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Solution of Partial Differential Equations and Their Application by Three Parameters of Integral Transform

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Abstract:

This article explains the Nour integral transform technique and denoted by "NO transform ", it's fundamental hypothesis and its power to solve the exact solution of partial differential operator equations (PDOE's) have been introduced and shown by the exact solution of various elementary engineering partial differential operator equations such as: wave partial equation, heat partial equation, Laplace’s partial equation, telegraph partial equation, and Klein-Gordon partial equation.

Keywords: (NO) transform, 1^{st} order differential equations, 2^{nd} order differential equations, Wave equation, Heat equation, Laplace’s equation, Telegraph equation, Klein-Gordan equation.

1. Introduction

PDE's introduce a special type of operator equations, with various partial differentiate of passive variables. PDE's degree is identified via the highest differentiate that shows in the operator equation using an arithmetical model that could to solve and evaluate the exact solution of PDE break off a function converts to identity when shifted into the equation .PDEs have been applied in different scientific offshoots, which yield from their might to express engineering applications in an arithmetical model that can be manipulated and find the exact solution via mathematical approach [1–3]. The Importance of PDEs necessitated applying the most effective arithmetical approaches as their exact solution [4–7]. Transform techniques’ might to transform applications from one domain to another to facilitate their exact solution has putted them as a preference in the domain of exact PDE solution. Authors have suggested many integral transforms to find exactly the solution of PDEs; every suggested transform has particular states when it shines [8–14]. The material of area of PDE's, on the other hand, has not yet improved from the revolutionary (Nour) technique. The NO transform is applied in this paper to evaluate the solution of first and second order PDE's, also some practical problems of operator equations, which are respected fundamental in the arithmetical engineering field.

There are many integral transforms with two or three parameters, such as: ZAM transform, Khalouta transform, Rishi transform [18].But what distinguishes our proposed transform is the ease of mathematical operations and algebraic simplification and this is clear from the examples and applications in the article. The main goal of this article is to solve partial differential equations and use this transform to find the exact solution for the most important and most common physical and engineering applications in the article.

2. Elementary Properties of NO Transform Technique

The Nour technique "NO technique" is defined as a function f (t) as [15]:

$$G(s,v,u) = No\,\{g(t)\} = (\tfrac{v}{s})^{\beta} \int_0^{\infty} e^{-ut}\, g(st)\, dt \,, \beta \in Z\,. \qquad (1)$$

And v, s, u are complex parameters, t ≥ 0 .

Standard as Convergence (I)

(NO) transform Technique as g (t) is a function exists, if it has exponential order and $\int_0^p |g(st)|\ dt$ exist ∀ p > 0. The convergence is required to be proven as sufficiently big u, therefore it is going to be assumed that s,v, $u > c$ and $c > 0$.

$$\left(\frac{v}{s}\right)^\beta \int_0^\infty |e^{-ut} g(st)|\, dt = \left(\frac{v}{s}\right)^\beta \left[\int_0^n |e^{-ut} g(st)|\, dt + \int_n^\infty |e^{-ut} g(st)|\, dt\right].$$

Where $n \in [0, \infty)$

$$\leq \frac{v^\beta}{s^{\beta+1}} \left[\int_0^n |g(t)|\, dt + \int_n^\infty e^{-\frac{u}{s}t} \ |g(t)| dt\right].$$

For: $\left[0 < \frac{v^\beta}{s^{\beta+1}} e^{-\frac{u}{s}t} \leq 1\right]$.

Applying NO Transform on partial.

$\leq \frac{v^\beta}{s^{\beta+1}} \left[\int_0^n |g(t)|\, dt + \int_n^\infty \ e^{-\frac{u}{s}t} M\, e^{ct} dt\right]$,

$=\frac{v^\beta}{s^{\beta+1}} [\int_0^n \ |g(t)| dt + M \frac{e^{-\left(\frac{u}{s}-c\right)t}}{\left(c-\frac{u}{s}\right)} |_n^\infty]$ (exponential order).

For $s, v, u > c$

$= \frac{v^\beta}{s^{\beta+1}} \left[\int_0^n \ |g(t)| dt + M \frac{e^{\left(c-\frac{u}{s}\right)n}}{\left(c-\frac{u}{s}\right)}\right]$.

The 1^{st} integral exists via assumption, and the 2^{nd} term is finite for v,s,u $> c$.

The integral $(\frac{v}{s})^\beta \int_0^\infty \ e^{-ut}\, g(st)\, dt$, converges absolutely and NO {g (t)} exists.

Standard for Convergence (II)

To ensure (I), NO {g (t)} exists if:

- g (t) is a function of exponential order on [0, b].
- g (t) is a limited function , piecewise, continuous function and has a bounded number of discontinuous requirements so that the integral $\int_b^0 \ |g(t)| dt$.

Where $(s, v, u) \to 0$ as $s, v, u \to \infty$.

Assuming $g(t)$ satisfy criterion (I), which implies $G(s, v, u)$ = NO{g (t)}will exist if $s, v, u \ \geq c$ for some c.

$|G(s, v, u)| = \left|\left(\frac{v}{s}\right)^\beta \int_0^\infty \ e^{-ut} g(st) dt\right| \leq \int_0^\infty \ |e^{-ut} g(st)| dt = G(s, v, u), \ \ s, v, u \to \infty, \frac{v^\beta}{s^{\beta+1}} e^{-ut} \to 0.$ For $t \geq 0$.

2.2 NO Technique Uniqueness [15]

Assume that functions f and g are functions of exponential kind b ,piecewise functions and continuous in $[0, \infty)$. If NO{g (t)}= NO{f(t)} where $s, v, u > c$, therefore f (t) = g(t) ∀ t≥ 0.

2.3 NO Transform of Partial Derivatives of $g(x,t)$

Integration by (u dv) method is applied to get NO technique as partial differentiate , as:

$No\left\{\frac{\partial g}{\partial t}(x,t)\right\} = (\frac{v}{s})^{\beta}\int_0^{\infty} e^{-ut}\frac{\partial g}{\partial t}(x,st)\,dt,$

let $u^* = e^{-ut}$, $dv^* = \frac{\partial g}{\partial t}(x,st)\,dt$,

$du^* = -ue^{-ut}\,dt$, $v^* = g(x,st)$,

$I = u^*v^* - \int v^*du^*$,

$= \left(\frac{v}{s}\right)^{\beta}\left([e^{-ut}g(x,st)]_0^{\infty} - \int_0^{\infty} g(x,st)\,(-ue^{-ut}\,dt)\right),$

$= u\left(\frac{v}{s}\right)^{\beta}\int_0^{\infty} e^{-ut}\,g(x,st)\,dt - \left(\frac{v}{s}\right)^{\beta}g(x,0)$,

$$No\left\{\frac{dg}{dt}(x,t)\right\} = u\;No\{g(x,u)\} - \left(\frac{v}{s}\right)^{\beta}g(x,0)\;. \tag{2}$$

$To\ Find\ No\left[\frac{\partial^2 g(x,st)}{\partial t^2}\right]$:

$let\ \ f = \left(\frac{\partial g}{\partial t}\right)$, $then\ by\ using\ Eq.(2)$.

$No\left[\frac{\partial^2 g(x,st)}{\partial t^2}\right] = No\left[\frac{\partial f(x,st)}{\partial t}\right] = u\,No\{f(x,u)\} - \left(\frac{v}{s}\right)^{\beta}f(x,0),$

$No\left[\frac{\partial^2 g}{\partial t^2}(x,st)\right] = u\left[u\,No\{g(x,u)\} - \left(\frac{v}{s}\right)^{\beta}g(x,0)\right] - \left(\frac{v}{s}\right)^{\beta}f(x,0),$

$= u^2\;No\{g(x,u)\} - u\left(\frac{v}{s}\right)^{\beta}g(x,0) - \left(\frac{v}{s}\right)^{\beta}\frac{\partial g}{\partial t}(x,0)$,

Integral the n[th] partial derivative:

$No\left\{\frac{d^n g}{dt^n}(x,st)\right\} = u^n N\{g(x,u)\} - \left(\frac{v}{s}\right)^{\beta}\left(\sum_{i=0}^{n-1} u^{n-i-1}\frac{\partial^{(i)}g(x,0)}{\partial t^{(i)}}\right).$

3. Exact Solutions of Elementary Partial Differential Operator Equations Applying NO Transform

The exact solutions for first order and second order partial differential operator equations, also the five basic mathematical models: wave partial operator equation, heat partial operator equation, Laplace's partial operator equation , telegraph partial operator equation, and Klein-Gordon operator equation are explained in this part.

Problem (1)

The linear 1^{st} order PDE:

$$g_x = 2g_t + g\ , \tag{3}$$

$with\ g(x,0) = 6e^{-3x}$.

And u is limited as $x,t > 0$.

Solution:

Let N be the NO transform of g.

Take NO transform to both sides of Eq.(3), gets.

$\frac{dN\,(x,u)}{dx} - 2u\,N\,(x,u) + 2\left(\frac{v}{s}\right)^{\beta} g(x,0) = N\,(x,u)$, this operator equation is a linear 1^{st} order ordinary operator equation.

$\frac{dN\,(x,u)}{dx} - (2u+1)\,N\,(x,u) = -12\left(\frac{v}{s}\right)^{\beta} \quad e^{-3x}$.

$$The\ integral\ factor\ is\ P(x) = e^{-\int \quad 2u+1dx} = e^{-(2u+1)x}.$$

And $N\,(x,u) = \frac{1}{p}\int \quad P.Q\ dx$.

$Then, N\,(x,u) = e^{(2u+1)x}\int \quad e^{-(2u+1)x}\left(-12\left(\frac{v}{s}\right)^{\beta} e^{-3x}\right) dx$,

$= e^{(2u+1)x}\left[-12\left(\frac{v}{s}\right)^{\beta}\int \quad e^{-x(2u+4)}\ dx\right]$,

$= e^{(2u+1)x}\left[-12\left(\frac{v}{s}\right)^{\beta}\left(\frac{-1}{2u+4} e^{-2(u+2)x} + c\right)\right]$,

$= e^{(2u+1)x}\left[\left(\frac{v}{s}\right)^{\beta}\frac{12}{2(u+2)} e^{-2(u+2)x} + c\right]$,

$= e^{-3x}\left(\left(\frac{v}{s}\right)^{\beta}\frac{6}{u+2}\right) + \left(ce^{(2u+1)x}\right)$.

Since N is limited, then c should be equal to 0. Take the invertible of NO transform technique gets.

$N\,(x,u) = 6e^{-3x} \quad e^{-2x}$,

$N\,(x,u) = 6e^{-(3x+2t)}$.

Problem (2)

Consider the partial Laplace's equation [16]:

$g_{xx} + g_{tt} = 0 \quad , g(x,0) = 0$. (4)

$g_{\,t}\,(x,0) = cos(x) \quad , x,t > 0$.

solution:

Suppose N(x,u) is the NO transform technique of g. Take NO transform technique to both sides of Eq. (4), gets:

$N''(x,u) + u^{2}\ N(x,u) - u\left(\frac{v}{s}\right)^{\beta} g(x,0) - \left(\frac{v}{s}\right)^{\beta}\frac{\partial g}{\partial t}(x,0) = 0$,

$N''(x,u) + u^{2}\,N(x,u) - u\left(\frac{v}{s}\right)^{\beta}(0) - \left(\frac{v}{s}\right)^{\beta} cos(x) = 0$,

$N''(x,u) + u^{2}\,N(x,u) = \left(\frac{v}{s}\right)^{\beta} cos(x)$.

The concluded equation is 2^{nd}order non-homogeneous ODE that has a particular solution as the following formula:

$N(x,u) = \frac{\frac{1}{u^{2}}\left(\frac{v}{s}\right)^{\beta} cos(x)}{\frac{1}{u^{2}}\,D^{2}+1}$,

$$N(x,u) = \frac{\frac{1}{u^2}\left(\frac{v}{s}\right)^{\beta} cos(x)}{\frac{1}{u^2}(-1)+1},$$

$$= \frac{\left(\frac{v}{s}\right)^{\beta} cos(x)}{u^2-1}. \qquad (5)$$

Where $D^2 = N^{''}(x,u)$.

Using the invertible NO technique to both sides of Eq. (5) creates the exact solution to Eq.(4)in the formula:

$$g(x,t) = sin\,h(t)\,cos(x) \quad .$$

Or

$$g(x,t) = \frac{1}{2}\ (e^t - e^{-t})\ cos(x) = \frac{1}{2}\, e^t\, cos(x) - \frac{1}{2}\, e^{-t}\, cos\,(x)\ .$$

Problem (3)

The partial wave equation [16]:

$$g_{xx} - 4g_{tt} = 0 \quad , \ g(x,o) = sin\,(\pi,0) \quad . \qquad (6)$$

$$g_t(x,o) = 0 \qquad , \qquad x,t > 0\ .$$

$solution$:

Using the NO transform technique to Eq. (6) and using the conditions provided, gets

$$N^{''}(x,u) - 4\left(u^2\, N(x,u) - \left(\frac{v}{s}\right)^{\beta}\ g(x,0) + \frac{\partial g(x,0)}{\partial t}\right) = 0\ ,$$

$$N^{''}(x,u) - 4u^2 N(x,u) + 4u\left(\frac{v}{s}\right)^{\beta}\ g(x,o) + 4\left(\frac{v}{s}\right)^{\beta}\ \frac{\partial g(x,o)}{\partial t}\ = 0\,,$$

$$N^{''}(x,u) - 4u^2 N(x,u) + 4u\ \left(\frac{v}{s}\right)^{\beta} sin(\pi x) = 0\,,$$

$$N^{''}(x,u) - 4u^2 N(x,u) = -4u\left(\frac{v}{s}\right)^{\beta} sin(\pi x)\,,$$

$$N(x,u) = \frac{-\frac{1}{u}\left(\frac{v}{s}\right)^{\beta} sin\,(\pi x)}{\frac{1}{4u^2}D^2 - 1},$$

$$= \frac{-\frac{1}{u}\left(\frac{v}{s}\right)^{\beta}\ sin\,(\pi x)}{\frac{-\pi^2}{4u^2}-1}\ ,$$

$$= \frac{-4u\left(\frac{v}{s}\right)^{\beta} sin\,(\pi x)}{-\pi^2 - 4u^2}\ ,$$

$$= \frac{-4u\left(\frac{v}{s}\right)^{\beta}}{-4\left(u^2 + \frac{1}{4}\pi^2\right)}\ sin(\pi x)\,,$$

$$= \frac{u\left(\frac{v}{s}\right)^{\beta}}{u^2 + \left(\frac{\pi}{2}\right)^2}\quad sin(\pi x)\ .$$

Using the invertible NO method of output the particular solution of Eq. (6) in the formula:

$$g(x,t) = cos\left(\frac{\pi}{2}t\right)(\pi x) \quad .$$

Problem (4)

Consider the linear telegraph partial equation [16]*:*

$g_{xx} = g_{tt} + 2g_t + g$. (7)

Subject to the initial conditions (I.C's)

$g(x,0) = e^x$, $g_t\,(x,0) = -2e^x$,

Solution:

Using the NO transform technique to Eq. (7), gets the following:

$$N''(x,u) - [u^2N(x,u) - u\left(\frac{v}{s}\right)^{\beta} g\,(x,o) - \left(\frac{v}{s}\right)^{\beta} \frac{\partial g(x,o)}{\partial t}] -$$

$2\left[uN(x,u) - \left(\frac{v}{s}\right)^{\beta} g(x,0)\right] - N(x,u) = 0$,

Providing the (I.C's) to the concluded equation gets:

$N''(x,u) - u^2N(x,u) + u\left(\frac{v}{s}\right)^{\beta} g(x,o) + \left(\frac{v}{s}\right)^{\beta} \frac{\partial g}{\partial t}\,(x,0) - 2u\,N(x,u) + 2\left(\frac{v}{s}\right)^{\beta} g(x,0) -$
$N(x,u) = 0$,

$N''(x,u) - N(x,u)(u^2 + 2u + 1) + u\left(\frac{v}{s}\right)^{\beta} e^x - 2\left(\frac{v}{s}\right)^{\beta} e^x + 2\left(\frac{v}{s}\right)^{\beta} e^x = 0$,

$N''(x,u) - N(x,u)(u^2 + 2u + 1) = -u\left(\frac{v}{s}\right)^{\beta} e^x$,

$\frac{1}{u^2+2u+1} N''(x,u) - N(x,u) = \frac{-u\left(\frac{v}{s}\right)^{\beta} e^x}{u^2+2u+1}$,

$\frac{1}{(u+1)^2} N''(x,u) - N(x,u) = \frac{-u\left(\frac{v}{s}\right)^{\beta} e^x}{(u+1)^2}$,

$N(x,u) = \frac{\frac{-u\left(\frac{v}{s}\right)^{\beta}}{(u+1)^2} e^x}{\frac{1}{(u+1)^2}D^2 - 1}$,

$= \frac{\frac{-u\left(\frac{v}{s}\right)^{\beta}}{(u+1)^2} e^x}{\frac{1}{(u+1)^2} - 1} = \frac{-u\left(\frac{v}{s}\right)\,\beta\; e^x}{(u+1)^2\left[\frac{1-(u+1)^2}{(u+1)^2}\right]}$,

$= \frac{-u\left(\frac{v}{s}\right)^{\beta} e^x}{u^2+2u} = \frac{-u\left(\frac{v}{s}\right)^{\beta} e^x}{-u\,(u+2)}$,

$N(x,u) = \frac{-\left(\frac{v}{s}\right)^{\beta} e^x}{u+2}$. (8)

Using the invertible of NO transform technique to Eq. (8) creates the exact solution to Eq. (7) in the formula:

$g(x,t) = e^{-2t}e^x = e^{x-2t}$.

Problem (5)

Consider the 2^{nd}order linear homogeneous Klein-Gordon operator equation [17]:

$u_{tt} = u_{xx} + u_x + 2u \quad ,0 < x < \infty \quad ,t > 0 .$ (9)

Subject to the initial conditions (I.C.'s):

$u(x,0) = e^x \ ,u_t\ (x,0) = 0 .$

Solution:

Using NO transform on Eq. (9), gets

$$u^2 N(x,u) - u \left(\frac{v}{s}\right)^{\beta} g(x,0) - \left(\frac{v}{s}\right)^{\beta} \frac{\partial g}{\partial t} (x,o) - N''(x,u)$$
$$-N'(x,u) - 2\,N(x,u) = 0\,,$$

$u^2\,N(x,u) - u \left(\frac{v}{s}\right)^{\beta} e^x - N''(x,u) - N'(x,u) - 2N(x,u) = 0\,,$

$N''(x,u) + N'(x,u) - N(x,u)(u^2 - 2) = -u \left(\frac{v}{s}\right) \beta\ e^x \ ,$

$\frac{1}{(u^2-2\ )} N''(x,u) + \frac{1}{(u^2-2\ )} N'(x,u) - N(x,u) = \frac{-u\left(\frac{v}{s}\right)^{\beta} e^x}{(u^2-2)},$

$N(x,u) = \frac{\frac{-u\left(\frac{v}{s}\right)^{\beta} e^x}{u^2-2}}{\frac{D^2}{(u^2-2)} + \frac{D}{(u^2-2)} - 1} \ ,$

$= \frac{\frac{-u\left(\frac{v}{s}\right)^{\beta} e^x}{(u^2-2)}}{\frac{1}{(u^2-2)}\left(1+1-(u^2-2)\right)} \ .$

$= \frac{-u\left(\frac{v}{s}\right)^{\beta} e^x}{2-u^2+2} \ ,$ (10)

Using the invertible of NO technique to Eq. (15) gets the exact solution to Eq. (9) in the formula:

$g(x,t) = e^x.$

4. Conclusion

The novel integral NO technique has been used to evaluate the solution of PDEs. The proofs that accompanied using the NO technique to PDE's and the exact solution of a practical problem consolidate the NO technique's capacity to efficiently lever and provide the exact solution for the PDEs, becoming it a powerful concurrent to another integral techniques in solving PDEs. This transform was applied to solve partial differential equations for most important applications, namely the heat ,wave , Laplace Klein-Gordon , and Telegraph equations .It was demonstrated through the solution steps that the advantage of this transform is simplicity of the arithmetic and algebraic operations and obtaining the exact solution is simpler steps.

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