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Suggesting Approximation and Exact Algorithms to Solve New Tri-Criteria Machine Scheduling Problems

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Abstract:

This study presents the multi criteria single-machine model. The machine scheduling problem (MSP) for n tasks on a single machine involves minimizing a function of three criteria: total completion time (C_j), maximum earliest (E_max), and tardiness (〖ΣT〗 _j), This is an NP-hard issue.

Within this work's theoretical section, we present the mathematical formulation of The presented topic then highlights the usefulness of the dominance rule (DR), which may be used to develop effective solutions. While in the practical part, one of the important exact methods; The proposed MSP tricriteria are solved by applying the Branch and Bound (BAB) method, which finds a set of efficient solutions for 1//F(ΣC_j ,ΣT_j ,E_max) up to n=100 jobs. The BAB approach finds the efficient solutions for the issue in an acceptable amount of time. In addition, we provide two heuristic approaches to address the problem in order to obtain appropriate approximations. The two proposed approaches' good performance is demonstrated by the practical experiments.

Keywords: Single machine problem, total Completion time, tardiness, maximum earliness, Branch and Bound.

1. Introduction

Combinatorial optimization issues include a subcategory known as machine scheduling problems (MSP).They are characterized as decision-making processes that may be applied often in manufacturing and a variety of service sectors. The goal of MSP, which deals with resource allocation to act across specified time periods, is to reduce one or more objectives [1]. Numerous domains, including computer science, manufacturing systems, transportation, industrial management, healthcare, agriculture, and many more, have been addressed by the scheduling theory [2]. Resources are referred to as machines, and tasks as jobs.

The practice of allocating scarce resources to a group of tasks across time is known as scheduling. The resources may be equipment at a workshop, personnel at a building site, runways at an airport, processing units within a computer system, and so forth . These assignments might involve carrying out computer programs, taking off and landing at an airport, working on building projects, or any combination of these [3]. Assigning resources to tasks in a way that maximizes one or more objectives is the aim of scheduling . Problem classes in factory scheduling come in a wide variety. These consist of flow shop, job shop, single machine, and parallel machine [4].

In recent years, the multicriteria scheduling problem has attracted a lot of interest. Nagar et al. [5] have published a thorough study of multicriteria. They demonstrate the resolution of two different types of issues. The first one addresses issues where there is minimal lexicographical order of criteria. Hierarchical minimization issues are shown by the research conducted by Smith [6,], [5], and [7]. The second kind of simultaneous method is divided into two categories: the first usually created all schedules that were efficient and chose the plan that produced between the two criteria, the optimal composite objective function value .The second purpose is to determine the total of these goals.The simultaneous minimizing of several objective function forms is taken into consideration in a number of scheduling issues. A detailed assessment of the most significant findings on multicriteria scheduling is presented by Hoogeveen (2005) [7]. Van Wassenhove and Gelders [8] conducted the first research in the simultaneous field. They looked at efficiency in terms of the maximum tardiness and total completion durations for single machine problems. Regarding multicriteria, For additional information on multicriteria (see [9,10,11]).

An SMSP problem using the multicriteria objectives function $1//(\sum C_j + R_L + T_{max})$ issue and is handled using BAB and a few heuristic techniques[12]. Several special cases are shown and shown to yield effective fixes for the issues at hand . They used exact and heuristic techniques to solve the $1//(\sum C_j + R_L + T_{max})$ issue in order to obtain good or optimum solutions [13].

Three improved algorithms from the Bat algorithm (BAT) were combined into a hyper-heuristic technique [14]. Depending on the results of each algorithm, the approach updates a particular implementation probability that is distributed for each employed algorithm repeatedly, and then selecting the algorithm to be utilized in the current iteration by random selection [15] [16].

Explore the $1//(\sum(E_j + T_j + C_j + U_j + V_j)$ issue, find the sequence that reduces this MOF as much as possible[17]. For this problem, they provide a BAB solution. Moreover, they utilize rapid LSMs, producing nearly ideal outcomes. The performance of exact and LSMs is assessed on a wide range of test issues; they report on computation experience [18] [19].

In this study, we investigate the issue of scheduling the number of tasks (n) on a single machine to minimize a multicriteria objective function, which may be expressed as follows: Each task will be done by a separate machine, which can only handle one job at a time. Each task j has a processing time and a due date. At time zero, Whole tasks are complete and ready for execution. The goal is to solve the $1//F(\Sigma C_j, \Sigma T_j, E_{max})$ issue using a collection of Pareto optimum solutions.

In Section two, we show some machine schedule problem concepts , The mathematical formulation of the SCSTE issue $1//F(\Sigma C_j, \Sigma T_j, E_{max})$, will be discussed in Section 3. Special cases for the SCSTE-problem are presented in Section 4. We demonstrate our suggested solutions for the SCSTE issue in Section 5. Section six introduces the practical and comparative results. We will finally provide the most relevant conclusions and recommendations in section seven.

2 Machine Schedule Problem Concept

2.1 Important Notations

This research employs many notations:

- n : number of jobs
- p_j: processing time of jobs j.
- d_j: due date of jobs j.
- C_j: completion time of job j, where $C_j = \Sigma_{k=1}^{j} p_k$
- ΣC_j: total completion time.

L_j: lateness of job j , $L_j = C_j - d_j$

T_j: tardiness of job j , $T_j = \max\{L_j, 0\}$

S_j: slack time of job j s.t. $S_j = d_j - p_j$

2.2 Important Definition Machine Scheduling Problem

In this study, several essential definitions are required:

Definition (1) [20]: Assume we have a collection of all schedules S for a scheduling issue P. A schedule σ∈S is feasible if it satisfies all of the requirements of the issue P.

Definition (2) [7]**:** A feasible schedule σ is considered Pareto optimal or efficient (non-dominated) in terms of the criteria $f \text{ and } g$ if there is no feasible schedule π such that both $f(\pi) \leq f(\sigma) \text{ and } g(\pi) \leq g(\sigma)$, with at least one of the inequality being tight.

Definition (3): (Shortest Processing Time (SPT) rule) [4]**:** Tasks are sorted in non-decreasing order of processing time (p_j) .This procedure is applied to solve the issue $1 // \Sigma C_j$.

Definition (4): Earliest Due Date (EDD) rule [7]: If the jobs are arranged in non-decreasing order of due date (d_j). This principle eliminates the issue $1 // T_{max}$.

Definition (5) [21]: In a multi-criteria resolution making challenge, the concept of "optimize" refers to a solution in which there is no way to enhance or develop one goal without hurting the other.

2.3 Dominance Rule (DR)

Applying numerous Dominance Rules (DRs) can decrease the current sequence. When assessing whether a node in the BAB method may be ignored before establishing its lower limit (LB), DRs can be useful since they frequently offer some or all of the path in order to obtain a satisfying result for the objective function. It is clear that DRs are especially useful in situations when a node may be ignored while having a lower LB than the best option. In the BAB technique, DRs are also useful for removing nodes that are dominated by others. As a result of these developments, the number of nodes required to discover the optimal solution has been significantly decreased.

Definition Emmon's Theorem (1) [13] :If $p_i \leq p_j$ and $d_i \leq d_j$, there is a suitable ordering in which job i comes before job j for the $1//\Sigma T_j$ issue.

Definition (6) [13]: The adjacency matrix of a graph G with n vertices is denoted by the matrix $A(G) = [a_{ij}]$,, whose i^{th} and j^{th} components are 1 in the case that V_i and V_j have at least one edge, and 0 otherwise, where:

$$a_{ij} = \begin{cases} 0, & if\ i = j\ or\ j \nrightarrow i \\ 1, & if\ j \longrightarrow j\ , \\ a_{ij}\ and\ \bar{a}_{ij}, & i \leftrightarrow j \end{cases}$$

3. Description of Tricriteria Scheduling $1//(\Sigma C_j, \Sigma T_j, E_{max})$ Problem

The set of jobs to be planned on a single machine is represented by N={ 1,2,3,...,n }. For every task $j \in N$, there is a processing time of positive integer $p_i$and a due date of positive integer d_j. Utilizing the three field classifications outlined by Graham et al. [3], the machine can only do one job at a time. Indicating the MSP is $1//(\Sigma C_j, \Sigma T_j, E_{max})$ .We aim to provide efficient solutions for a machine with a given schedule $S = (1,2, \dots, n)$ as follows:

$Z = min\{\Sigma C_j, \Sigma T_j, E_{max}\}$

Subject to

$$\left.\begin{array}{ll} C_1 \geq p_{s(1)}, & j = 1,2,\dots,n \\ C_j = C_{(j-1)} + p_{s(j)}, & j = 1,2,\dots,n \\ L_j = C_j - d_{s(j)}, & j = 1,2,\dots,n \\ T_j \geq C_j - d_{s(j)}, & j = 1,2,\dots,n \\ E_{max} = min(T_j, p_{s(j)}), & j = 1,2,\dots,n \\ T_j \geq 0, \quad E_{max} \geq 0, & j = 1,2,\dots,n \end{array}\right\} \dots (SCSTE)$$

Solving this SCSTE-problem is difficult and identify the efficient solution set because of the presence of the ΣT_j function.

4-Specail Cases for SCSTE-Problem

Case (1): For problem (SCSTEm) if $p_j = p$ and $d_j = d, \forall j$ then we have unique solution if:

1- $d \leq C_j, \forall j$ then $(\Sigma C_j, \Sigma T_j, Emax) = \left(p\frac{n(n+1)}{2}, p\frac{n(n+1)}{2} - nd, 0\right)$

2- $d > C_j, \forall j$ then $(\Sigma C_j, \Sigma T_j, Emax) = \left(p\frac{n(n+1)}{2}, 0, d - p\right)$

Proof: since $p_j = p$, $then\ C_j = jp$, $then$

$$\Sigma C_j = P\frac{n(n+1)}{2} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

For $d_j = d \quad if$:

1- $d \leq C_j, \forall j$ then $L_j = C_j - d = jp - d,\ then$

$T_j = \max\{jp - d, 0\} = jp - d \quad \forall j \ \ since \ \ d \leq C_j$

$$\Sigma T_j = \Sigma jp - \Sigma d = p\frac{n(n+1)}{2} - nd \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)$$

$Ej = \max\{d - C_j, 0\} = 0, \forall j$

$$\therefore E_{max} = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (3)$$

From (1),(2) and (3) we obtain :

$(\Sigma C_j, \Sigma T_j, Emax) = \left(p\frac{n(n+1)}{2}, p\frac{n(n+1)}{2} - nd, 0\right)$

2- $d > C_j, \forall j$ then $T_j = \max\{jp - d, 0\} = 0, \forall j \ then$:

$$\Sigma T_j = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (4)$$

$E_j = \max\{d - C_j, 0\} = d - C_j = d - jp$

$$E_{max} = d - p \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (5)$$

$for\ j = 1$

From (1) , (4) and (5) we obtain :

$(\Sigma C_j, \Sigma T_j, Emax) = \left(p\frac{n(n+1)}{2}, 0, d - p\right)$

Case (2): For the problem (SCSTEm) if $d_j \geq C_j, \forall j$, then the problem changed to $(\Sigma C_j, Emax)$ and this problem has a unique solution if SPT and MST are identical.

Proof: : since $d_j \geq C_j$, $\forall j$, $then\ T_j = 0, \forall j, then\ \ \Sigma T_j = 0\ \ then$

$(\Sigma C_j, \Sigma T_j, Emax) = (\Sigma C_j, Emax)$)

Now if SPT and MST are identical s.t SPT=MST = σ, $then$

$\Sigma C_j(\sigma) \leq \Sigma C_j(\pi)\ and\ \ Emax(\sigma) \leq Emax(\pi)$

In the same time , where π is any sequence

The $\left(\Sigma C_j(\sigma), \Sigma C_j(\pi)\right)$ $is\ an\ efficient\ solution\ for\ the\ problem$ (SCSTEm)

Now suppose that $\pi \neq \sigma$ but it give another efficient solution then $\Sigma C_j(\pi) \leq \Sigma C_j(\sigma)\ and\ \ Emax(\pi) < Emax(\sigma)$ and that is contradiction ! , then this problem has unique efficient solution.

Case (3): For the problem (SCSTEm) if $p_j = p\ \ for\ \ \forall j$, then we may obtain an efficient solution if $\sigma = EDD$.

Proof: : since $p_j = p$, $then\ C_j = jp\ \ and$

$$\Sigma C_j = p\frac{n(n+1)}{2} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (6)$$

$E_j = \max\{d_j - C_j, 0\} = \max\{d_j - jP, 0\}$

If $d_j = p\ \ and\ jP \geq d_j\ , then\ all\ jobs\ is\ late\ and\ \ E_j = 0\ , \forall\ \ and$

$$Emax = 0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (7)$$

Then the problem change to $\left(\Sigma C_j, \Sigma T_j, 0\right) = \left(p\frac{n(n+1)}{2},\ \Sigma T_j, 0\right) = \left(p\frac{n(n+1)}{2},\ \Sigma T_j\right)$

If we apply $\sigma = EDD$ for the problem , then the $\Sigma T_j(\sigma) \leq \ \Sigma T_j(\pi)$ where π any sequence and since $p\frac{n(n+1)}{2}$ is fixed then $(p\frac{n(n+1)}{2}\ , \Sigma T_j(\sigma)$ has an efficient solution .

Case (4): For the problem (SCSTEm) if $d_j = d\ \ , \ \forall j$:

1- If $d \leq \ C_j\ , \forall j$ then$\left(\Sigma C_j\ , \Sigma T_j\ , Emax\right) = \left(\Sigma C_j, n(n-d)\ , 0\right) = \left(\Sigma C_j\ , n(n-d)\right)$

This problem can solved by SPT rule .

2- If $d > \ C_j\ , \forall j$ then$\left(\Sigma C_j\ , \Sigma T_j\ , Emax\right) = \left(\Sigma C_j\ , 0\ , d - p\ \right) = \left(\Sigma C_j\ , d - p\right)$

This problem can solved by SPT rule .

Proof: 1- If $d \leq \ C_j\ , \forall j\ all\ jobs\ are\ late$

$\therefore L_j = C_j - d\ \ \ , \forall j\ \ then$

$T_j = max\{C_j -\ d, 0\ \} = C_j - d\ \ then$

$$\Sigma T_j = n(n-d) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (8)$$

$E_j = max\{d - C_j\ , 0\ \} = 0\ , \forall j, then$

$$Emax = 0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (9)$$

From (8) and (9) we obtain :

$$\left(\Sigma C_j\ , \Sigma T_j\ , Emax\right) = \left(\Sigma C_j\ , n(0-d)\right)$$

Since $n(0-d)$ is constant , then this problem can solved by SPT rule .

2- If $d > \ C_j\ , \forall j\ \ , then\ \ T_j = 0\ , \forall j\ \ ,\ then\ all\ jobs\ are\ early$

$$\Sigma T_j = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (10)$$

$E_j = max\{d - C_j, 0\} = d - C_j$ *then*

$Emax = d - C_j = d - P_j$ ………………………………………… (11)

From (10) and (11) we obtain:

$(\Sigma C_j, \Sigma T_j, Emax) = (\Sigma C_j, 0, d - p_j) = (\Sigma C_j, d - p_j)$

Since $d - p_j$ is constant , then this problem can solved by SPT rule .

5. Propose methods for solving SCSTE -Problem

5.1 Exact method for SCSTE-problem

In this part, we propose to find a set of Pareto optimal solutions for SCSTE-problem by applying the traditional Branch and Bound (BAB) method, which we may also refer to as BAB. The following are the BAB steps:

Algorithm : BAB Method

Step (1):**INPUT** n, $p_i$and d_j $for\ j$ for $j = 1,2, \dots, n$ . .

Step (2): **SET** S = ϕ, define $F(\sigma) = \left(\Sigma C_{\sigma(j)}, \Sigma T_j(\sigma), E_{max}(\sigma)\right)$, for any σ.

Step (3): Apply the $\sigma = SPT$ rule to get the upper bound (UB). Compute $F(\sigma)$, which equals1,2,…,n for this order. Finally, at the search tree's parent node, put the upper bound $UB = F(\sigma)$.

Step (4): Calculate a lower bound LB(δ) for each node of the BAB search tree and each partial sequence of tasks The formula for LB(δ) is equal to the sum of the costs of the sequence jobs (δ) and the cost of the sequence jobs generated by sequencing the tasks in the SPT rule.

Step (5): Take a branch from any node when $LB \leq UB$.

Step (6): After obtaining a set of solutions at the last level of the search tree, we filter S by adding δ to the set S if $F(\delta)$ indicates the outcome, unless it is predominated by the previously acquired efficient solutions in S. This method is known as filtering S.

Step (7): **STOP**.

The BAB can solve SCSTE-problems up to n = 100 in an acceptable period. In this segment, we also provide an additional BAB that relies on BAB to decrease the quantity of opened nodes, hence saving time and raising the number of n for the issues resolved.

4.2 . Heuristic Methods for SCSTE -problem

First, use the heuristic technique where the SPT rule handles the problem and calculates the objective function. Subsequently, arrange the remaining jobs according on the SPT rule and compute the objective function. This process continues until n sequences are achieved. The primary stages of SCSTEmax are as follows:

Algorithm(2): SCSTEmax Heuristic Method

Step (1): INPUT n, p_j and d_j , $j = 1,2, \dots, n$, $\delta = \emptyset$

Step (2): Organize jobs according to SPT rule (σ_1) and compute $F_{11}(\sigma_1)$ $\delta = \delta \cup \{F_{11}(\sigma_1)\}$;

Step (3):FOR $j = 2, \dots, n$, put job i in the first position of σ_{j-1} to obtain σ_j and calculate

$F_{2j}(\sigma_j)\ \delta = \delta \cup \{ F_{1j}(\sigma_j)\}$.

END;

Step (4): Arrange tasks in Emax rule (π_1) , and calculate $F_{21}\ (\pi_1)\ \ \delta = \delta \cup \{ F_{21}\ (\pi_1)\}$; .

Step (5): FOR $j = 2, \dots, n$, put job j in the first position of π_{j-1} to obtain π_j and calculate

$F_{2j}(\pi_j)$.

END;

Step (6): Filter set δ to provide a set of effective SCSTE -issue solutions

Step (7): OUTPUT The list of efficient solutions δ .

Step (8): STOP.

Finding a sequence sort with least p_j *and* d_j that does not conflict with DR and computing the goal function encapsulate the idea underlying the next heuristic strategy. The main steps of DR-SCSTEmax are as follows:

Algorithm (3): DR_ SCSTEmax Heuristic Method

Step (1): INPUT**:** n, p_j and d_j , $j = 1,2, \dots, n$, $\delta = \emptyset$

Step (2): Utilize theorem (1) to determine the DR adjacency matrix A , where $N = \{1,2, \dots, n\}, \delta = \emptyset$ **.**

Step (3): Determine a sequence σ_1 with a minimum p_j that does not conflict with DR (matrix A); in the event that ∃ more than one job break tie arbitrarily, $\delta = \delta \cup \{\sigma_1\}$.

Step (4): Determine a sequence σ_2 with a minimum p_j that does not conflict with DR (matrix A); in the event that ∃ more than one job break tie arbitrarily, $\delta = \delta \cup \{\sigma_2\}$.

Step (5): From δ, determine which sequence set $\delta^{`}$.

Step (6): Compute $F(\delta)$.

Step (7): OUTPUT The list of efficient solutions δ .

Step (8): END.

6.Applying for suggesting methods for solving SCSTE - problem

The outcomes of using CEM and BAB to solve the SCSTE problem, n=3: 10 are displayed in table (1).

Table(1): BAB and CEM Summaries regarding the SCSTE-Problem, n = 3:10.

n	CEM			BAB(DR)		
	OP	TIME	NES	MOF	TIME	NES
	AV(F)	AT/S	ANES	AV(F)	AT/S	ANES
3	(33.5, 0.2,15.9)	R	2.4	(33.5,0.2,15.9)	R	2.4
4	(41.0, 1.4,15.9)	R	4.2	(41.0,1.4,15.9)	R	4.2
5	(69.9, 8.2,10.9)	R	6.0	(70.1,8.3,11.0)	R	5.6
6	(85.9,12.0,11.4)	R	12.8	(85.2,12.5,11.9)	R	11.4
7	(123.1,28.1,11.4)	R	16.2	(122.6,27.9,11.8)	R	15.0

8	(186.9,67.2,12.3)	R	15.2	(186.4,67.2,12.7)	R	14.6
9	(205.4,86.6,12.1)	R	28.4	(205.3,86.9,12.1)	R	26.4
10	(250.1,106.1,11.2)	470.1	23.4	(251.4,107.6,11.5)	R	19.6
AV	(124.4,38.7,12.6)		13.5	(124.4,39,12.8)	R	12.4

Table (1) shows that BAB yields findings that are more accurate than CEM since its find all the solutions for SCSTE –problems.

Table(2): Summary between **SCSTEmax and SCSTEmax** (DR) with CEM for SCSTE -problem, **n**=3:10.

Note that the Heuristic DR-SCSTEmax yields better results from SCSTEmax than CEM for the SCSTE-problem for n=3:10.

n	CEM			SCSTEmax			SCSTEmax (DR)		
	OP	TIME	NES	MOF	TIME	NES	MOF	TIME	NES
	AV(F)	AT/S	ANES	AV(F)	AT/S	ANES	AV(F)	AT/S	ANES
3	(33.5, 0.2,15.9)	R	2.4	(34.5,1.1,16.0)	R	2.2	(31.3,0.2,17.2)	R	1.6
4	(41.0, 1.4,15.9)	R	4.2	(42.2,1.5,15.7)	R	3.2	(40.9,1.2,15.8)	R	2.0
5	(69.9, 8.2,10.9)	R	6.0	72.9,11.3,11.8) (	R	4.6	(70.7,7.8,11.8)	R	2.0
6	(85.9,12.0,11.4)	R	12.8	90.5,16.5,12.1) (	R	4.8	(85.6,12.6,12.7)	R	2.0
7	(123.1,28.1,11.4)	R	16.2	126.7,34.5,13.) (5	R	6.0	(127.3,30.7,11.4)	R	2.0
8	(186.9,67.2,12.3)	R	15.2	193.9,77.6,13.) (6	R	6.6	(193.7,73.5,12.7)	R	2.0
9	(205.4,86.6,12.1)	R	28.4	214.2,98.9,12.) (6	R	8.0	221.1,101.1,11.) (3	R	2.0
10	250.1,106.1,11.) (2	470.1	23.4	265.1,121.7,1) (1.7	R	7.4	272.3,126.9,10.) (9	R	2.0
AV	(221.1,38.7,12.6)		13.5	(130,45.3,31.3)		5.35	(130.3,44.2,12,9)		1.9

Table(3): Summary between **SCSTEmax and SCSTEmax** (DR) with BAB for SCSTE -problem, **n**=11:100.

n	BAB			SCSTEmax			SCSTEmax (DR)		
	OP	TIME	NES	MOF	TIME	NES	MOF	TIME	NES
	AV(F)	AT/S	ANES	AV(F)	AT/S	ANES	AV(F)	AT/S	ANES
11	(230.4,80.9,13.2)	R	21.4	250.4,103.2,14.1) (	R	7.6	(250.0,97.2,12.1)	R	2.0
15	481.3,258.7,12.0) (	R	27.4	501.9,284.7,13.5) (	R	9.4	515.1,291.4,12.5) (	R	2.0
20	850.0,542.3,10.2) (	1.1	26.6	882.7,580.9,14.3) (	R	12.2	923.9,615.0,10.3) (	R	2.0

30	1967.4,1477.9,10) (9.	8.4	81.8	2079.9,1597.0,12) (7.	R	13.4	2276.7,1782.7,9.) (4	R	2.0
40	3444.3,2772.9,12) (3.	18.1	83.4	3601.6,2942.5,15) (8.	R	17.4	3952.3,3277.7,11) (1.	R	2.0
50	5015.8,4203.2,13) (3.	52.4	101.2	5154.3,4353.9,18) (1.	R	19.6	5744.6,4925.1,11) (1.	R	2.0
60	7750.7,6715.5,12) (7.	82.9	90.0	8044.2,7023.0,15) (7.	R	22.4	8903.7,7864.4,10) (9.	R	2.0
100	19766.2,18044.1,) (9.9	301.4	70.6	20275.0,18582.4,) (16.6	R	24.8	22793.8,21071.2,) (9.0	R	2.0
AV	(4938.2,4261.9 ,11.8)	77.3	62.8	(5098.7,4433.4 ,15.1)		15.8	(5670.0,4990.5 ,10.8)		2.0

Table (4) displays a comparison of the outcome between DR-SCSTEmax and SCSTEmax for SCSTE- problem for **n** =500, 600, 700, 800,900,1000,2000,4000, 6000.

Table(4): Summary between **SCSTEmax and SCSTEmax** (DR) for SCSTE -problem, **n**=500:6000.

N	**SCSTEmax**			**SCSTEmax(DR)**		
	MOF	**TIME**	**NES**	**MOF**	**TIME**	**NES**
	AV(F)	**AT/S**	**ANES**	**AV(F)**	**AT/S**	**ANES**
500	(494483.1,485662.4,20.5)	5.0	40.6	(489637.6,480723.2,-0.0)	8.8	1.0
600	(687118.0,676637.2,16.5)	6.6	40.4	(703866.4,693302.7,0.1)	10.0	1.2
700	(964428.0,952090.8,19.6)	8.7	37.8	(952763.4,940322.0,0.2)	13.0	1.0
800	(1242049.6,1227947.1,16.5)	10.6	40.2	(1229932.2,1215754.4,-0.0)	16.3	1.0
900	(1557441.8,1541591.9,20.8)	13.7	41.8	(1542681.6,1526728.8 ,-0.0)	27.9	1.0
1000	(1952579.8,1934879.7,17.5)	16.2	35.2	(1931876.6,1914101.6 ,-0.0)	38.2	1.0
2000	(7803844.5,7768631.1,18.8)	76.2	40.0	(7732594.4,7697288.8,-0.0)	227.4	1.0
4000	31088267.7,31017471.3,18.6) (	466.7	39.8	(30803586.0,30732709.4,-0.0)	949.5	1.0
6000	70266317.8,70159912.3,16.7) (	1438.9	38.2	(69610745.4,69504238.6,-0.0)	2933.1	1.0
AV	(1823674.4,53400457.3,20.2)	763.5	39.3	(11999742.6,11783879.8,0.15)	469.3	1.0

7. Conclusions and Future Works

1. From this study, we find the efficient solution up to n = 100 tasks by using the BAB algorithm with DR. Applying the BAB algorithm yields results that are compared with CEM.

2. For the SCSTE-problem, we propose two effective and easy heuristic methods: SCSTEmax and SCSTEmax (DR), which exhibit good performance.

3. From Table 1 we notice that the program shows very accurate results and this is evidence that the program shows exact results , from tables 2,3,4 notice that SCSTEmax (DR) shows better results than the other results.

4.As a future research, we propose to discover effective and approximate solutions for the SCSTE−problem for $n > 100$ by the application of local search algorithms (e.g., genetic, Bees, simulated annealing, particle swarm optimization, etc.).

5.As further work, we advise using $1//(\Sigma C_j + \Sigma T_j + E_{max})$ to find efficient (optimal) solutions .

6. As further work, we advise using $(\Sigma C_j , \Sigma E_j T_{max})$ to find the efficient solution.

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