$$h_{ij} = a_{ij} - \frac{1}{4}b_ib_j - \frac{1}{\alpha^2}y_iy_j. \tag{62}$$

Differentiating equation (16) with respect to $\beta$, we have

$$\frac{\partial p_0}{\partial \beta} = 0.$$

We know from (51) that $\beta = 0$ over the hypersurface $M^{n-1}$ so put the value of $\beta = 0$ into above equation and (25), we get

$$\frac{\partial p_0}{\partial \beta} = 0$$

$$\gamma_1 = \frac{3}{8\alpha}$$

$$m_i = b_i.$$

Using the values of $p, p_1, \gamma_1$ and $m_i$ in (24), hv-torsion tensor on the hypersurface $M^{n-1}$, becomes

$$C_{ijk} = \frac{4(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j) + 3b_ib_jb_k}{16\alpha}. \tag{63}$$

Substituting the value of $C_{ijk}$ from (63) into (43) and using (50), we get

$$M_{\beta\gamma} = C_{ijk}B^i_\beta B^j_\gamma N^k$$

$$M_{\alpha\beta} = C_{ijk}B^i_\alpha B^j_\beta N^k$$

$$= \left[\frac{4(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j) + 3b_ib_jb_k}{16\alpha}\right] B^i_\alpha B^j_\beta N^k$$

$$= \frac{(h_{ij}B^i_\alpha B^j_\beta)b_kN^k}{4\alpha}.$$

Using (41), we get

$$= \left[\frac{h_{\alpha\beta}b_kN^k}{4\alpha}\right].$$

Using (59) in above expression, we get

$$= \frac{1}{4\alpha}\left(\sqrt{\frac{4b^2}{4-b^2}}N_k\right)h_{\alpha\beta}N^k$$

$$= \frac{1}{4\alpha}\left(\sqrt{\frac{4b^2}{4-b^2}}\right)h_{\alpha\beta}N_kN^k.$$

We know that $N_kN^k = 1$. Use this fact in above expression, we get

$$M_{\alpha\beta} = \frac{1}{4\alpha}\left(\sqrt{\frac{4b^2}{4-b^2}}\right)h_{\alpha\beta}. \tag{64}$$

Again, substituting the value of $C_{ijk}$ from (63) into (45) and using (50) and (41), we get

$$M_\beta = C_{ijk}B^i_\beta N^jN^k$$

$$\therefore M_\alpha = C_{ijk}B^i_\alpha N^jN^k$$

$$= \left[\frac{4(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j) + 3b_ib_jb_k}{16\alpha}\right] B^i_\alpha N^jN^k$$

$$M_\alpha = 0. \tag{65}$$

Substituting the value of $M_\alpha$ from (65) into (48) as follows:

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta$$

$$H_{\beta\gamma} - H_{\gamma\beta} = 0 \times H_\gamma - 0 \times H_\beta$$

$$H_{\beta\gamma} - H_{\gamma\beta} = 0$$

$$H_{\beta\gamma} = H_{\gamma\beta}$$

$$\therefore H_{\alpha\beta} = H_{\beta\gamma},$$

which shows that $H_{\alpha\beta}$ is symmetric.

**Theorem 4.4** *Let $(M, F)$ be a Finsler manifold, where $F = \sqrt{\alpha(\alpha+\beta)}$, $n \in N$, is a square root metric and $M^{n-1}$ be its associated hypersurface. Then the hypersurface $M^{n-1}$ will be hyperplane of first kind if and only if*

$$2b_{ij} = b_i c_j + b_j c_i.$$

Moreover we show that second fundamental tensor $H_{\alpha\beta}$ of $M^{n-1}$ is proportional to it's angular metric tensor $h_{\alpha\beta}$. That is, $H_{\alpha\beta} = \frac{c_0 2b}{\sqrt{4-b^2}} h_{\alpha\beta}$.

*Proof.* Let us differentiate (50) with respect to $\beta$, we get

$$b_{i|\beta} B^i_\alpha + b_i B^i_{\alpha|\beta} = 0. \tag{66}$$

Put the value of $B^i_{\alpha|\beta}$ from (46) and $b_{i|\beta} = b_{i|j} B^j_\beta + b_{i|j} N^j H_\beta$ into (66), we get

$$b_{i|j} B^j_\beta B^i_\alpha + b_{i|j} N^j H_\beta B^i_\alpha + b_i H_{\alpha\beta} N^i = 0. \tag{67}$$

We know that

$$b_{i|j} = -b_h C^h_{ij}.$$

Put the value of $b_h$ from (59) into above expression as follows:

$$b_{i|j} = -b_h C^h_{ij}$$

$$b_{i|j} = -\sqrt{\frac{4b^2}{4-b^2}} N_h C^h_{ij}$$

$$b_{i|j} B^i_\alpha N^j = -\sqrt{\frac{4b^2}{4-b^2}} N_h C^h_{ij} B^i_\alpha N^j$$

$$= -\sqrt{\frac{4b^2}{4-b^2}} M_\alpha$$

$$= -\sqrt{\frac{4b^2}{4-b^2}} \times 0 \quad \text{(using (65))}$$

$$= 0.$$

Using $b_{i|j} B^i_\alpha N^j = 0$ and (59) into (67) and then using the fact that $N_i N^i = 1$, we get

$$b_{i|j}B^j_\beta B^i_\alpha + \sqrt{\frac{4b^2}{4-b^2}}H_{\alpha\beta} = 0. \tag{68}$$

It is obvious that $b_{i|j}$ is symmetric. Now contracting (68) with $v^\beta$ first and then with $v^\alpha$ respectively and using (38), (49) and (65), we get

$$b_{i|j}B^i_\alpha y^j + \sqrt{\frac{4b^2}{4-b^2}}H_\alpha = 0 \tag{69}$$

$$b_{i|j}y^i y^j + \sqrt{\frac{4b^2}{4-b^2}}H_0 = 0. \tag{70}$$

We know from the Lemma 3.6 and Lemma 3.7, a hypersurface $M^{n-1}$ is a hyperplane of first kind if and only if normal curvature vanishes, i.e., $H_0 = 0$. Using the value $H_0 = 0$ into (70) we find that hypersurface $M^{n-1}$ is a hyperplane of first kind if and only if $b_{i|j}y^i y^j = 0$. This $b_{i|j}$ is the covariant derivative of with respect to Cartan connection $C\Gamma$ of Finsler space $F$, it may depend on $y^i$. Moreover $\nabla_j b_i = b_{ij}$ is the covariant derivative of $b_i$ with respect to Riemannian connection $\Gamma^i_{jk}$ constructed from $a_{ij}(x)$, therefore $b_{ij}$ dose not depend on $y^i$. We shall consider the difference $b_{i|j} - b_{ij}$ of above covariant derivatives in further discussion. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \Gamma^i_{jk}$ is given by equation (28). Since $b_i$ is a gradient vector, from (26) and (27) we have

$$E_{ij} = b_{ij}, F_{ij} = 0, F^i_j = 0. \tag{71}$$

Using (71) into (28), we get

$$\begin{aligned} D^i_{jk} &= b_{jk}B^i + b_{0k}B^i_j + b_{0j}B^i_k - b_{0m}g^{im}B_{jk} \\ &-A^m_k C^i_{jm} - A^m_j C^i_{km} + A^m_s C_{jkm}g^{is} \\ &+\lambda^s(C^m_{sk}C^i_{jm} + C^m_{sj}C^i_{km} - C^i_{ms}C^m_{jk}). \end{aligned} \tag{72}$$

Using (51), (53), (54) and (55) into (29) to (24), we get

$$B_k = (n+1)(2n+1)b_k + \frac{n+1}{\alpha}y_k, B^i = bb^i + by^i \tag{73}$$

$$B_{ij} = \frac{(n+1)\{a_{ij}\alpha^2 - y_i y_j + 2n(n+1)b_i b_j \alpha\}}{2\alpha^3} \tag{74}$$

$$B^i_j = 0 \tag{75}$$

$$A^m_k = 0, \lambda^m = B^m b_{00}. \tag{76}$$

Using tensor contraction operation with (74) and (75) by $y^j$, we get $B_{i0} = 0$, $B^i_0 = 0$. Further contracting equation (76) by $y^k$ and using the fact that $B^i_0 = 0$, we get $A^m_0 = B^m b_{00}$. Contracting equation (72) by $y^k$ and using the facts $B_{i0} = 0$, $B^i_0 = 0$, $A^m_0 = B^m b_{00}$ and $C^m_{s0} = 0$, $C^i_{0m} = 0$, $C^m_{j0} = 0$ obtained by contracting (74), (75), (76) and (42), we get

$$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - b_{00}B^m C^i_{jmi} \tag{77}$$

$$D^i_{00} = bb^i b_{00} + by^i b_{00}. \tag{78}$$

Multiplying (74) by $b_i$ and then using (51), (70) and (72), we get

$$b_i D^i_{j0} = bb_{j0} + bb_j b_{00} - bb_i b^m C^i_{jm} b_{00}. \tag{79}$$

Now multiplying (75) by $b_i$ and then using (51) we get

$$b_i D_{00}^i = \frac{4b^2}{4-b^2} b_{00}. \tag{80}$$

From (63) and (65) it is clear that

$$b^m b_i C_{jm}^i B_\alpha^j = \frac{4b^2}{4-b^2} M_\alpha = 0. \tag{81}$$

Contracting the expression $b_{i|j} = b_{ij} - b_r D_{ij}^r$ by $y^i$ and $y^j$ respectively and then using (80), we get

$$b_{i|j} y^i y^j = b_{00} - b_r D_{00}^r = \frac{4b^2}{4-b^2} b_{00}.$$

Put $b_{i|j} = b_{ij} - b_r D_{ij}^r$ into (66) and (67) and then using (76), (50) and (78) and the value of $b_{i|j} y^i y^j$ above, (69) and (70) can be written as

$$\sqrt{\frac{4b^2}{4-b^2}} b_{i0} B_\alpha^i + bH_\alpha = 0 \tag{82}$$

$$\sqrt{\frac{4b^2}{4-b^2}} b_{00} + bH_0 = 0. \tag{83}$$

From the (80) it is clear that the condition $H_0 = 0$ is equivalent to $b_{00} = 0,$ where $b_{ij}$ is independent of $y^i$. Since $y^i$ satisfy (51), the condition can be written as $b_{ij} y^i y^j = (b_i y^i)(c_j y^j)$ for some $c_j(x)$, so that we have

$$2b_{ij} = b_i c_j + b_j c_i. \tag{84}$$

Thus we shown that a Finslerian hypersurface $M^{n-1}$ will be hyperplane of first kind if and only if $2b_{ij} = b_i c_j + b_j c_i$.

Now we try to show that second fundamental tensor $H_{\alpha\beta}$ of the hypersurface $M^{n-1}$ is proportional to its angular metric tensor $h_{\alpha\beta}$.

For that, contracting (84) and using the fact that $b_i y^j = 0$, we get $b_{00} = 0$. This implies that the condition $b_{00} = 0$ and $2b_{ij} = b_i c_j + b_j c_i$ are equivalent.

Multiplying (84) by $B_\alpha^i$ and then $B_\beta^j$ and using (50) and (51), we have

$$2b_{ij} B_\alpha^i = \left(b_i B_\alpha^i\right) c_j + b_j c_i B_\alpha^i$$

$$2b_{ij} B_\alpha^i = 0c_j + b_j c_i B_\alpha^i$$

$$2b_{ij} B_\alpha^i B_\beta^j = \left(b_j B_\beta^j\right) c_i B_\alpha^i$$

$$2b_{ij} B_\alpha^i B_\beta^j = 0c_i B_\alpha^i$$

$$2b_{ij} B_\alpha^i B_\beta^j = 0$$

$$b_{ij} B_\alpha^i B_\beta^j = 0.$$

Again, multiplying (84) by $B_\alpha^i$ and $y^j$ and then using (50), we have

$$2b_{ij} B_\alpha^i y^j = \left(b_i B_\alpha^i\right) c_j y^j + \left(b_j y^j\right) c_i B_\alpha^i$$

$$2b_{ij} B_\alpha^i y^j = 0 \times c_j y^j + 0 \times c_i B_\alpha^i$$

$$2b_{ij} B_\alpha^i y^j = 0$$

$$b_{ij}B^i_\alpha y^j = 0$$

$b_{i0}B^i_\alpha = 0.$ (contraction by $y^j$ is taking place)

Again, consider (84)

$$2b_{ij} = b_i c_j + b_j c_i$$

$2b_{ij}y^j = b_i(c_j y^j) + (b_j y^j)c_i$ (multiplying by $y^j$ both sides)

$2b_{i0} = b_i c_0 + 0 \times c_i$ (contraction by $y^j$ is taking place)

$$2b_{i0} = b_i c_0$$

$2b_{i0}b^i = (b_i b^i)c_0$ (multiplying by $b^i$ both sides)

$2b_{i0}b^i = b^2 c_0$ ($\because b^2 = b_i b^i$)

$$b_{i0}b^i = \frac{b^2 c_0}{2}.$$

Using this in (79) gives $H_\alpha = 0$. Now using (72) and (73) and using $b_{00} = 0$ and $b_{ij}B^i_\alpha B^j_\beta = 0$, we get $\lambda^m = 0$, $A^i_j B^j_\beta = 0$ and $B_{ij}B^i_\alpha B^j_\beta = \frac{1}{2\alpha}h_{\alpha\beta}$. Thus using (55), (56), (57), (61) and (69), we get

$$b_r D^r_{ij} B^i_\alpha B^j_\beta = -\frac{c_0 4b^2}{4-b^2}h_{\alpha\beta}. \quad (85)$$

Thus using the relation $b_{i|j} = b_{ij} - b_r D^r_{ij}$ and (85), (68) reduces to

$$-\frac{c_0 4b^2}{4-b^2}h_{\alpha\beta} + \sqrt{\frac{4b^2}{4-b^2}}H_{\alpha\beta} = 0 \quad (86)$$

$$H_{\alpha\beta} = \frac{c_0 2b}{\sqrt{4-b^2}}h_{\alpha\beta}.$$

Thus we shown that the second fundamental tensor $H_{\alpha\beta}$ of $M^{n-1}$ is proportional to its angular metric tensor $h_{\alpha\beta}$.

**Theorem 4.5** *Let $(M,F)$ be a Finsler manifold, where $F = \sqrt{\alpha(\alpha+\beta)}$, $n \in N$, is a square root metric and $M^{n-1}$ be its associated hypersurface.Then the hypersurface $M^{n-1}$ will be hyperplane of second kind if and only if*

$$b_{ij} = e b_i b_j.$$

*Proof.* We know from Lemma 3.8, hypersurface $M^{n-1}$ is a hyperplane of second kind if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Now we consider these two sufficient conditions one by one.

1. If $H_{\alpha\beta} = 0$, then (86) becomes

$$-\frac{c_0 4b^2}{4-b^2}h_{\alpha\beta} + \sqrt{\frac{4b^2}{4-b^2}} \times 0 = 0$$

$$-\frac{c_0 4b^2}{4-b^2}h_{\alpha\beta} = 0$$

$$c_0 = 0$$

$$c_0 = c_i y^i = 0$$

there exist a function $e(x)$ such that $c_i(x) = e(x)b_i(x)$ and this $c_i(x) = e(x)b_i(x)$ forces $c_0$ to vanish as follows:

$$c_0 = c_i(x)y^i$$
$$= e(x)b_i(x)y^i$$
$$= e(x)\big(b_i(x)y^i\big)$$
$$= e(x) \times 0 \quad (\text{asalongthehypersurface} b_i y^i = 0)$$
$$\therefore c_0 = 0.$$

2. Again, if $H_\alpha = 0$, then Lemma 3.6 and Lemma 3.7 imply $H_0 = 0$.

We have already shown above $H_0 = 0$ is equivalent to

$$2b_{ij} = b_i c_j + b_j c_i.$$

Now we combine case 1 and case 2. For that, put the value of $c_i(x) = e(x)b_i(x)$ obtained in case 1 to equation obtained in case 2, we get

$$2b_{ij} = b_i e(x)b_j(x) + b_j e(x)b_i(x)$$
$$= e\big(b_i b_j + b_j b_i\big)$$
$$= e \times 2b_i b_j$$
$$\therefore 2b_{ij} = 2eb_i b_j$$
$$b_{ij} = eb_i b_j. \tag{87}$$

Thus we shown that the hypersurface $M^{n-1}$ of the Finsler manifold $(M, F)$ will be hyperplane of second kind iff $b_{ij} = eb_i b_j$.

**Theorem 4.6** *Let $(M, F)$ be a Finsler manifold, where $F = \sqrt{\alpha(\alpha+\beta)}$, $n \in N$, is a square root metric and $M^{n-1}$ be its associated hypersurface. Then the hypersurface $M^{n-1}$ will not be hyperplane of third kind.*

*Proof.* We know from sufficient conditions of Lemma 3.9 a hypersurface becomes a hyperplane of third kind if $H_\alpha = 0$, $H_{\alpha\beta} = 0$ and $M_{\alpha\beta} = 0$. Now we consider these three sufficient conditions one by one.

1. If $H_\alpha = 0$, then we get the condition $2b_{ij} = b_i e(x)b_j(x) + b_j e(x)b_i(x)$, which has already been proved above and is termed as the condition of hyperplane of first kind.

2. If $H_{\alpha\beta} = 0$, then we get the condition $b_{ij} = eb_i b_j$, which has already been proved above and is termed as the condition of hyperplane of second kind.

3. Now put $M_{\alpha\beta} = 0$ in (64), we get

$$0 = \frac{1}{4\alpha}\left(\sqrt{\frac{4b^2}{4-b^2}}\right)h_{\alpha\beta},$$

which implies that no condition could be deduced to satisfy $M_{\alpha\beta} = 0$, i.e., it is impossible to find a condition under which a hypersurface becomes a hyperplane of third kind, as term on the R.H.S. of the above equation can never be zero.

Finally, we shown that hypersurface $M^{n-1}$ is not a hyperplane of third kind.

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