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K - Sharp Ordering on Intuitionistic Fuzzy Matrices

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Abstract:

Aim of this article is to develop the concept of k-Sharp ordering as a generalization of k-minus ordering [1]. A relationship between the k-minus ordering and k-Sharp ordering is estabilished. We derive some properties of intuitionistic fuzzy matrices (IFM) under k-sharp ordering. In general k-minus ordering need not imply k-sharp ordering and this is illustrated with suitable example.

Keywords: Intuitionistic Fuzzy Matrices (IFMs), Ordering, k-g inverses, k-regular.

1. Introduction

In this paper, we are concerned with fuzzy matrices over the fuzzy algebra $F = [0,1]$ is defined by the max - min operation $a + b = max\{a,b\}$ and $a \cdot b = min\{a,b\}$ for all $a,b \in F$. Let $F_{m \times n}$ be the collection of all $m \times n$ fuzzy matrices in the fuzzy algebra $\{F : F = [0,1]\}$. If there exist X such that $AXA = A$, then the matrix $A \in F_{m \times n}$ is said to be regular, X is known as a generalized (g^-) inverse of A. In [2], Kim and Roush developed a fuzzy matrix theory analogous to Boolean matrices and also investigated the inverse of Boolean matrices . Meenakshi and Gandhimathi [3], investigated the regularity of intuitionistic fuzzy matrices. Padder and Murugadas explored idempotent IFMs as well as T-type idempotent IFMs in [4]. In [5], using the concept of fuzzy sets, Atanassov introduced and developed the notion of intuitionistic fuzzy sets.. Ben Israel and Greville [6], discussed the idea of generalized inverses. As a continuation of the work on fuzzy matrices in [7], Pal and Khan developed basic properties of intuitionistic fuzzy matrices. The minus ordering on matrices is defined by Meenakshi and Inbam in terms of their generalised inverses. Moreover, they established fuzzy matrices space ordering in[9], which is a partial ordering on the set of all idempotent matrices in fuzzy matrices. Sriram and Murugadas explored minus ordering on fuzzy matrices in [10]. In [11], Cen published T-ordering and the relationship between T-ordering in fuzzy matrices. Poongodi, Padmavathi, Vinitha, and Hema investigated the idea of ordering for k-regular Interval Valued Fuzzy Matrices in [12], as a generalization of the minus ordering for regular fuzzy matrices. Cho investigated the consistency of fuzzy matrix equations in [13]. As a generalization of the regular fuzzy matrix, Meenakshi and Jenita developed the idea of k - regular fuzzy matrix [14]. The concept of generlized inverse of intuitionistic fuzzy matrices is introduced by Khan and Paul [15]. Pradhan and Pal [16], offer a method for computing the inverse of an intuitionistic fuzzy matrix using the generalized inverses of the original matrix's blocks. In [17], Meenakshi and Jenita addressed the k-g inverses of k-regular fuzzy matrices. In [18], Jenita and Karuppusamy examined the k-regularity of fuzzy and block Intuitionistic fuzzy matrices. The idea of generalized regular block intuitionistic fuzzy matrices was presented in [19]. Special types of inverses and their characterisation were covered in [20, 21]. [22, 23], are excellent resources for learning more about fuzzy matrix and its applications. Jenita, Karuppusamy, and Thangamani proposed the concept of k - regular intuitionistic fuzzy matrices in [24], as a generalisation of regular intuitionistic fuzzy matrices. Several inverses of k-regular intuitionistic fuzzy matrices were investigated in [25]. The k-sharp ordering for k-regular intuitionistic fuzzy matrices is described in this study as a generalisation of the k-minus ordering, and some of its properties related to k-g inverses are investigated.

2. Preliminaries

The matrix operations on intuitionistic fuzzy matrices as stated in [3] will be followed.

$$A + B = \left(\left\langle \max\{a_{ij\mu}, b_{ij\mu}\}, \min\{a_{ijv}, b_{ij\vartheta}\}\right\rangle\right),$$
$$AB = \left(\left\langle \max_k \min\{a_{ik\mu}, b_{kj\mu}\}, \min_k \max\{a_{ikv}, b_{kjv}\}\right\rangle\right)$$

Define the order relation on $(IFM)_{m\times n}$ as follows,

$$A \leq B \Leftrightarrow a_{ij\mu} \leq b_{ij\mu} \text{ and } a_{ijv} \geq b_{ijv}, \text{ for all } i \text{ and } j.$$

Definition 2.1 [1]
For $A \in (IFM)_n^-$ and $B \in (IFM)_n$, the k-minus ordering denoted as $<_k^-$ is defined as $A <_k^- B \Leftrightarrow A^kU = B^kU, U \in A\{1_r^k\}$ and $VA^k = VB^k, V \in A\{1_\ell^k\}$.

Definition 2.2 [25]

If there exist the matrix $X \in (IFM)_n$, such that $A^kXA = A^k$, for some positive integer k, then the matrix $A \in (IFM)_n$ is said be right k-regular. Right k-g-inverse of A is called X. Let $A_r\{1^k\} = \{X/A^kXA = A^k\}$.

Definition 2.3 [25]
If there exists a matrix $Y \in (IFM)_n$ such that $AYA^k = A^k$, for some integer k, then the matrix $A \in (IFM)_n$, is said be left k-regular. Left k-g-inverse of A is called Y. Let $A_\ell\{1^k\} = \{Y/AYA^k = A^k\}$. In general, right k-regular is different from left k-regular.

Definition 2.4 [9]

For $A \in F_{m\times n}^\sharp$ and $B \in F_{m\times n}$, the sharp ordering denoted as $\overset{\sharp}{<}$ is defined as $A \overset{\sharp}{<} B \Leftrightarrow A^\sharp A = A^\sharp B$ and $AA^\sharp = BA^\sharp$.

Lemma 2.1 [3]
For A and $B \in (IFM)_{m\times n}, R(B) \subseteq R(A) \Leftrightarrow B = XA$ for some $X \in (IFM)_m, C(B) \subseteq C(A) \Leftrightarrow B = AY$ for some $Y \in (IFM)_n$.

Lemma 2. 2[16]
For $A \in (IFM)_{m\times n}$ and $B \in (IFM)_{n\times p}, R(AB) \subseteq R(B), C(AB) \subseteq C(A)$.

Lemma 2. 3[24]
For $A, B \in (IFM)_n$, and for a positive integer k, then the following statements hold.
(i) If A is right k - regular and $R(B) \subseteq R(A^k)$, then $B = BXA$ for each right k - g inverse X of A.
(ii) If A is left k - regular and $C(B) \subseteq C(A^k)$, then $B = AYB$ for each left k - g inverse Y of A.

Lemma 2. 4[9]
For $A \in (IFM)_n^-$ and $B \in (IFM)_n$, the following are equivalent.
(i) $A <_k^- B$
(ii) $A^k = B^k UA = AVB^k$ for some $U, V \in A\{1^k\}$.

3. **k-sharp ordering on Intuitionistic Fuzzy Matrices**

This section deals a special type of ordering involving Drazin inverses named as k-Sharp ordering for k-regular Intuitionistic Fuzzy Matrices.

$$(IFM)_n^- = \{A \in (IFM)_n / A \text{ has } k-g \text{ inverse }\}, (IFM)_n^{\tilde{D}} = \{A \in (IFM)_n / A_{\tilde{D}} \text{ exists }\}.$$

$A_{\tilde{D}}$ is the Drazin inverse of A.
For $k = 1$, the following result reduces to the result of sharp-ordering on fuzzy matrices [9].

Definition 3.1
For $A \in (IFM)_n$, the Drazin inverse of A denoted by X is the solution of the following equation

$$A^k = A^k XA = AXA^k \text{ for some positive integer } k$$
$$X = XAX$$
$$AX = XA$$

The smallest positive integer k for which (1) holds is called index of A. For k-regular IFM's it is called regularity index. The group inverse is a particular case of Drazin inverse with index one. Already we have proved that, the Drazin inverse is a k-g inverse but the k-g inverse need not imply the Drazin inverse [25]. Also, if the Drazin inverse of a matrix exists, it is unique.

Definition 3.2
For $A \in (IFM)_n$, the group inverse of A, denoted as $A^\#$ is a commuting semi-inverse of A, that is, $AA^\# A = A, A^\# AA^\# = A^\#$ and $AA^\# = A^\# A$.

Definition 3.3
For $A \in (IFM)_{m\times n}^\#$ and $B \in (IFM)_{m\times n}$, the sharp ordering denoted as $\overset{\#}{<}$ is defined as $A \overset{\#}{<} B \Leftrightarrow A^\# A = A^\# B$ and $AA^\# = BA^\#$.

Definition 3.4
For $A \in (IFM)_{m\times n}^-$ and $B \in (IFM)_{m\times n}$, the minus ordering denoted as $<^-$ is defined as $A <^- B \Leftrightarrow AX = BX$, and $XA = XB$, for some $X \in A\{1\}$.

Thus Sharp ordering is the speical case of minus ordering. In general, minus order need not imply Sharp order.

Example 3.1

Let $A = \begin{bmatrix} \langle 1,0\rangle & \langle 1,0\rangle \\ \langle 1,0\rangle & \langle 0,1\rangle \end{bmatrix}, X = \begin{bmatrix} \langle 0,1\rangle & \langle 1,0\rangle \\ \langle 1,0\rangle & \langle 1,0\rangle \end{bmatrix}$, and $B = \begin{bmatrix} \langle 0.1,0.1\rangle & \langle 1,0\rangle \\ \langle 1,0\rangle & \langle 0,1\rangle \end{bmatrix}$.

$$A_\mu X_\mu A_\mu = \left\{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = A_\mu. \text{ and}$$
$$A_\vartheta X_\vartheta A_\vartheta = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = A_\vartheta.$$

Therefore $AXA = \begin{bmatrix} \langle 1,0\rangle & \langle 1,0\rangle \\ \langle 1,0\rangle & \langle 0,1\rangle \end{bmatrix} = A. X$ is a g-inverse of A.

$$A_\mu X_\mu = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } B_\mu X_\mu = \begin{bmatrix} 0.1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$
$$A_\mu X_\mu = B_\mu X_\mu$$
$$A_\vartheta X_\vartheta = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } B_\vartheta X_\vartheta = \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$
$$A_\vartheta X_\vartheta = B_\vartheta X_\vartheta.$$

Therefore, $AX = \begin{bmatrix} \langle 1,0\rangle & \langle 1,0\rangle \\ \langle 0,1\rangle & \langle 1,0\rangle \end{bmatrix} = BX.$

$$X_\mu A_\mu = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } X_\mu B_\mu = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0.1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$
$$X_\mu A_\mu = X_\mu B_\mu$$
$$X_\vartheta A_\vartheta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } X_\vartheta B_\vartheta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$
$$X_\vartheta B_\vartheta = X_{\hat\vartheta} B_{\hat\vartheta}.$$

Therefore, $XA = \begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle \\ \langle 1,0\rangle & \langle 1,0\rangle \end{bmatrix} = XB.$

Hence A is a minus ordering of Intuitionistic Fuzzy Matrices.

$X_\mu A_\mu X_\mu = \left\{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right\}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = X_\mu$. and $X_\vartheta A_\vartheta X_\vartheta = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = X_\vartheta$.

Therefore $XAX = \begin{bmatrix} \langle 0,1\rangle & \langle 1,0\rangle \\ \langle 1,0\rangle & \langle 1,0\rangle \end{bmatrix} = X.$

$$AX = \begin{bmatrix} \langle 1,0\rangle & \langle 1,0\rangle \\ \langle 0,1\rangle & \langle 1,0\rangle \end{bmatrix} \text{ and } XA = \begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle \\ \langle 1,0\rangle & \langle 1,0\rangle \end{bmatrix}.$$

$$AX = \begin{bmatrix} \langle 1,0\rangle & \langle 1,0\rangle \\ \langle 0,1\rangle & \langle 1,0\rangle \end{bmatrix} \text{ and } XA = \begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle \\ \langle 1,0\rangle & \langle 1,0\rangle \end{bmatrix}.$$

Here, $AX \neq XA$ Therefore, minus Ordering need not imply Sharp Ordering.

Definition 3.5

For $A \in (IFM)_n^{\tilde{D}}$ and $B \in (IFM)_n$, the k-sharp ordering denoted as $<_k^{\tilde{D}}$ is defined as $A <_k^{\tilde{D}} B \Leftrightarrow$
$A^k A_{\tilde{D}} = B^k A_{\tilde{D}}$
and $A_{\tilde{D}} A^k = A_{\tilde{D}} B^k$
$A_{\tilde{D}}$ is the Drazin inverse of A. Since the Drazin inverse is unique, the right k-g inverse and left k-g inverse are same.

Remark 3.1

Thus, k-sharp ordering is the special case of k-minus ordering.
In general, k-minus order need not imply k-sharp order.

Example 3.2

Let $A = \begin{bmatrix} \langle 0.7,0\rangle & \langle 0.6,0.4\rangle \\ \langle 0.5,0.2\rangle & \langle 0,0.5\rangle \end{bmatrix}$,

$$A_\mu^2 = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \\ 0 & 0.4 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \\ 0 & 0.4 \\ 0.2 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \\ 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix} \neq A_\mu \text{ and}$$

Therefore, $A^2 = \begin{bmatrix} \langle 0.7,0\rangle & \langle 0.6,0.4\rangle \\ \langle 0.5,0.2\rangle & \langle 0.5,0.4\rangle \end{bmatrix} \neq A$

For the permutation matrices $P_1 = \begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 1,0\rangle \end{bmatrix}$ and $P_2 = \begin{bmatrix} \langle 0,1\rangle & \langle 1,0\rangle \\ \langle 1,0\rangle & \langle 0,1\rangle \end{bmatrix}$.

$$A_\mu P_{1\mu} A_\mu = \left\{ \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} \neq A_\mu$$
$$\text{and } A_\vartheta P_{1\vartheta} A_\vartheta = \left\{ \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.5 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.5 \end{bmatrix} = \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix} \neq A_\vartheta$$

Therefore, $AP_1A = \begin{bmatrix} \langle 0.7,0\rangle & \langle 0.6,0.4\rangle \\ \langle 0.5,0.2\rangle & \langle 0.5,0.4\rangle \end{bmatrix} \neq A$.

$$A_\mu P_{2\mu} A_\mu = \left\{ \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0.6 & 0.7 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.6 \\ 0.5 & 0 \end{bmatrix} \neq A_\mu$$
$$\text{and } A_\vartheta P_{2\vartheta} A_\vartheta = \left\{ \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.5 \end{bmatrix}$$
$$= \begin{bmatrix} 0.4 & 0 \\ 0.5 & 0.2 \end{bmatrix} \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.4 \\ 0.2 & 0.5 \end{bmatrix} \neq A_\vartheta.$$

Therefore, $AP_2A = \begin{bmatrix} \langle 0.6,0.2\rangle & \langle 0.6,0.4\rangle \\ \langle 0.5,0.2\rangle & \langle 0,0.5\rangle \end{bmatrix} \neq A$

Therefore, A is not regular. A is 2 regular.

$$\text{For } X = \begin{bmatrix} \langle 0.7,0\rangle & \langle 0.6,0.4\rangle \\ \langle 0.5,0.2\rangle & \langle 0.5,0.4\rangle \end{bmatrix}$$

$$A_\mu^2 X_\mu A_\mu = \left\{\begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}\begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}\right\}\begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}\begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = A_\mu^2$$

and $A_\vartheta^2 X_\vartheta A_\vartheta = \left\{\begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix}\right\}\begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.5 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.5 \end{bmatrix} = \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix} = A_\vartheta^2.$$

Therefore, $A^2XA = \begin{bmatrix} \langle 0.7,0\rangle & \langle 0.6,0.4\rangle \\ \langle 0.5,0.2\rangle & \langle 0.5,0.4\rangle \end{bmatrix} = A^2$

$$\text{and } A_\mu X_\mu A_\mu^2 = \left\{\begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix}\begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}\right\}\begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}\begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = A_\mu^2$$

and

$$A_\vartheta X_{\hat{\vartheta}} A_\vartheta^2 = \left\{\begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.5 \end{bmatrix}\begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix}\right\}\begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix} = A_{\hat{\vartheta}}^2$$

Therefore, $AXA^2 = \begin{bmatrix} \langle 0.7,0\rangle & \langle 0.6,0.4\rangle \\ \langle 0.5,0.2\rangle & \langle 0.5,0.4\rangle \end{bmatrix} = A^2.$

Hence, $A^2XA = AXA^2 = A^2$. X is 2-g inverse of A.

Here,

$$X_\mu A_\mu X_\mu = \left\{\begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}\begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix}\right\}\begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}\begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} = X_\mu$$

and $X_\vartheta A_{\hat{\vartheta}} X_{\hat{\vartheta}} = \left\{\begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.5 \end{bmatrix}\right\}\begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix} = X_v.$$

Therefore, $XAX = \begin{bmatrix} \langle 0.7,0\rangle & \langle 0.6,0.4\rangle \\ \langle 0.5,0.2\rangle & \langle 0.5,04\rangle \end{bmatrix} = X.$

Therefore, $AX = \begin{bmatrix} \langle 0.7,0\rangle & \langle 0.6,0.4\rangle \\ \langle 0.5,0.2\rangle & \langle 0.5,0.4\rangle \end{bmatrix} = XA.$

Hence, X is a Drazin inverse of A.

For $B = \begin{bmatrix} \langle 0.7,0\rangle & \langle 0.6,0.4\rangle \\ \langle 0.5,0.2\rangle & \langle 0.6,0.2\rangle \end{bmatrix}$

$$B_\mu^2 = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.6 \end{bmatrix}\begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.6 \end{bmatrix}$$

and $B_\vartheta^2 = \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.2 \end{bmatrix} = \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.2 \end{bmatrix}$

Therefore, $B^2 = \begin{bmatrix} \langle 0.7,0 \rangle & \langle 0.6,0.4 \rangle \\ \langle 0.5,0.2 \rangle & \langle 0.6,0.2 \rangle \end{bmatrix} = B, B$ is regular.

Here,

Therefore, $A^2X = \begin{bmatrix} \langle 0.7,0 \rangle & \langle 0.6,0.4 \rangle \\ \langle 0.5,0.2 \rangle & \langle 0.5,0.4 \rangle \end{bmatrix} = B^2X.$

Therefore, $XA^2 = \begin{bmatrix} \langle 0.7,0 \rangle & \langle 0.6,0.4 \rangle \\ \langle 0.5,0.2 \rangle & \langle 0.5,0.4 \rangle \end{bmatrix} = XB^2.$

Hence, A is 2-Sharp ordering.

Example 3.3

From Example (3.2), A is 2-Sharp ordering.

Here, $A = \begin{bmatrix} \langle 0.7,0 \rangle & \langle 0.6,0.4 \rangle \\ \langle 0.5,0.2 \rangle & \langle 0,0.5 \rangle \end{bmatrix}$.

$$A\mu X_\mu A_\mu = \left\{ \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} \right\} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix} \neq A_\mu$$

and $A_\vartheta X_{\hat{\vartheta}} A_\vartheta = \left\{ \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix} \right\} \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.5 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.5 \end{bmatrix} = \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.4 \end{bmatrix} \neq A_{\hat{v}}.$$

Therefore, $AXA = \begin{bmatrix} \langle 0.7,0 \rangle & \langle 0.6,0.4 \rangle \\ \langle 0.5,0.2 \rangle & \langle 0.5,0.4 \rangle \end{bmatrix}$. $AXA \neq A$.

Therefore, k-Sharp Ordering need not imply Sharp Ordering.

Example 3.4

For $A = \begin{bmatrix} \langle 0.4,0 \rangle & \langle 0.1,0.2 \rangle \\ \langle 0.2,0.1 \rangle & \langle 0,0.4 \rangle \end{bmatrix}$,

$$A_\mu^2 = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0 \end{bmatrix} \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.1 \end{bmatrix} \neq A_\mu \text{ and}$$
$$A_\vartheta^2 = \begin{bmatrix} 0 & 0.2 \\ 0.1 & 0.4 \end{bmatrix} \begin{bmatrix} 0 & 0.2 \\ 0.1 & 0.4 \end{bmatrix} = \begin{bmatrix} & 0.2 \\ 0.1 & 0.2 \end{bmatrix} \neq A_\vartheta.$$

Therefore, $A^2 = \begin{bmatrix} \langle 0.4,0 \rangle & \langle 0.1,0.2 \rangle \\ \langle 0.2,0.1 \rangle & \langle 0.1,0.2 \rangle \end{bmatrix} \neq A$

For the permutation matrices $P_1 = \begin{bmatrix} \langle 1,0 \rangle & \langle 0,1 \rangle \\ \langle 0,1 \rangle & \langle 1,0 \rangle \end{bmatrix}$ and $P_2 = \begin{bmatrix} \langle 0,1 \rangle & \langle 1,0 \rangle \\ \langle 1,0 \rangle & \langle 0,1 \rangle \end{bmatrix}$.

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$$A_\mu P_{1\mu} A_\mu = \left\{\begin{bmatrix}0.4 & 0.1\\0.2 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right\}\begin{bmatrix}0.4 & 0.1\\0.2 & 0\end{bmatrix}$$
$$= \begin{bmatrix}0.4 & 0.1\\0.2 & 0\end{bmatrix}\begin{bmatrix}0.4 & 0.1\\0.2 & 0\end{bmatrix} = \begin{bmatrix}0.4 & 0.1\\0.2 & 0.1\end{bmatrix} \neq A_\mu$$
and $A_\vartheta P_{1\vartheta} A_\vartheta = \left\{\begin{bmatrix}0 & 0.2\\0.1 & 0.4\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right\}\begin{bmatrix}0 & 0.2\\0.1 & 0.4\end{bmatrix}$
$$= \begin{bmatrix}0 & 0.2\\0.1 & 0.4\end{bmatrix}\begin{bmatrix}0 & 0.2\\0.1 & 0.4\end{bmatrix} = \begin{bmatrix}0 & 0.2\\0.1 & 0.2\end{bmatrix} \neq A_\vartheta$$

Therefore, $AP_1A = \begin{bmatrix}\langle 0.4,0\rangle & \langle 0.1,0.2\rangle\\ \langle 0.2,0.1\rangle & \langle 0.1,0.2\rangle\end{bmatrix} \neq A$.

$$A_\mu P_{2\mu} A_\mu = \left\{\begin{bmatrix}0.4 & 0.1\\0.2 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right\}\begin{bmatrix}0.4 & 0.1\\0.2 & 0\end{bmatrix}$$
$$= \begin{bmatrix}0.1 & 0.4\\0 & 0.2\end{bmatrix}\begin{bmatrix}0.4 & 0.1\\0.2 & 0\end{bmatrix} = \begin{bmatrix}0.2 & 0.1\\0.2 & 0\end{bmatrix} \neq A_\mu$$
and $A_\vartheta P_{2\vartheta} A_\vartheta = \left\{\begin{bmatrix}0 & 0.2\\0.1 & 0.4\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right\}\begin{bmatrix}0 & 0.2\\0.1 & 0.4\end{bmatrix}$
$$= \begin{bmatrix}0.2 & 0\\0.4 & 0.1\end{bmatrix}\begin{bmatrix}0 & 0.2\\0.1 & 0.4\end{bmatrix} = \begin{bmatrix}0.1 & 0.2\\0.1 & 0.4\end{bmatrix} \neq A_\vartheta.$$

Therefore, $AP_2A = \begin{bmatrix}\langle 0.2,0.1\rangle & \langle 0.1,0.2\rangle\\ \langle 0.2,0.1\rangle & \langle 0,0.4\rangle\end{bmatrix} \neq A$

Therefore, A is not regular. A is 2 regular.

For $X = \begin{bmatrix}\langle 0.4,0\rangle & \langle 0,0.3\rangle\\ \langle 0.2,0.1\rangle & \langle 0.1,0.2\rangle\end{bmatrix}$

$$A_\mu^2 X_\mu A_\mu = \left\{\begin{bmatrix}0.4 & 0.1\\0.2 & 0.1\end{bmatrix}\begin{bmatrix}0.4 & 0\\0.2 & 0.1\end{bmatrix}\right\}\begin{bmatrix}0.4 & 0.1\\0.2 & 0\end{bmatrix}$$
$$= \begin{bmatrix}0.4 & 0.1\\0.2 & 0.1\end{bmatrix}\begin{bmatrix}0.4 & 0.1\\0.2 & 0\end{bmatrix} = \begin{bmatrix}0.4 & 0.1\\0.2 & 0.1\end{bmatrix} = A_\mu^2$$

and $A_\vartheta^2 X_{\hat\vartheta} A_\vartheta = \left\{\begin{bmatrix}0 & 0.2\\0.1 & 0.2\end{bmatrix}\begin{bmatrix}0 & 0.3\\0.1 & 0.2\end{bmatrix}\right\}\begin{bmatrix}0 & 0.2\\0.1 & 0.4\end{bmatrix}$

$$= \begin{bmatrix}0 & 0.2\\0.1 & 0.2\end{bmatrix}\begin{bmatrix}0 & 0.2\\0.1 & 0.4\end{bmatrix} = \begin{bmatrix}0 & 0.2\\0.1 & 0.2\end{bmatrix} = A_{\hat\nu}^2.$$

Therefore, $A^2XA = \begin{bmatrix}\langle 0.4,0\rangle & \langle 0.1,0.2\rangle\\ \langle 0.2,0.1\rangle & \langle 0.1,0.2\rangle\end{bmatrix} = A^2$ and $A_\mu X_\mu A_\mu^2 = \left\{\begin{bmatrix}0.4 & 0.1\\0.2 & 0\end{bmatrix}\begin{bmatrix}0.4 & 0\\0.2 & 0.1\end{bmatrix}\right\}\begin{bmatrix}0.4 & 0.1\\0.2 & 0.1\end{bmatrix}$

$$= \begin{bmatrix}0.4 & 0.1\\0.2 & 0\end{bmatrix}\begin{bmatrix}0.4 & 0.1\\0.2 & 0.1\end{bmatrix} = \begin{bmatrix}0.4 & 0.1\\0.2 & 0.1\end{bmatrix} = A_\mu^2$$

and $A_\vartheta X_\vartheta A_\vartheta^2 = \left\{\begin{bmatrix}0 & 0.2\\0.1 & 0.4\end{bmatrix}\begin{bmatrix}0 & 0.3\\0.1 & 0.2\end{bmatrix}\right\}\begin{bmatrix}0 & 0.2\\0.1 & 0.2\end{bmatrix}$
$$= \begin{bmatrix}0 & 0.2\\0.1 & 0.3\end{bmatrix}\begin{bmatrix}0 & 0.2\\0.1 & 0.2\end{bmatrix} = \begin{bmatrix}0 & 0.2\\0.1 & 0.2\end{bmatrix} = A_\vartheta^2.$$

Therefore, $AXA^2 = \begin{bmatrix}\langle 0.4,0\rangle & \langle 0.1,0.2\rangle\\ \langle 0.2,0.1\rangle & \langle 0.1,0.2\rangle\end{bmatrix} = A^2$.

Hence, $A^2XA = AXA^2 = A^2. X$ is 2-g inverse of A.

For $B = \begin{bmatrix}\langle 0.4,0\rangle & \langle 0.1,0.2\rangle\\ \langle 0.2,0.3\rangle & \langle 0.3,0.1\rangle\end{bmatrix}$

$$B_\mu^2 = \begin{bmatrix}0.4 & 0.1\\0.2 & 0.3\end{bmatrix}\begin{bmatrix}0.4 & 0.1\\0.2 & 0.3\end{bmatrix} = \begin{bmatrix}0.4 & 0.1\\0.2 & 0.3\end{bmatrix}$$

and $B_\vartheta^2 = \begin{bmatrix}0 & 0.2\\0.3 & 0.1\end{bmatrix}\begin{bmatrix}0 & 0.2\\0.3 & 0.1\end{bmatrix} = \begin{bmatrix}0 & 0.2\\0.3 & 0.1\end{bmatrix}$

Therefore, $B^2 = \begin{bmatrix}\langle 0.4,0\rangle & \langle 0.1,0.2\rangle\\\langle 0.2,0.3\rangle & \langle 0.3,0.1\rangle\end{bmatrix} = B, B$ is regular.

Here,

$$A_\mu^2 X_\mu = \begin{bmatrix}0.4 & 0.1\\0.2 & 0.1\end{bmatrix}\begin{bmatrix}0.4 & 0\\0.2 & 0.1\end{bmatrix} = \begin{bmatrix}0.4 & 0.1\\0.2 & 0.1\end{bmatrix}$$

and $B_\mu^2 X_\mu = \begin{bmatrix}0.4 & 0.1\\0.2 & 0.3\end{bmatrix}\begin{bmatrix}0.4 & 0\\0.2 & 0.1\end{bmatrix} = \begin{bmatrix}0.4 & 0.1\\0.2 & 0.1\end{bmatrix}$

.Therefore, $A_\mu^2 X_\mu = B_\mu^2 X_\mu$,

$A_\vartheta^2 X_\vartheta = \begin{bmatrix}0 & 0.2\\0.1 & 0.2\end{bmatrix}\begin{bmatrix}0 & 0.3\\0.1 & 0.2\end{bmatrix} = \begin{bmatrix}0 & 0.2\\0.1 & 0.2\end{bmatrix}$

and $B_\vartheta^2 X_\vartheta = \begin{bmatrix}0 & 0.2\\0.3 & 0.1\end{bmatrix}\begin{bmatrix}0 & 0.3\\0.1 & 0.2\end{bmatrix} = \begin{bmatrix}0 & 0.2\\0.1 & 0.2\end{bmatrix}$

. Therefore, $A_\vartheta^2 X_\vartheta = B_\vartheta^2 X_\vartheta$.

Therefore, $A^2 X = \begin{bmatrix}\langle 0.4,0\rangle & \langle 0.1,0.2\rangle\\\langle 0.2,0.1\rangle & \langle 0.1,0.2\rangle\end{bmatrix} = B^2 X$.

$X_\mu A_\mu^2 = \begin{bmatrix}0.4 & 0\\0.2 & 0.1\end{bmatrix}\begin{bmatrix}0.4 & 0.1\\0.2 & 0.1\end{bmatrix} = \begin{bmatrix}0.4 & 0.1\\0.2 & 0.1\end{bmatrix}$

and $X_\mu B_\mu^2 = \begin{bmatrix}0.4 & 0\\0.2 & 0.1\end{bmatrix}\begin{bmatrix}0.4 & 0.1\\0.2 & 0.3\end{bmatrix} = \begin{bmatrix}0.4 & 0.1\\0.2 & 0.1\end{bmatrix}$

.Therefore, $X_\mu A_\mu^2 = X_\mu B_\mu^2$,

$X_\vartheta A_\vartheta^2 = \begin{bmatrix}0 & 0.3\\0.1 & 0.2\end{bmatrix}\begin{bmatrix}0 & 0.2\\0.1 & 0.2\end{bmatrix} = \begin{bmatrix}0 & 0.2\\0.1 & 0.2\end{bmatrix}$

and $X_\vartheta B_\vartheta^2 = \begin{bmatrix}0 & 0.3\\0.1 & 0.2\end{bmatrix}\begin{bmatrix}0 & 0.2\\0.3 & 0.1\end{bmatrix} = \begin{bmatrix}0 & 0.2\\0.1 & 0.2\end{bmatrix}$

. Therefore, $X_v A_\vartheta^2 = X_v B_v^2$.

Therefore, $A^2 X = \begin{bmatrix}\langle 0.4,0\rangle & \langle 0.1,0.2\rangle\\\langle 0.2,0.1\rangle & \langle 0.1,0.2\rangle\end{bmatrix} = B^2 X$.

$$A_\mu X_\mu = \begin{bmatrix}0.4 & 0.1\\0.2 & 0\end{bmatrix}\begin{bmatrix}0.4 & 0\\0.2 & 0.1\end{bmatrix} = \begin{bmatrix}0.4 & 0.1\\0.2 & 0\end{bmatrix}$$
$$X_\mu A_\mu = \begin{bmatrix}0.4 & 0\\0.2 & 0.1\end{bmatrix}\begin{bmatrix}0.4 & 0.1\\0.2 & 0\end{bmatrix} = \begin{bmatrix}0.4 & 0.1\\0.2 & 0.1\end{bmatrix}.$$

Therefore, $A_\mu X_\mu \neq X_\mu A_\mu$.

$$A_\vartheta X_\vartheta = \begin{bmatrix}0 & 0.2\\0.1 & 0.4\end{bmatrix}\begin{bmatrix}0 & 0.3\\0.1 & 0.2\end{bmatrix} = \begin{bmatrix}0 & 0.2\\0.3 & 0.2\end{bmatrix}$$
$$X_\vartheta A_\vartheta = \begin{bmatrix}0 & 0.3\\0.1 & 0.2\end{bmatrix}\begin{bmatrix}0 & 0.2\\0.1 & 0.4\end{bmatrix} = \begin{bmatrix}0 & 0.2\\0.1 & 0.2\end{bmatrix}.$$

Therefore, $A_\vartheta X_\vartheta \neq X_\vartheta A_\vartheta$. Hence, $AX \neq XA$.

Therefore, Drazin inverse does not exist and hence A and B are not in 2-Sharp ordering.

Thus k-minus order need not imply k-Sharp order.

Lemma 3.1

For $A, B \in (IFM)_n^-$. If $A <_k^- B$ with $R(A) \subseteq R(B^k)$ and $C(A) \subseteq C(B^k)$, then $B\{1^k\} \subseteq A\{1^k\}$

Proof:

By Lemma [2.3],

$A <_k^- B \Rightarrow A^k = B^k UA = AVB^k$ for some $U, V \in A\{1^k\}$

If B is right k-regular with $R(A) \subseteq R(B^k)$, then by Lemma [2.3], $A = AVB$ for each right k-g inverse V of B.

If B is left k-regular with $C(A) \subseteq C(B^k)$, then by Lemma [2.3], $A = BUA$ for some left k-g inverse U of B.

For $X \in B\{1_r^k\}$,

$$\begin{aligned} A^k XA &= (AVB^k)X(BUA) \\ &= AV(B^k XB)UA \\ &= AVB^k UA \\ &= A^k UA = A^k. \end{aligned}$$

For $Y \in B\{1_\ell^k\}$,

$$\begin{aligned} AYA^k &= (AVB)Y(B^k UA) \\ &= AV(BYB^k)UA \\ &= AVB^k UA \\ &= AVA^k = A^k. \end{aligned}$$

Hence $B\{1^k\} \subseteq A\{1^k\}$.

Lemma 3.2

For $A \in (IFM)_n^{\tilde{D}}$ and $B \in (IFM)_n$, the following are equivalent:

(i) $A <_k^{\tilde{D}} B$

(ii) $A^k = B^k A_{\tilde{D}} A = AA_{\tilde{D}} B^k, A_{\tilde{D}}$ is the Drazin inverse of A.

Proof:

$$(i) \Rightarrow (ii)$$

$$A <_k^{\tilde{D}} B \Leftrightarrow A^k A_{\tilde{D}} = B^k A_{\tilde{D}}$$

$$\text{and } A_{\tilde{D}} A^k = A_{\tilde{D}} B^k$$

$A_{\tilde{D}}$ is the Drazin inverse of A.

$$A^k = A^k A_{\tilde{D}} A = B^k A_{\tilde{D}} A$$

$$A^k = AA_{\tilde{D}} A^k = AA_{\tilde{D}} B^k$$

$$(ii) \Rightarrow (i)$$

$$A_{\tilde{D}} \text{ exists } \Rightarrow A^k A_{\tilde{D}} A = AA_{\tilde{D}} A^k = A^k$$

$$A_{\tilde{D}} AA_{\tilde{D}} = A_{\tilde{D}}$$

$$AA_{\tilde{D}} = A_{\tilde{D}} A$$

and $A_{\tilde{D}}$ is unique.

$$\begin{aligned} A^k A_{\tilde{D}} &= (B^k A_{\tilde{D}} A) A_{\tilde{D}} \\ &= B^k (A_{\tilde{D}} A A_{\tilde{D}}) \\ &= B^k A_{\tilde{D}} \\ A_{\tilde{D}} A^k &= A_{\tilde{D}} (A A_{\tilde{D}} B^k) \\ &= (A_{\tilde{D}} A A_{\tilde{D}}) B^k \\ &= A_{\tilde{D}} B^k \end{aligned}$$

Corollory 3.1

For $A \in F_n^D$ and $B \in F_n$, the following are equivalent

(i) $A <_k^D B$

(ii) $A^k = B^k A_D A = A A_D B^k, A_D$ is the Drazin inverse of A.

Lemma 3.3

For $A, B \in (IFM)_n^{\tilde{D}}$, if $A <_k^{\tilde{D}} B$ then $R(A^k) \subseteq R(B^k), C(A^k) \subseteq C(B^k)$ and $A^k = A^k A_{\tilde{D}} B = B A_{\tilde{D}} A^k, A_{\tilde{D}}$ is the Drazin inverse of A.

Proof:

By Lemma [3.2],

$A^k = B^k A_{\tilde{D}} A = A A_{\tilde{D}} B^k$ By Lemma [3.1], If B is right k-regular and $R(A^k) \subseteq R(B^k)$,then $A^k = A^k A_{\tilde{D}} B, A_{\tilde{D}}$ is a right k-g inverse of B.

If B is left k-regular and $C(A^k) \subseteq C(B^k)$, then $A^k = B A_{\tilde{D}} A^k, A_{\tilde{D}}$ is a left k-g inverse of B.

Corollory 3.2

For $A, B \in F_n^D$, if $A <_k^D B$, then $R(A^k) \subseteq R(B^k), C(A^k) \subseteq C(B^k)$ and $A^k = A^k A_D B = B A_D A^k, A_D$ is the Drazin inverse of A.

Lemma 3.4

For $A \in (IFM)_n^{\tilde{D}}, B \in (IFM)_n$ the following are equivalent:

(i) $A <_k^{\tilde{D}} B$,

(ii) $A^k B^k = (A^k)^2 = B^k A^k$

Proof:

Since $A_{\tilde{D}}$ exists, $A^{k+1} A_{\tilde{D}} = A_{\tilde{D}} A^{k+1} = A^k$.

$A_{\tilde{D}} A A_{\tilde{D}} = A_{\tilde{D}}$ and $A_{\tilde{D}} A = A A_{\tilde{D}}$.

(i) ⇒ (ii)

By Lemma [3.2], $A <_k^{\tilde{D}} B \Leftrightarrow A^k = B^k A_{\tilde{D}} A = A A_{\tilde{D}} B^k$

$$\begin{aligned} A^k B^k &= A^{k+1} A_{\tilde{D}} B^k \\ &= A^k (A A_{\tilde{D}} B^k) \\ &= A^k A^k = (A^k)^2 \\ B^k A^k &= B^k A_{\tilde{D}} A^{k+1} = (B^k A_{\tilde{D}} A) A^k = A^k A^k = (A^k)^2 \end{aligned}$$

$(ii) \Rightarrow (i)$

$$
\begin{aligned}
A^k A_{\widetilde{D}} &= (A^{k+1} A_{\widetilde{D}}) A_{\widetilde{D}} \\
&= A^k (A A_{\widetilde{D}}) A_{\widetilde{D}} \\
&= (A^{k+1} A_{\widetilde{D}})(A A_{\widetilde{D}}) A_{\widetilde{D}} \\
&= A^k (A A_{\widetilde{D}})(A A_{\widetilde{D}}) A_{\widetilde{D}} \\
&= A^k (A A_{\widetilde{D}})^2 A_{\widetilde{D}} \\
&= \vdots \\
&= A^k (A A_{\widetilde{D}})^k A_{\widetilde{D}} \\
&= A^k (A^k)(A_{\widetilde{D}})^k A_{\widetilde{D}} \\
&= (A^k)^2 (A_{\widetilde{D}})^k A_{\widetilde{D}} \\
&= B^k A^k (A_{\widetilde{D}})^{k+1} \\
&= B^k A^{k-1} A_{\widetilde{D}}^{k-1} (A_{\widetilde{D}} A A_{\widetilde{D}}) \\
&= B^k A^{k-1} A_{\widetilde{D}}^{k} \\
&= B^k A^{k-2} A_{\widetilde{D}}^{k-2} (A_{\widetilde{D}} A A_{\widetilde{D}}) \\
&= B^k A^{k-2} A_{\widetilde{D}}^{k-1} \\
&= \vdots \\
&= B^k A_{\widetilde{D}}
\end{aligned}
$$

Similarly, $A_{\widetilde{D}} A^k = A_{\widetilde{D}} B^k$. Therefore $A <_k^{\widetilde{D}} B$. Hence the Proof.

Corollary 3.3

For $A \in F_n^D, B \in F_n$ the following are equivalent

(i) $A <_k^D B$

(ii) $A^k B^k = (A^k)^2 = B^k A^k$

Theorem 3.1

For $A \in (IFM)_n^{\widetilde{D}}$ and $B \in (IFM)_n$ then $A <_k^{\widetilde{D}} B \Leftrightarrow A^k B^k = B^k A^k$ and $A <_k^- B$

Proof:

By Lemma [3.4] and the Definition of k-minus ordering and k-sharp ordering,

$$A <_k^{\widetilde{D}} B \Rightarrow A <_k^- B \text{ and } A^k B^k = B^k A^k = (A^k)^2$$

Conversely $A <_k^- B \Rightarrow A^k = B^k U A = A V B^k, U, V \in A\{1^k\}$ (By Lemma [2.8])

$$
\begin{aligned}
A^k B^k &= (A V A^k) B^k \\
&= A V B^k A^k \\
&= A^k A^k = (A^k)^2 \\
B^k A^k &= B^k (A^k U A) \\
&= A^k B^k U A \\
&= A^k A^k = (A^k)^2
\end{aligned}
$$

By the Lemma [3.4],

$$A^k B^k = B^k A^k = (A^k)^2 \Rightarrow A <_k^{\widetilde{D}} B.$$

Corollory 3.4
For $A \in F_n^D, B \in F_n$ then $A <_k^D B \Leftrightarrow A^k B^k = B^k A^k$ and $A <_k^- B$.

Lemma 3.5
For $A, B \in (IFM)_n^{\tilde{D}}$ if $A <_k^{\tilde{D}} B$ and $B\{1^k\} \subseteq A\{1^k\}$ then $A^k = A^k B_{\tilde{D}} B = B B_{\tilde{D}} A^k$

Proof:
By Lemma [3.3]

$$A^k = A^k A_{\tilde{D}} B = B A_{\tilde{D}} A^k$$

$B\{1^k\} \subseteq A\{1^k\}$ and the existence of Drazin inverse is unique
$\Rightarrow$ the Drazin inverse of B is the Drazin inverse of A.
Hence $A^k = A^k B_{\tilde{D}} B = B B_{\tilde{D}} A^k$

Corollary 3.5
For $A, B \in F_n^D$, if $A <_k^D B$ and $B\{1^k\} \subseteq A\{1^k\}$, then $A^k = A^k B_D B = B B_D A^k$.

Lemma 3.6
For $A \in (IFM)_n$, if $A^\sharp$ exists then A^m has a group inverse for any positive integer $m > 1$.

Proof:
By using $AA^\sharp = A^\sharp A$, we have
$\left(A^\sharp\right)^m$ is a group inverse on A^m.
Hence $(A^m)^\sharp$ exists for all $m > 1$.

Theorem 3.2
For $A \in (IFM)_n$, if G is the Drazin inverse of A with index k, then for any $s \geq k, A^s$ has group inverse and G^s is the group inverse of A^s.

Proof:
Suppose A has Drazin inverse G with index k, then $A^{k+1} G = A^k, G^2 A = G$ and $AG = GA$.
Claim: G^s is the group inverse of A^s.

$$\begin{aligned} A^k G^k A^k &= A^k G G^{k-1} A^k \\ &= A^k G A G^{k-1} A^{k-1} \\ &= A^k G^{k-1} A^{k-1} \\ &=\vdots \\ &= A^k G A = A^k. \end{aligned}$$

$A^k G^k = G^k A^k$ follows from $AG = GA$.
Now, $G^k A^k G^k = G^k G^k A^k$

$$\begin{aligned} &= (G^k)^2 A^k \\ &= (G^2)^k A^k \\ &= (G^2 A)^k = G^k. \end{aligned}$$

Thus G^k is the group inverse of A^k.
By using Lemma [3.6], for any $s \geq k, A^s$ has group inverse and G^s is the group inverse of A^s.

Theorem 3.3

For $A \in (IFM)_n^{\widetilde{D}}$ and $B \in (IFM)_n, A <_k^{\widetilde{D}} \Rightarrow A^k \overset{\sharp}{<} B^k$

Proof:

$A <_k^{\widetilde{D}} \Rightarrow A^k G = B^k G$ and $GA^k = GB^k, G$ is the Drazin inverse of A.

Since G is the Drazin inverse of A,

$$A^{k+1}G = A^k$$
$$GAG = G \text{ and } AG = GA$$

By Theorem [3.2], if G is the Drazin inverse of A, then G^k is the group inverse of A^k.

Claim: $A^k G^k = B^k G^k$ and $G^k A^k = G^k B^k, G^k$ is the group inverse of A^k.

$$\begin{aligned} A^k G^k &= A^k G G^{k-1} \\ &= B^k G G^{k-1} = B^k G^k \\ G^k A^k &= G^{k-1} G A^k \\ &= G^{k-1} G B^k = G^k B^k. \end{aligned}$$

Hence $A^k \overset{\sharp}{<} B^k$

4. Conclusion

Ordering principles are essential for rating and categorising real-world situations. This study extends the concept of sharp ordering to k-regular fuzzy matrices and k-regular intuitionistic fuzzy matrices.

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