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Designing of Single Sampling Plan by Attributes Under the Conditions of Borel Distribution

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Abstract:

Acceptance sampling is one of the widely used methodologies in the industry to sentence about the quality of a lot or lots based on random samples. In this modern industrial era, the manufacturing process is well monitored, hence the occurrence of defects is rare. However, the lots formed from a process, in practice, have quality variations, which occur due to random fluctuations. Therefore, the proportion of nonconforming units in the lots cannot be eliminated completely. In this situation, the appropriate distribution to model the number of defects is the Borel distribution. This paper aims to develop a single sampling plan by attributes when the number of defects follows the Borel distribution. Further, the operating characteristic function (OC) of the sampling plan is derived and comprehensive analysis of the performance of the proposed sampling plan is described through its OC curves. The procedure for determining the plan parameters using unity values with operating ratio as a measure of discrimination is discussed. The methodology for obtaining an optimal plan is also presented through a numerical illustration. And a comparative study between SSP under Borel distribution and Zero Truncated Poisson distribution has been done.

Keywords: Single Sampling Plans by attribute, Operating Characteristic curves, Borel distribution, unity values.

1. Introduction

Quality control is an effective method for determining whether any materials, processes, machines, or final products are deprived of quality. Statistical quality control ensures the quality of products. High-quality goods contribute to increased consumer loyalty and satisfaction. Additionally, it aids in lowering risk factors and replacement costs for damaged goods. Acceptance sampling is indeed a quality-control approach that allows a manufacturer to assess the overall product lot's quality through statistical analysis and testing of randomly chosen samples. The fundamental form of acceptance sampling is single sampling by attribute which relates to dichotomous scenarios. Two parameters that describe single sampling plan (SSP) by attributes are the sample size (n) and acceptance number (c) and it must be chosen by considering the needs of both the producer and the consumer.

Specific works of SSP can be seen in Schilling and Neubauer (2009) which explains the comprehensive study on the determination of plan parameters of SSP by attributes. Hald (1967) and Schilling (1982) described a method for selecting the parameters of a single sampling plan under the conditions of Poisson distribution based on the operating ratio and the unity values. S. Sampath (2009) explained the properties of SSP when there is impreciseness and randomness in proportion of defective units in a production lot. Veerakumari and Azarudheen (2017) constructed a SSP under the condition of Intervened Poisson Distribution through unity value approach and the efficiency of the developed plan is evaluated through operating characteristic curves and numerical illustration. Kaviyarasu and Devika (2018) introduced a two-parameter discrete probability distribution to establish a sampling plan over Generalized Poisson Distribution through an attribute acceptance sampling plan. The performance of operation characteristics curves for different sets of plan parameters of SSP is explained by Vijayaraghavan et al. (2007) under the conditions of Gamma Prior and Poisson sampling distributions.

In certain situations when the number of defects are very rare and cannot be eliminated completely, the nonconforming units will follow Borel distribution. Borel distribution has many applications in the fields of queuing theory, public sectors, manufacturing etc. The performance of the estimation methods like the method of maximum likelihood, method of moments and the Bayes estimation for the parameters of misclassified size biased Borel distribution are analysed by Trivedi and Patel (2016). Wanas and Khuttar (2020) introduced the necessary and sufficient conditions for the power series whose coefficients are probabilities of the Borel distribution to be in the family of analytic functions which are defined in the open unit disk. Daly and Shneer (2021) explained the properties of Borel distribution with two objectives, viz., to create the tools required to use Stain's approach for probability approximation to problems where the Borel distribution is an approximating object and the second object is to determine the concentration inequalities for the Borel Distribution. By making use of Borel Distribution S R Swamy et al. (2021) introduced a new family of normalized analytic and bi-univalent functions which are associated with Horadam polynomials in the open unit disk.

This article aims at developing a single sampling plan by attributes when the number of defects follows the Borel distribution. The aspect of Borel distribution is given in the second segment. In the third segment, scheming of single sampling plan under the conditions of Borel distribution is given. Fourth segment provides the operating characteristics function and operating characteristics curve analysis. Optimization of parameters under Borel distribution and a numerical illustration are done in the fifth segment. Last segment provides a comparative study between SSP under Borel distribution and Zero Truncated Poisson distribution.

2. Characteristics of Borel Distribution

If a Galton-Watson branching process has a common Poisson offspring distribution with mean μ, then the total number of individuals in the branching process has Borel distribution with parameter μ. In such a case, according to Daly and Shneer (2021), the total descendants X satisfies

$$X \stackrel{\mathrm{st}}{=} 1 + \sum_{i=1}^{\varepsilon} X_i \qquad (2.1)$$

Where $\stackrel{\mathrm{st}}{=}$ denotes equality in distribution, $X_1, X_2 \ldots$ are IID copies of X, and $\varepsilon \sim Po(\lambda)$ has a Poisson distribution with mean $\lambda < 1$, independent of the X_i. This random variable X is said to have a Borel distribution with parameter λ, written X ~ Borel (λ). The corresponding mass function is

$$P(X = k) = \frac{e^{-(\lambda k)}(\lambda k)^{(k-1)}}{k!} \qquad (1.2)$$

For $k \in N = \{1, 2 \ldots\}$.

The mean and variance of the Borel Distribution with parameter λ are given by,

$$\mu = E(X) = (1-\lambda)^{-1} \qquad (2.3)$$

and

$$\sigma^2 = Var(X) = \lambda (1-\lambda)^{-3} \qquad (2.2)$$

3. Constructing SSP Under the Conditions of Borel Distribution

A single sampling plan by attributes is described by the parameters lot size (N), sample size (n) and acceptance number (c). Let us consider a lot of size N units and a random sample of size n from the lot. If the number of defective units in the sample (X=x) is less than the acceptance number (c), then we accept the lot otherwise we reject the lot. The Analysis of Operating Characteristic Curves (OC) function is one of the best performance indicators for evaluating the efficacy of sampling plans. The OC function of SSP is defined as,

$$P_a\,(p) = P[x \leq c] \qquad (3.1)$$

where p is the lot quality, given as the proportion defective or fraction defective.

The operating characteristics function of SSP under the condition of Borel distribution is given by the following,

$$P_a\,(p) = \sum_{k=1}^{c} P(X = k|\,\lambda) = \sum_{k=1}^{c} \frac{e^{-(\lambda k)}(\lambda k)^{(k-1)}}{k!} \qquad (3.2)$$

where λ = np.

Hence the probability of acceptance can be obtained from

$$Pa\,(p) = \sum_{k=1}^{c} P(X = k|\,\lambda) = k = 1c\ \ e^{-(\lambda k)}(\lambda k)(k-1)\ \ k! \qquad (3.3)$$

for the given values of n, c and p. The operating characteristics curve of the plan can be attained by plotting P_a (p) against p. In order to verify the features of SSP under the condition of Borel Distribution, three combinations of operating characteristics curves are drawn corresponding to the parameters (50, 1), (100, 3) and (150, 3) which is shown in

Figure 1. From

Figure 1 it is observable that for every value of n and c the probability of acceptance decreases as the proportion defective increases which clearly says that lot with low quality is accepted with low probability and lot with high quality (i.e., less proportion defective) is accepted with high probability of acceptance. A detailed OC curve analysis has been done for different values of n and c to examine the impact of the plan parameters. On the basis of

$$Pa\,(p) = \sum_{k=1}^{c} P(X = k|\,\lambda) = k = 1c\ \ e^{-(\lambda k)}(\lambda k)(k-1)\ \ k! \qquad (3.4)$$

a set of two OC curves for the proposed single sampling plan is shown in Figure 2 and Figure 3, for the parameters (n=50, 100, 150, 200, c=1), and (n=100, c=1, 3, 5, 7). Figure 2 provides the OC curves for fixed value of c (c = 1) and for different values of n (n = 50, 100, 150, 200). Consequences of sample size n can be obtained from Figure 2. At a fixed value of c = 1, it clearly demonstrates that, for given c, as sample size grows, the chance of acceptance drops at all levels of p, hence assuring consumer protection against unsatisfactory products. Also, it clearly demonstrates that, usage of small sample size will protect the producer's interests. For instance, for (n = 50, c = 1) gives the probability of acceptance as 0.95 at p = 0.001 and as 0.60 at p = 0.01 whereas at (n = 200, c = 1) gives only the probability of acceptance as 0.81 at p = 0.001 and 0.13 at p = 0.01. In Figure 3, different sets of OC curves are constructed for fixed n values in order to examine the consequences of acceptance number c. There are four curves which represent the cases of c = 1, 3, 5, 7. The probability of acceptance will increase for a fixed n as c increases. It is clear from the OC curves given in Figure 3 that, the plan with parameters (n=100, c=1), the probability of acceptance will be 0.99 at p = 0.0001 and 0.90 at p = 0.001 whereas for the plan parameter (n=100, c=5), the probability of acceptance is 1 at p = 0.0001 and is 0.99 at p = 0.001. That is when c > 1, it ensures protection for producer for high values of acceptance number because when p is low the probability of acceptance increases and when proportion defective increases probability of acceptance reduces. The protection for the customer is increasing with decreasing acceptance number, but producer's protection increases with increasing values of c.

4. Optimization of Parameters Under the Conditions of Borel Distribution

In the selection of plan parameters in a sampling plan, a common approach is to let the OC curves pass through the acceptable quality level (AQL, denoted by p_1) and the limiting quality level (LQL, denoted by p_2) quality points, which are connected to producer's risk (α) and consumer's risk (β), respectively. By accepting good quality lots with probability 1 - α and rejecting low quality lots with probability β, the sampling plan created by considering these two quality factors will ensure protection for both the producer and the consumer. The primary objective in determining the plan parameters is to satisfy both the conditions of producer and the consumer. To accomplish this for the specified magnitude (α, β, p_1, p_2), the following requirements must be satisfied.

i. $P_a(p_1) = 1 - \alpha$ (4.1)

ii. $P_a(p_2) = \beta$ (4.2)

The plan parameters n and c for the given (α, β, p_1, p_2) can be find by using unity value approach. As mentioned by Schilling and Neubauer (2009), unity values np_1 and np_2 are those values which satisfies the conditions

$Pa(p1) = 1 - \alpha$ (4.3)

And

$Pa(p2) = \beta$ (4.4).

By applying the unity value approach, the unity values are calculated and tabulated for (c, P_a (p)) in **Error! Reference source not found.**. The parametric values for different combinations of p_1, p_2, α and β of sampling plans are obtained. **Error! Reference source not found.** also includes the operating ratios OR = np_2/np_1 corresponding to (α = 0.10, β = 0.05) which will make the sampling plans easier to determine. The following process can be used to establish the plan parameters.

1) Measure the operating ratio OR = p_2 / p_1 by specifying the values (α, β, p_1, p_2) (Table 1).
2) As to assure both the risks, select a value of OR from the table that is equal to or slightly below the desired value.
3) Divide np_1 by p_1 or np_2 by p_2 to get the sample size, which is n.
4) The operating ratio OR acquired in Step (I) is used to determine the acceptance number c.

Numerical Illustration

Let us consider a single sampling plan with the given sets of values for (α, β, p_1, p_2) say, α=0.05, β = 0.10, p_1 = 0.01, p_2 = 0.07. Then the operating ratio for the given values is obtained as OR = 7 (OR = p_2 / p_1). Select the acceptance number c, the unity values np1 and np2, and the closer operating ratio for the given values of (α, β, p_1, p_2) using **Error! Reference source not found.**. The values obtained from the table are np_1 = 0.321, np_2 = 2.531 and c = 3. The sample sizes obtained are as follows, n = 32 (n = np_1/p_1 = 32) and n = 36 (n = np_2/p_2 = 36.15). Choose for the higher n of the two sample sizes and therefore, the necessary sampling plan for the above criteria is (36, 3).

From the unity value table (**Error! Reference source not found.**), the p values for the construction of OC curves are obtained. For specified value of c, select the corresponding set of unity values which are associated with the specified values of probability of acceptance. Proportion defective (p) values can be obtained by dividing the np values with n corresponding to each value of P_a (p). And the corresponding OC curve for the sampling plan is shown in Figure 4.

From the above numerical illustration, the optimum parameters for the given strength of values are n = 36 and c = 3. The values of p with respect to P_a (p) are obtained as $p = \frac{np}{n}$ and are given in *Table 2*, The operating characteristic curve for this sampling plan under the conditions of Borel Distribution is given in Figure 4.

5. Evaluation of AOQL for SSP Under the Conditions of Borel Distribution

Instead of destructing the product that could not meet the quality specifications, we can allocate it to rectifying inspection. 100% inspection of rejected lot is done in the rectifying inspection. Also, the defective units are replaced with good ones or equivalent screening is assumed. After the rectifying inspection, the expected quality of the outgoing product is called average outgoing quality (AOQ) of the respective lot. The formula for AOQ is given by

$$AOQ = p.\ P_a\,(p) \tag{5.1}$$

where P_a (p) is the probability of acceptance and p is the proportion defective. Average outgoing quality limit (AOQL) is the maximum value of AOQ. Hence after rectification inspection, AOQL will be the worst quality of a lot in terms of percent defective. By plotting AOQ value against percent defective (*Figure* 5) one can obtain the AOQ curve, and AOQL will be the point which is the maximum point of AOQ curve. The AOQL for the above obtained SSP (n=36, c=3) is 0.0160, which implies that for any product after rectifying inspection will be no worse than 1.6 percent defective under the considered sampling plan.

Comparative Study

Comparison of single sampling plans under different distributions can be done through analysing operating characteristic curves. Now considering the comparison of SSP under Borel distribution with SSP under Zero Truncated Poisson distribution (ZTPD) (*Figure 6)*. While comparing it is noted that the probability of acceptance is low for higher values of proportion defective p under Borel distribution conditions. For instance, SSP under Borel distribution with c = 3, n = 50 and p = 0.003, the probability of acceptance is 0.993. Hence the producer has a risk of 0.7% of good lots being rejected. But when proportion defective p = 0.05, the probability of acceptance is 0.104. Hence the consumer's risk is 10.4%. But for the same specifications SSP under ZTPD, for p = 0.003, the probability of acceptance is 0.999 and for p = 0.05 the probability of acceptance is 0.735. Therefore, the producer's risk is 0.1% and consumer's risk is 73.5%. Hence it is clear that the values obtained for the sampling plan under the condition of Borel distribution is less than that of the values obtained for the sampling plan under the conditions of ZTPD. So, we can conclude that the single sampling plan under the conditions of Borel distribution will ensures protection for both producer and consumer.

Conclusion

For many businesses, including manufacturers, distributors, transportation firms, financial services firms, healthcare providers, and government entities, quality control and quality improvement have become crucial business strategies. A company can outperform their competitors if they can satisfy consumers by enhancing and managing quality. In this paper, the operating characteristic function is obtained for the single sampling plan under the conditions of Borel distribution. In order to measure the performance of the SSP under Borel distribution we have computed the OC curves and analysis is made on the obtained OC curves. The paper also comes up with the strategy for constructing and choosing the plan parameters by using unity value approach. A table is designed for the given values of parameters (α, β, p_1, p_2). A numerical illustration and evaluation of AOQL for SSP under specified parameters are discussed. The comparative study of SSP under ZTPD and SSP under Borel distribution strongly agrees that the SSP under the conditions of Borel Distribution is providing more protection for both producers and consumers.

Table 1 Unity values for different acceptance number (c)

		Probability of Acceptance							Operating Ratios
		0.99	0.95	0.75	0.5	0.25	0.10	0.05	OR (0.05,0.10)
Acceptance Number	C = 1	0.001	0.051	0.281	0.691	1.381	2.301	2.991	45.11
	C = 2	0.081	0.201	0.561	1.001	1.651	2.481	3.121	15.52
	C = 3	0.171	0.321	0.711	1.131	1.751	2.531	3.141	7.88
	C = 4	0.241	0.411	0.801	1.211	1.791	2.541	3.151	6.18
	C = 5	0.311	0.481	0.861	1.251	1.811	2.551	3.151	5.30
	C = 6	0.361	0.531	0.901	1.281	1.821	2.551	3.151	4.80
	C = 7	0.401	0.581	0.941	1.301	1.831	2.551	3.151	4.39
	C = 8	0.441	0.611	0.961	1.321	1.831	2.551	3.151	4.17
	C = 9	0.471	0.641	0.981	1.331	1.841	2.551	3.151	3.97
	C=10	0.491	0.671	1.001	1.341	1.841	2.551	3.151	3.80

Table 2 p values for the defined parametric values of SSP under the conditions of Borel Distribution

$P_a(p)$	0.99	0.95	0.75	0.50	0.25	0.10	0.05
p	0.0047	0.0090	0.0197	0.0314	0.0486	0.0703	0.0872

Figure 1 OC curves for SSP under the conditions of Borel distribution

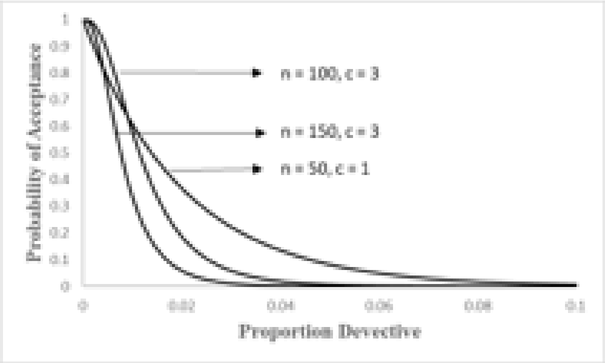


Figure 2 OC curves for SSP under the conditions of Borel distribution for fixed c

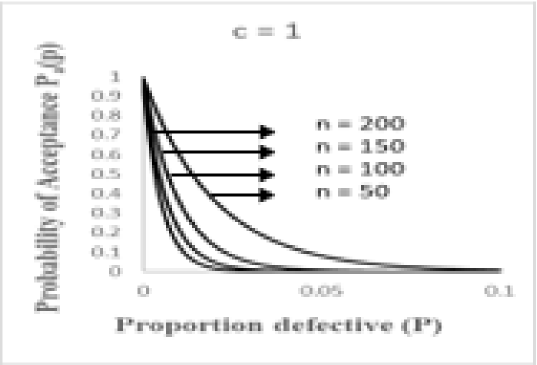


Figure 3 OC curves for SSP under the conditions of Borel distribution for fixed n

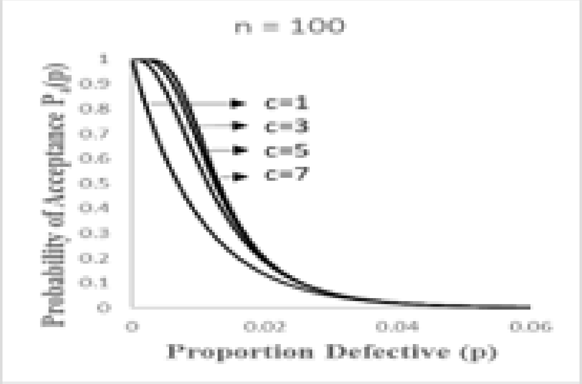


Figure 4 OC curve of single sampling plan (36, 3) under the conditions of Borel distribution

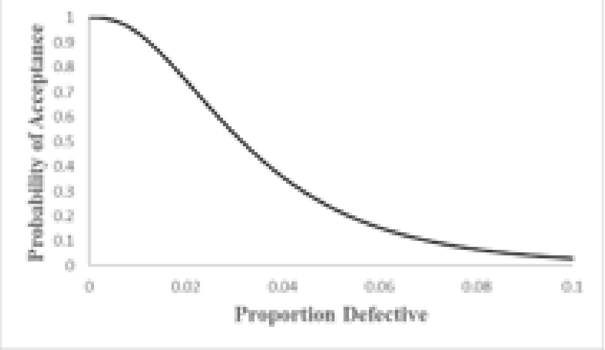


Figure 5 AOQ curve of single sampling plan (36, 3) under the conditions of Borel distribution

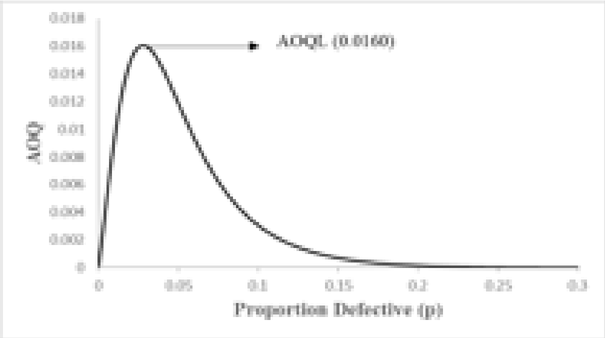
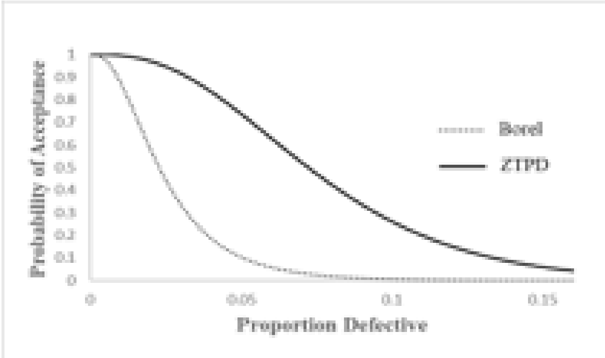


Figure 6 OC curves of SSP under the conditions of Borel distribution and Zero Truncated Poisson distribution



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