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Some Exacts Solutions of Two Dimensional Unsteady Incompressible Magnetohydrodynamic Flow of Viscous Fluid by Ansatz Method

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Abstract:

Development of geometrical and topological methods in fluid flows, will give more insight into wide range of problems. Our purpose in this paper is to study and obtain certain invariants related to vorticity field and magnetic field in the case of fluid and magneto hydrodynamic flows. This paper aims at presenting some exact solutions for two dimensional incompressible hydromagnetic flow of viscous fluid. The governing nonlinear partial differential equations for an incompressible MHD flow of viscous fluid and topology conservation equation are obtained. The equations are solved by assuming appropriate vector potential. Velocity and vorticity vector are obtained for particular cases and verified the topology conservation equation. The topology of vorticity vector field is conserved even when Lorentz force term exist in magnetohydrodynamic flow under the condition that the magnetic field is the extremal field for the magnetic energy functional throughout the flow. Velocity vectors are demonstrated graphically for each case and graphs are given for various low and high viscous fluids.

Keywords: Viscous fluid,Unsteady flow,incompressible, gnetohydrodynamic flow.

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1. Introduction

Topological theories have an impact on fluid dynamics, especially in magneto hydrodynamics, from a kinematic perspective, as fluid flows in general include the continuous deformation of any transported scalar or vector field. Furthermore, it is relevant to the neutral surface structure of a fluid (for example, the ocean), where the equation of state depends nonlinearly on several parameters (such heat and salinity). Topological fluid dynamics is a relatively new field of mathematics that studies the topological characteristics of flows with complex trajectories and how these properties relate to fluid motions.

The relationship between field structure and helicity integrals was thoroughly examined by Berger and Field (1984). The topological features of two-dimensional periodic systems in a magnetic field were explored by Kohmoto (1985). Using topological methods, Boyland et al. (2003) investigated the motion of systems of point vortices on the infinite plane, in singly periodic arrays, and in doubly periodic lattices. Volkov (2006) examined the processes involved in the creation of three-dimensional jet flows within large-scale vortex structures in a sealed cubic chamber.

The critical topological length of material lines rises either exponentially or linearly depending on the stirring protocol, according to Philip Boyland's 2013 description of topological kinematics related to the stirring by rods of a two-dimensional fluid. According to Yeates and Hornig's (2014) investigation, a topological flux function—a solitary scalar function—is both required and sufficient to ascertain the topology. The nonstationary Navier-Stokes system topology and shape optimization problem with minimization of dissipated energy in the fluid flow domain was introduced by Abdelwahed and Hassine (2014). later on. Ka-Luen Cheun (2014) constructed three types of accurate solutions for the two-dimensional incompressible magnetohydrodynamics (MHD) equations.

In order to solve the incompressible Navier-Stokes equations, Xie et al. (2021) developed the lattice kinetic scheme (LKS), a topology optimization approach for flow channel design. Missiato (2022) addressed the vorticity connected to the MHD and highlighted the significance of the Lorentz and Coriolis forces. because the Lorentz force creates a strong relationship between vorticity, vortex, and MHD. Using topological approaches, Boyland et al. (2003) investigated the motion of systems of point vortices on the infinite plane, in singly periodic arrays, and in doubly periodic lattices.

These days, the potential ansatz method has been used to generate a large number of accurate solutions of the Navier-Stokes equations. By using the ansatz approach, Joseph (2020) was able to determine the conservation of field line topology for both viscous and inviscid flows. In the presence of a uniform transverse magnetic field, Salas (2011) examined an incompressible electrically conducting fluid flow across a moving semi-infinite vertical cylinder.

Development of geometrical and topological methods in fluid flows, which are universal in nature, will give more insight into wide range of problems and thus able to analyze any problem coming under the purview of this in which we are particularly interested. Our purpose in this paper is to study and obtain certain invariants related to vorticity field and magnetic field in the case of fluid and magneto hydrodynamic flows by ansatz method. Our primary focus is on vorticity conservation in viscous incompressible flows. We have examined several scenarios in which the topology of the vorticity vector field is conserved after examining the necessity for topology conservation.

Objectives

Governing equations

Given an arbitrary vector field S and a material surface Topology, Truesdell (1954) established a necessary and sufficient condition showing that, in a continuous motion, the vorticity field in viscous fluid flows is conserved.

$$\frac{\partial S}{\partial t} + (\omega.\nabla)S - (S.\nabla)\omega + S(\nabla.\omega) = 0 \qquad (1.1)$$

Furthermore, for vector tube S to exist, the following conditions must be satisfied:

$$S \times \left(\frac{\partial S}{\partial t} + (\omega.\nabla)S - (S.\nabla)\omega\right) = 0 \qquad (1.2)$$

Here, ω is the generating vector field.

An arbitrary vector field's topological conservation condition is satisfied by

$$\frac{\partial S}{\partial t} + (\omega.\nabla)S - (S.\nabla)\omega = \lambda S \qquad (1.3)$$

Here, λ is a scalar function.

2.Elementary derivation of MHD Equations

The dynamics of an electrically conducting fluid with a magnetic field (such as liquid metal or ionized
plasma) is known as magnetohydrodynamics (MHD). It is an amalgam of electromagnetics and fluid dynamics.

Induction equation

Without the displacement current, Maxwell's equations govern the electromagnetic fields $\vec{E}$ and $\vec{B}$.

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \qquad (2.1)$$

$$\nabla . \vec{B} = 0 \qquad (2.2)$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} \qquad (2.3)$$

where μ_0 is the permeability, $\vec{J}$ is the current density, $\vec{B} = \mu_0 \vec{H}$ is the magnetic field.

Subsequently, using the Maxwell's equations, we can derive the viscous flow induction equation as

$$\frac{\partial \vec{H}}{\partial t} = \nabla \times \left(\vec{u} \times \vec{H}\right) + \eta \nabla^2 \vec{H} \qquad (2.4)$$

where $\eta = \frac{1}{\mu\sigma}$ is the magnetic diffusivity.

In a smooth domain of the Euclidean space $\Omega \subseteq R^3$, the basic governing equations for unsteady incompressible magneto hydrodynamic flow of viscous fluid is given by

$$\nabla . \vec{u} = 0 \qquad (2.5)$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{u} + \frac{1}{\rho}\left(\vec{J} \times \vec{B}\right) \qquad (2.6)$$

From equ. (2.3) we get $\nabla \times \vec{H} = \vec{J}$ (2.7)

Equation (2.6) becomes

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{u} + \frac{1}{\rho}\left(\left(\nabla \times \vec{H}\right) \times \vec{H}\right) \qquad (2.8)$$

Where $\vec{u}(x, y, t)$ is a time dependent vector field (or) velocity field, $\nu = \frac{\mu}{\rho}$ is the coefficient of kinematic viscosity ($\nu > 0$and $p(x, y, t)$ is the pressure function. where

$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ - Hamilton operator

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$- Laplace operator and

The vorticity equation for the incompressible magneto hydrodynamic viscous fluid flow is derived using the curl of equations (1.5) as follows:

$$\frac{\partial \vec{\varpi}}{\partial t} + \nabla \times (\vec{\varpi} \times \vec{u}) = \nu \nabla^2 \vec{\varpi} + \frac{\nabla \times \left(\left(\nabla \times \vec{H}\right) \times \vec{H}\right)}{\mu_0 \rho} \qquad (2.9)$$

where $\vec{\varpi} = \nabla \times \vec{u}$(or) curl$\vec{u}$

We must figure out whether or not topology conservation of vorticity field lines takes place in the context of incompressible magneto hydrodynamic viscous flows. The vorticity vector field must concurrently fulfill equations (1.3) and (2.8) if such a flow exists. Equations (2.8) and (2.9) are satisfied by the exact solutions for magneto hydrodynamic viscous fluid flows with topology-conserving vorticity fields that are obtained in the following section.

3.Ansatz approach

It has been feasible to solve a number of nonlinear partial differential equations by translating them into ordinary differential equations. Our goal is the same here: to transform the partial differential equations governing magnetohydrodynamic viscous fluid flows into ordinary differential equations so that the precise solutions can be found by solving them. It is assumed that there is a velocity potential so as to readily satisfy the equation (2.5). We consider the velocity potential in ansatz form and obtain a series of precise answers. In Cartesian coordinates, the vector potential's ansatz form is obtained as follows:

$A = e^{at}(u(mx+ny), v(mx+ny), 0)$ and

$$\vec{H} = e^{at}(u(mx+ny), v(mx+ny), 0) \tag{3.1}$$

Where m and n are arbitrary constants, u and v are functions to be determined.

The velocity field can be obtained by taking the curl of equation (2.5) as

$$\vec{u} = \nabla \times A = e^{at}\begin{pmatrix} 0 \\ 0 \\ mv'(mx+ny) - nu'(mx+ny) \end{pmatrix} \tag{3.2}$$

Clearly, the velocity field satisfies equation (2.5).

Now, the corresponding vorticity field is

$$\vec{\omega} = \nabla \times v = e^{at}\begin{pmatrix} mnv''(mx+ny) - n^2u''(mx+ny) \\ mnu''(mx+ny) - m^2v''(mx+ny) \\ 0 \end{pmatrix} \tag{3.3}$$

If equation (2.9) is satisfied by the equation above, then this is the solution to the Navier-Stokes equation.

Substituting equation (3.3) in equation (2.9), we get

$$e^{at}\begin{pmatrix} nF_1(mx+ny) - mF_2(mx+ny) \\ nF_1(mx+ny) - mF_2(mx+ny) \\ 0 \end{pmatrix} = 0 \tag{3.4}$$

where $\delta = m^2 + n^2$ and

$F_1(mx+ny) = au''(mx+ny) - \nu\delta u'^{v}(mx+ny)$

$$F_2(mx+ny) = av''(mx+ny) - \nu\delta v'^{v}(mx+ny) \tag{3.5}$$

Hence the vector potential given by equation (3.1) will be the solution if equation (3.4) is satisfied. The corresponding sufficient condition is

$$F_1(mx+ny) = F_2(mx+ny) = 0. \tag{3.6}$$

Hence, the solution of equation (3.5) becomes

$$u(mx+ny) = a_1e^{\alpha(mx+ny)} + a_2e^{-\alpha(mx+ny)} + a_3(mx+ny) + a_4 \tag{3.7}$$

Where a_1, a_2, a_3, a_4 are arbitrary constants and

$$\alpha = \sqrt{\frac{a}{\delta\nu}} \tag{3.8}$$

In order to derive explicit periodic solutions for the hydro magnetic viscous flows, we must solve equation (3.8) for the parameter 'a'.(α imaginary) Solving we get

$$a = \delta\nu \tag{3.9}$$

Substitute equation (3.9) in equation (3.7) and solving , we get

$$u(mx+ny) = c_1\cos(mx+ny) + c_2\sin(mx+ny) \tag{3.10}$$

Similarly, we can solve the exact solutions of the remaining equations in equation (3.5) as

$$v(mx + ny) = c_3\cos(mx + ny) + c_4\sin(mx + ny) \qquad (3.11)$$

Where c_1, c_2, c_3, c_4 are arbitrary parameters.

Substituting the values of u, v in equation (3.1), we obtain the velocity potential as

$$A = e^{at}\begin{pmatrix} c_1\cos(mx + ny) + c_2\sin(mx + ny) \\ c_3\cos(mx + ny) + c_4\sin(mx + ny) \\ 0 \end{pmatrix} \qquad (3.12)$$

Accordingly, the precise solution to the Navier-Stokes equation which satisfies equations (2.5) and (2.8) is provided by

$$\vec{u} = e^{at}\begin{pmatrix} 0 \\ 0 \\ c_5\cos(mx + ny) + c_6\sin(mx + ny) \end{pmatrix} \qquad (3.13)$$

Where

$c_5 = mc_4 - nc_2,\ c_6 = nc_1 - mc_3,$

and corresponding vorticity filed is obtained as

$$\vec{\omega} = e^{at}\begin{pmatrix} n(-c_5\sin(mx + ny) + c_6\cos(mx + ny)) \\ m(c_5\sin(mx + ny) - c_6\cos(mx + ny)) \\ 0 \end{pmatrix} \qquad (3.14)$$

Geometrical representation of the vector field is given through figures (1-4) for different low viscous fluids like Acetone, Methanol, water and ethanol (1995, 1998).

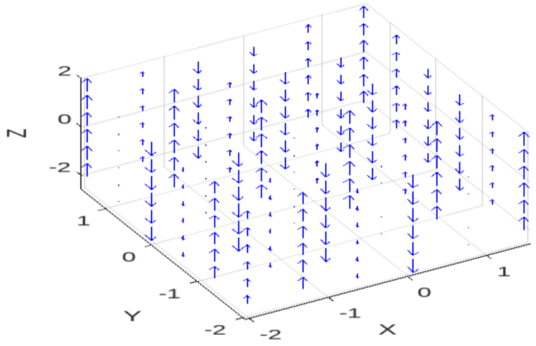

Figure 1. Quiver plot for Acetone
(m=2.0;n=2.0;c_1=1.0;c_2=2.0;c_3= 3.0;c_4= 4.0;ν=0.302;t=2.0)

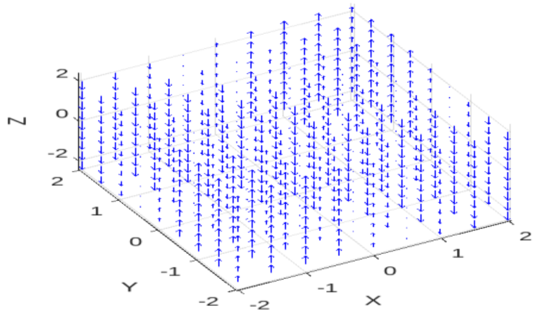

Figure 2. Quiver plot for Methanol
(m=1.0;n=1.0;c_1=1.0;c_2=2.0;c_3=1.0;c_4=1.0; ν=0.6;t=1.0)

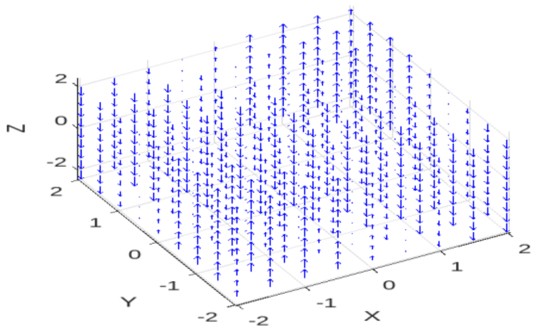

Figure 3. Quiver plot for Water
(m=1.0;n=1.0;c_1=1.0;c_2=2.0;c_3=1.0;c_4=1.0; ν=1.0;t=1.0)

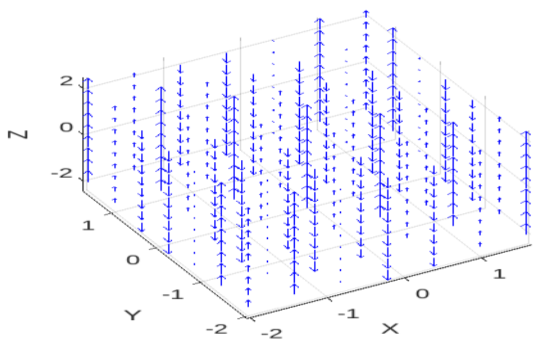

Figure 4. Quiver plot for Ethanol
(m=2.0;n=2.0;c_1=2.0;c_2=1.0;c_3=3.0;c_4=2.0; ν =1.2;t=3.0)

From the quiver plot (1-4), we can conclude that velocity profile obtained is periodic for various low viscous fluids.

Graphical visualization of the vector field is given through figures (5-8) for different high viscous fluids like Olive oil Motor oil, maple syrup and honey (1995, 1998).

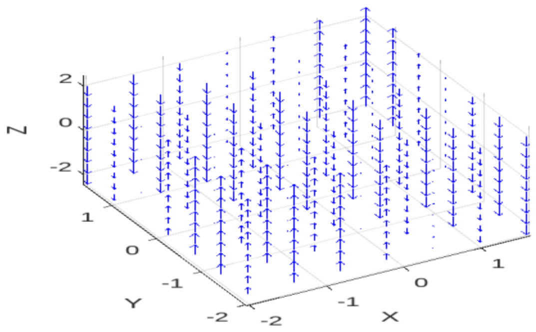

Figure 5. Quiver plot for Olive oil *($m=1.0;n=1.0;c_1=1.0;c_2=2.0;c_3=1.0;c_4=1.0;\nu=84.0;t=1.0$)*

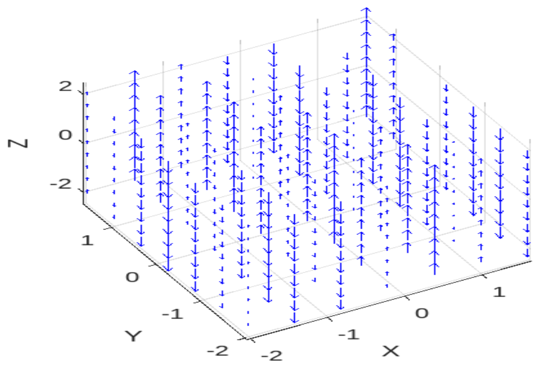

Figure 6. Quiver plot for Motor oil *($m=2.0;n=1.0;c_1=2.0;c_2=1.0;c_3=2.0;c_4=1.0;\nu=540.0;t=0.1$)*

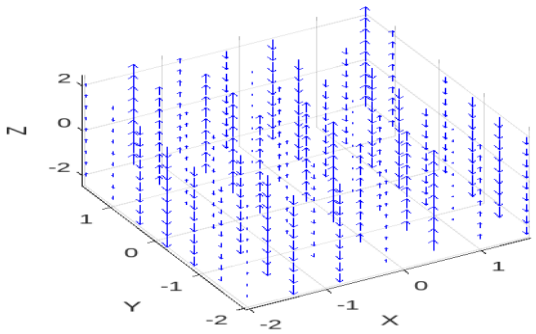

Figure 7. Quiver plot for Maple syrup *($m = 2.0; n = 1.0; c_1 = 2.0; c_2 = 1.0; c_3 = 2.0; c_4 = 1.0; \nu = 3200.0; t = 0.001$)*

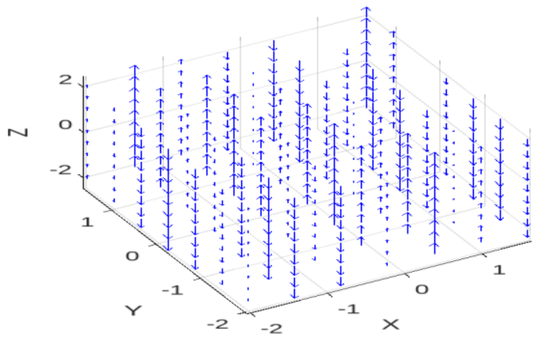

Figure 8. Quiver plot for Honey ($m = 2.0;n = 1.0;c_1=2.0;c_2=1.0;c_3=2.0;c_4=1.0;\nu=5400.0; t=0.001$)

The effects of viscosity on several high viscosity fluids are shown in Figure (5-8). It is apparent that at varying viscosities, the velocity profile is periodic. This flow has to satisfy equation (1.3) for vorticity conservation. When the vorticity and velocity fields are substituted into this formula, the equation will be satisfied if $\lambda = \delta\nu$. As a result, we obtained the accurate periodic solution for the fluid flow with incompressible couple stress, and if $\lambda = \delta\nu$, the vorticity field is topologically conserved.

Particular case 1:

An explicit example is obtained by letting all the parameters as unity; the velocity field will be zero. Hence, the velocity vector does not exists in this particular case.

Particular case 2:

By assuming a steady state incompressible magneto hydrodynamic viscous flow, the corresponding velocity field is obtained as follows

$$\vec{u} = \begin{pmatrix} 0 \\ 0 \\ c_5 \cos(mx + ny) + c_6 \sin(mx + ny) \end{pmatrix} \quad (3.15)$$

Where

$c_5 = mc_4 - nc_2,\ c_6 = nc_1 - mc_3,$

and vorticity filed is

$$\vec{\omega} = \begin{pmatrix} n(-c_5 \sin(mx + ny) + c_6 \cos(mx + ny)) \\ m(c_5 \sin(mx + ny) - c_6 \cos(mx + ny)) \\ 0 \end{pmatrix} \quad (3.16)$$

Conclusion

In the current perusal, the exact solution of unsteady flow of magneto hydrodynamic incompressible viscous fluid is obtained. Certain invariants related to vorticity field and magnetic field in the case of fluid and magneto hydrodynamic flows by ansatz method are obtained. The conservation of vorticity vector field in viscous incompressible flows is studied in detail. The topology of vorticity field is conserved even when the Lorentz force term exist in magneto hydrodynamic flow. Also, from the geometrical representation, we can conclude that, the velocity profile is periodic for low as well as high viscous fluids. In addition, exact solution of steady viscous flow is obtained.

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