Advances in Nonlinear Variational Inequalities
ISSN: 1092-910X
Vol 28 No. 2 (2025)

New Results on Third Hankel Determinant for Certain Subclass of Bi-Univalent Functions

Zainab Sadiq Jafar*[1], Waggas Galib Atshan [2]
[1] Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq
[2] Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaniyah, Iraq; waggas.galib@qu.edu.iq
*Correspondence: zainabssadiq76@uomustansiriyah.edu.iq

Article History:

Received: *30-07-2024*

Revised: *17-09-2024*

Accepted: *24-09-2024*

Abstract:

This paper introduces new discoveries on the third Hankel determinant for the subclass C(β, δ) of bi-univalent functions in the open unit disk area D. The objective of this course is to ascertain the boundaries of the Hankel determinant of order 3, represented as H_3 (1). Furthermore, these calculations determine new boundaries for the third Hankel determinant inside the C(β, δ) family.

Keywords: Univalent Function, bi-univalent function, coefficients bounds, Analytic function, Hankel determinant.

1. Introduction

Consider $\mathcal{A}$ to be a collection of functions f that are analytic in the open unit disk $\mathcal{D}$, defined as $\mathcal{D} = \{z: z \in \mathbb{C}, |z| < 1\}$. An analytic function f belonging to the class $\mathcal{A}$ has a Taylor series expansion that may be expressed in the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \,, (z \epsilon \mathcal{D}). \quad (1)$$

The class of every functions in $\mathcal{A}$ that are univalent in $\mathcal{D}$ is indicated by the symbol S. The Koebe One Quarter Theorem [11] states that for each function f in class S, the domain $\mathcal{D}$ will contain a disk with a radius of $\frac{1}{4}$. Clearly, for any function f in the class S, we have an inverse function f^{-1} fulfills $f^{-1}\big(f(z)\big) = z, (z \in \mathcal{D})$ and $f^{-1}\big(f(w)\big) = w, \Big(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\Big)$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right)w^4 + \cdots \quad (2)$$

A function f from the class Σ is deemed to be bi-univalent in the domain $\mathcal{D}$ if both $f(z)$ with $f^{-1}(z)$ are univalent in $\mathcal{D}$.

Several authors [14,17,18,20,28-30] have conducted research on $H_2(2)$ for different categories of functions and have determined its optimal upper limit. The absolute value of the difference between a_3 and a_2 squared, denoted as $|a_3 - a_2^2|$, is referred to as the Fekete-Szegö functional $H_2(1)$. The functional $\left|a_3 - \mu a_2^2\right|$ was extended to encompass both real and complex values of μ. Fekete and Szegö calculated the precise estimates of $|a_3 - \mu a_2^2|$ for a class of univalent functions $f \in S$ and certain real values of μ. This estimate is sometimes referred to as the functional $|a_2 a_4 - a_3^2|$ analogous to $H_2(2)$. The Hankel determinant $H_3(1)$ was also investigated by several authors (Refs. [21,27,31,32]). The primary objective of our research is to study the class $C(\beta, \delta)$ in relation to the Hankel determinant $H_3(1)$.

Consider the class P of analytic functions p that are normalized by the condition:

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \ \mathrm{Re}(p(z)) > 0, z \in \mathcal{D}.$$

In 1976, Noonan and Thomas [22] introduced the q^{th} Hankel determinant of f for $n \geq 1$ such that $q \geq 1$ as follows:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \ \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, (a_1 = 1).$$

For $q = 2$ and $n = 1$, it is established that the function $H_2(1)$ may be represented as $a_3 - a_2^2$. The second Hankel determinant $H_2(2)$ is the absolute value of the expression $|a_2 a_4 - a_3^2|$ for the classes of bi-starlike and bi-convex ([3,4,5,6,7,8,9,12,23]). Al-Ameedee et al. [1] examined the second Hankel determinant for particular subclasses of bi-univalent functions. Furthermore, Atshan et al.[2] examined the Hankel determinant of m-fold symmetric bi-univalent functions utilising a novel operator. Fekete and Szegö [13] investigated the Hankel determinant of function f as

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_1 a_3 - a_2^2.$$

An earlier investigation was conducted to determine the value of $|a_3 - \mu a_2^2|$, where $a_1 = 1$ and $\mu \in \mathbb{R}$. In addition, as an illustration, individuals who have a value of $|a_3 - \mu a_2^2|$ can refer to [15]. The third Hankel determinant and these functions have been examined in the context of the functional described in [7,10,19,24,25,26,33].

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}, \ (a_1 = 1) \ \text{and} \ (n = 1, q = 3).$$

By utilizing the triangle inequality for $H_3(1)$, we may deduce

$$|H_3(1)| \leq |a_3||a_2 a_4 - a_3^2| - |a_4||a_4 - a_2 a_3| + |a_5||a_3 - a_2^2|. \quad (3)$$

First, some preliminary lemmas.

Lemma 1 ([11]). Let's consider the class P, which comprises all analytic functions $p(z)$ can be expressed as

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad (4)$$

Such that $\mathrm{Re}(p(z)) > 0 \ \forall z \in \mathcal{D}$. Thus $|p_n| \leq 2, \ \forall n = 1,2,\cdots$.

Lemma 2 ([16]). If a function p belongs to the set P and is defined by equation (4), then

$$2p_2 = p_1^2 + (4 - p_1^2)x$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z.$$

2. Main Results

Definition 1. A function f belonging to the class Σ is said to be in a class $C(\beta, \delta)$, if it satisfies the following conditions:

$$\mathrm{Re}\left(\delta z\left(\frac{f'(z)}{f(z)} + \frac{f''(z)}{f'(z)} + zf'''(z)\right)\right) > \beta, \tag{5}$$

$$\mathrm{Re}\left(\delta w\left(\frac{g'(w)}{g(w)} + \frac{g''(w)}{g'(w)} + wg'''(w)\right)\right) > \beta, \tag{6}$$

where $(0 < \beta \leq 1)$, $\delta > 0, z, w \in \mathcal{D}$ and $g = f^{-1}$.

Theorem 1. Let $f(z)$ be a function given by equation (1), belonging to the class $C(\beta, \delta)$, where $0 \leq \beta < 1, \delta > 0$. Next, we possess

$$|a_2a_4 - a_3^2| \leq \frac{(1-\beta)^2}{\delta^2}\left[\frac{16}{117}(1-\beta)^2 + \frac{4}{117}\right]. \tag{7}$$

Proof. Based on equations (5) and (6), we may deduce

$$\delta z\left(\frac{f'(z)}{f(z)} + \frac{f''(z)}{f'(z)} + zf'''(z)\right) = \beta + (1-\beta)p(z) \tag{8}$$

and

$$\delta w\left(\frac{g'(w)}{g(w)} + \frac{g''(w)}{g'(w)} + wg'''(w)\right) = \beta + (1-\beta)q(w), \tag{9}$$

where $(0 \leq \beta < 1; p, q \in P)$, $z,\ w \in \mathcal{D}$ and $g = f^{-1}$.

Given the function $\mathfrak{a}$ and $\mathfrak{b}$, defined on the domain $\mathcal{D}$ and mapping to the range $\mathcal{D}$, with $\mathfrak{a}(0) = \mathfrak{b}(0) = 0, |\mathfrak{a}(z)| < 1$, and $|\mathfrak{b}(w)| < 1$, we consider the functions p and q belonging to the set P,with

$$p(z) = \frac{1 + \mathfrak{a}(z)}{1 - \mathfrak{a}(z)} = 1 + \sum_{n=1}^{\infty} e_n z^n$$

and

$$q(w) = \frac{1 + \mathfrak{b}(w)}{1 - \mathfrak{b}(w)} = 1 + \sum_{n=1}^{\infty} d_n w^n.$$

$$\beta + (1-\beta)p(z) = 1 + \sum_{n=1}^{\infty}(1-\beta)\, e_n z^n \tag{10}$$

and

$$\beta+(1-\beta)\mathrm{q}(w)=1+\sum_{\mathrm{n}=1}^{\infty}(1-\beta)\,d_{\mathrm{n}}w^{\mathrm{n}}. \tag{11}$$

Given that $f\in\Sigma$ has the Maclurian series specified by (1) and that its inverse, $g=f^{-1}$, can be represented using the expansion provided by (2), we can observe that

$$\begin{aligned}\delta z\left(\frac{f'(z)}{f(z)}+\frac{f''(z)}{f'(z)}+zf'''(z)\right)\\ &=\delta+3\delta\mathrm{a}_2z+\delta(14\mathrm{a}_3-5\mathrm{a}_2^2)z^2+3\delta\left(13\mathrm{a}_4-7\mathrm{a}_2\mathrm{a}_3+3\mathrm{a}_2^3\right)z^3\\ &+\delta\left(84\mathrm{a}_5-36\mathrm{a}_2\mathrm{a}_4-20\mathrm{a}_3^2+52\mathrm{a}_2^2\mathrm{a}_3-17\mathrm{a}_2^4\right)z^4+\cdots\end{aligned} \tag{12}$$

and

$$\begin{aligned}&\delta w\left(\frac{g'(w)}{g(w)}+\frac{g''(w)}{g'(w)}+wg'''(w)\right)\\ &=\delta-3\delta\mathrm{a}_2w+\delta(23\mathrm{a}_2^2-14\mathrm{a}_3)w^2+3\delta\left(-54\mathrm{a}_2^3+58\mathrm{a}_2\mathrm{a}_3-13\mathrm{a}_4\right)w^3\\ &+\delta\left(468\mathrm{a}_2\mathrm{a}_4+1003\mathrm{a}_2^4-1764\mathrm{a}_2\mathrm{a}_3+232\mathrm{a}_3^2-84\mathrm{a}_5+208\mathrm{a}_2^2\mathrm{a}_3\right)w^4\\ &+\cdots.\end{aligned} \tag{13}$$

It follows from (10) and (11), together with (12) and (13), that

$$3\delta\mathrm{a}_2=(1-\beta)\mathrm{u}_1, \tag{14}$$

$$\delta\left(14\mathrm{a}_3-5\mathrm{a}_2^2\right)=(1-\beta)\mathrm{u}_2, \tag{15}$$

$$3\delta(13\mathrm{a}_4-7\mathrm{a}_2\mathrm{a}_3+3\mathrm{a}_2^3)=(1-\beta)\mathrm{u}_3, \tag{16}$$

$$\delta\left(84\mathrm{a}_5-36\mathrm{a}_2\mathrm{a}_4-20\mathrm{a}_3^2+52\mathrm{a}_2^2\mathrm{a}_3-17\mathrm{a}_2^4\right)=(1-\beta)\mathrm{u}_4 \tag{17}$$

and

$$-3\delta\mathrm{a}_2=(1-\beta)\mathrm{v}_1, \tag{18}$$

$$\delta\left(23\mathrm{a}_2^2-14\mathrm{a}_3\right)=(1-\beta)\mathrm{v}_2, \tag{19}$$

$$3\delta\left(-54\mathrm{a}_2^3+58\mathrm{a}_2\mathrm{a}_3-13\mathrm{a}_4\right)=(1-\beta)\mathrm{v}_3, \tag{20}$$

$$\delta\left(468\mathrm{a}_2\mathrm{a}_4+1003\mathrm{a}_2^4-1764\mathrm{a}_2\mathrm{a}_3+232\mathrm{a}_3^2-84\mathrm{a}_5+208\mathrm{a}_2^2\mathrm{a}_3\right)=(1-\beta)\mathrm{v}_4. \tag{21}$$

From (14) and (18), we have

$$\frac{(1-\beta)\mathrm{u}_1}{3\delta}=\mathrm{a}_2=\frac{-(1-\beta)\mathrm{v}_1}{3\delta}, \tag{22}$$

It follows that its

$$\mathrm{u}_1=-\mathrm{v}_1, \tag{23}$$

Subtracting (15) from (19) and (16) from (20), we get

$$\mathrm{a}_3=\frac{(1-\beta)^2\mathrm{u}_1^2}{9\delta^2}+\frac{(1-\beta)(\mathrm{u}_2-\mathrm{v}_2)}{28\delta} \tag{24}$$

and

$$a_4 = \frac{4(1-\beta)^3 u_1^3}{351\delta^3} + \frac{65(1-\beta)^2 u_1(u_2 - v_2)}{2184\delta^2} + \frac{(1-\beta)(u_3 - v_3)}{78\delta}. \quad (25)$$

Thus, by applying (22), (24) and (25), we find that

$$a_2a_4 - a_3^2 = \frac{1}{504\delta^3}(1-\beta)^3 u_1^2(u_2 - v_2) - \frac{1}{117\delta^4}(1-\beta)^4 u_1^4 + \frac{1}{234\delta^2}(1-\beta)^2 u_1(u_3 - v_3) - \frac{1}{784\delta^2}(1-\beta)^2(u_2 - v_2)^2. \quad (26)$$

Next, according to Lemma 2 and (23), we have

$$u_2 - v_2 = \frac{4 - u_1^2}{2}(x - y) \quad (27)$$

and

$$u_3 - v_3 = \frac{u_1^3}{2} + \frac{(4 - u_1^2)u_1}{2}(x + y) - \frac{(4 - u_1^2)u_1}{4}(x^2 + y^2) + \frac{4 - u_1^2}{2}[(1 - |x|^2)z - (1 - |y|^2)w]. \quad (28)$$

For some x,y, z with w such that $|x| \leq 1, |y| \leq 1$ with $|w| \leq 1$.

Since $p \in P$, it follows that the absolute value of $|u_1| \leq 2$. Assuming $u_1 = u$, we can presume, without any negative impact on the whole analysis, that u belongs to the interval [0,2]. Thus, by replacing the equations (27) and (28) in equation (26), with the condition that $\sigma = |x| \leq 1$ and $\xi = |y| \leq 1$, we obtain

$$|a_2a_4 - a_3^2| \leq F_1 + F_2(\sigma + \xi) + F_3(\sigma^2 + \xi^2) + F_4(\sigma + \xi)^2 = F(\sigma, \xi),$$

where

$$F_1 = F_1(\beta, u) = \frac{(1-\beta)^2 u^4}{117\delta^2}\left(\frac{(1-\beta)^2}{\delta^2} + \frac{1}{4}\right) \geq 0,$$

$$F_2 = F_2(\beta, u) = \frac{(1-\beta)^2(4 - u^2)u^2}{9\delta^2}\left(\frac{(1-\beta)}{112\delta} + \frac{1}{52}\right) \geq 0,$$

$$F_3 = F_3(\beta, u) = \frac{(1-\beta)^2(4 - u^2)u}{468\delta^2}\left(\frac{u}{2} - 1\right) \leq 0$$

and

$$F_4 = F_4(\beta, u) = \frac{(1-\beta)^2(4 - u^2)^2}{3136\delta^2} \geq 0.$$

Our objective is to optimize the function $F(\sigma, \xi)$ over the closed square $[0,1] \times [0,1]$ for values of u that range from 0 to 2. Given that $F_3 \leq 0$ with $F_3 + 2F_2 \geq 0$, we may deduce that the value of u is between 0 and 2. Additionally, the expression $E_{\sigma,\sigma}E_{\xi,\xi} - (E_{\sigma,\xi})^2 < 0$.

hence, the function E is incapable of possessing a local maximum inside the confines of a closed square. Now, we examine the highest value of E along the perimeter of a closed square. When the standard deviation $\sigma = 0$ with $0 \leq \xi \leq 1$, the following condition holds:

$$E(0,\xi) = \vartheta(\xi) = F_1 + F_2\xi + (F_3 + F_4)\xi^2.$$

Currently, we will examine the following two examples:

Case 1. The formula $F_3 + F_4 \geq 0$ holds. Given the constraints, where $0 \leq \xi \leq 1$, and any constant value of u, where $0 \leq u < 2$, it is clear that

$$\vartheta'(\xi) = F_2 + 2(F_3 + F_4)\xi > 0,$$

The function $\vartheta(\xi)$ is monotonically increasing. Hence, when u is a constant number within the range of $[0,2]$, the highest value of $\vartheta(\xi)$ is obtained when ξ equals 1, with

$$\max \vartheta(\xi) = \vartheta(1) = F_1 + F_2 + F_3 + F_4.$$

Case 2. Let $F_3 + F_4 < 0$. Since $2(F_3 + F_4) + F_2 \geq 0$ for $0 < \xi < 1$ with $0 < u < 2$, it is clear that $2(F_3 + F_4) + F_2 < 2(F_3 + F_4)\xi + F_2 < F_2$ and so $\vartheta(\xi) > 0$. Therefore, the maximum of $\vartheta(\xi)$ occurs at $\xi = 1$ such that $0 \leq \xi \leq 1$, we get

$$F(1,\xi) = \phi(\xi) = (F_3 + F_4)\xi^2 + (F_2 + 2F_4)\xi + F_1 + F_2 + F_3 + F_4.$$

Therefore, considering the instances of $F_3 + F_4$, we obtain

$$\max \phi(\xi) = \phi(1) = F_1 + 2F_2 + 2F_3 + 4F_4.$$

Since $\vartheta(1) \leq \phi(1)$, we get $\max\big(F(\sigma,\xi)\big) = F(1,1)$ on the boundary of square $[0,1] \times [0,1]$. The function $\mathcal{G}$, defined on the open interval $(0,1)$, is given by the following:

$$\mathcal{G}(u) = \max\big(F(\sigma,\xi)\big) = F(1,1) = F_1 + 2F_2 + 2F_3 + 4F_4.$$

Now, putting F_1, F_2, F_3 and F_4 in the function $\mathcal{G}$, we obtain

$$\mathcal{G}(u) = (1-\beta)^2[C + D],$$

where

$$C = \frac{u^4}{3}\left[\frac{(1-\beta)^2}{39\delta^2} + \frac{1}{156}\right]$$

and

$$D = \frac{(4-u^2)}{2}\left[\frac{u^2}{78} + \frac{(1-\beta)u^2}{252\delta} - \frac{u}{117} + \frac{(4-u^2)}{392}\right].$$

Through basic computations, it is determined that $\mathcal{G}(u)$ has a positive correlation with u.

Therefore, the maximum value of $\mathcal{G}(u)$ is achieved when u is equal to 2 and

$$\max \mathcal{G}(u) = \mathcal{G}(2) = \frac{(1-\beta)^2}{\delta^2}\left[\frac{16}{117}(1-\beta)^2 + \frac{4}{117}\right].$$

This evidently completes the proof of the above Theorem.

Theorem 2. Let $f(z) \in C(\beta, \delta), 0 \leq \beta < 1, \delta > 0$. Then, we have

$$|a_2a_3 - a_4| \leq \begin{cases} \frac{8}{39\delta}(1-\beta)\left[\frac{(1-\beta)^2}{\delta^2} + \frac{1}{4}\right], & n \leq u \leq 2 \\ \frac{2}{39\delta}(1-\beta), & 0 \leq u \leq n, \end{cases} \quad (29)$$

where

$$n = \frac{c_3 \pm \sqrt{c_3^2 - 12c_2(c_1 - c_2)}}{3(c_1 - c_2)},$$

$$c_1 = \frac{(1-\beta)}{39\delta}\left[\frac{(1-\beta)^2}{\delta^2} + \frac{1}{4}\right],$$

$$c_2 = \frac{(1-\beta)}{\delta}\left[\frac{(1-\beta)}{56\delta} + \frac{1}{52}\right],$$

and

$$c_3 = \frac{1}{78\delta}(1-\beta).$$

Proof. From (22), (24) and (25), we obtain

$$|a_2a_3 - a_4| = \left|\frac{9(1-\beta)^3u_1^3}{351\delta^3} - \frac{39(1-\beta)^2u_1(u_2 - v_2)}{2184\delta^2} - \frac{(1-\beta)(u_3 - v_3)}{78\delta}\right|.$$

Lemma 2. States that we may make the assumption, without any limitations, that u belongs to the interval [0,2]. Here, $u_1 = u$, thus for $\varsigma = |x| \leq 1$ with $\zeta = |y| \leq 1$, we obtain

$$|a_2a_3 - a_4| \leq \mathcal{J}_1 + \mathcal{J}_2(\varsigma + \zeta) + \mathcal{J}_3(\varsigma^2 + \zeta^2) = \mathcal{J}(\varsigma, \zeta),$$

where

$$\mathcal{J}_1(\beta, u) = \frac{(1-\beta)u^3}{39\delta}\left(\frac{(1-\beta)^2}{\delta^2} + \frac{1}{4}\right) \geq 0,$$

$$\mathcal{J}_2(\beta, u) = \frac{(1-\beta)(4-u^2)u}{4\delta}\left(\frac{(1-\beta)}{28\delta} + \frac{1}{39}\right) \geq 0$$

and

$$\mathcal{J}_3(\beta, u) = \frac{(1-\beta)(4-u^2)}{156\delta}\left(\frac{u}{2} + 1\right) \geq 0.$$

By employing the identical methodology as Theorem 2, we determine that the highest value is achieved when ς equals 1 and ζ equals 1in the closed square [0,2],

$$\phi(u) = \max(\mathcal{J}(\varsigma, \zeta)) = \mathcal{J}_1 + 2(\mathcal{J}_2 + \mathcal{J}_3).$$

By replacing the value of $\mathcal{J}_1, \mathcal{J}_2$ and $\mathcal{J}_3$ in the function $\phi(u)$, we obtain

$$\phi(\mathrm{u}) = \mathfrak{c}_1\mathrm{u}^3 + \mathfrak{c}_2\mathrm{u}(4-\mathrm{u}^2) + \mathfrak{c}_3(4-\mathrm{u}^2),$$

where

$$\mathfrak{c}_1 = \frac{(1-\beta)}{39\delta}\left[\frac{(1-\beta)^2}{\delta^2} + \frac{1}{4}\right],$$

$$\mathfrak{c}_2 = \frac{(1-\beta)}{\delta}\left[\frac{(1-\beta)}{56\delta} + \frac{1}{52}\right]$$

and

$$\mathfrak{c}_3 = \frac{1}{78\delta}(1-\beta).$$

We have

$$\phi'(\mathrm{u}) = 3(\mathfrak{c}_1 - \mathfrak{c}_2)\mathrm{u}^2 - 2\mathfrak{c}_3\mathrm{u} + 4\mathfrak{c}_2,$$

$$\phi''(\mathrm{u}) = 6(\mathfrak{c}_1 - \mathfrak{c}_2)\mathrm{u} - 2\mathfrak{c}_3.$$

If $\mathfrak{c}_1 - \mathfrak{c}_2 > 0$, then it means that $\mathfrak{c}_1$ is greater than $\mathfrak{c}_2$. Next, we note that the derivative of $\phi'(\mathrm{u}) > 0$. Thus, the function $\phi(\mathrm{u})$ exhibits monotonically growing behavior within the confined interval [0,2]. Therefore, the function $\phi(\mathrm{u})$ attains its greatest value at $\mathrm{u} = 2$, when

$$|\mathrm{a}_2\mathrm{a}_3 - \mathrm{a}_4| \le \phi(2) = \frac{8(1-\beta)}{39\delta}\left[\frac{(1-\beta)^2}{\delta^2} + \frac{1}{4}\right],$$

if $\mathfrak{c}_1 - \mathfrak{c}_2 < 0$, let $\phi'(\mathrm{u}) = 0$, then we receive

$$\mathrm{u} = n = \frac{\mathfrak{c}_3 \pm \sqrt{\mathfrak{c}_3^2 - 12\mathfrak{c}_2(\mathfrak{c}_1 - \mathfrak{c}_2)}}{3(\mathfrak{c}_1 - \mathfrak{c}_2)},$$

when $n < \mathrm{u} \le 2$. As a result, we find $\phi'(\mathrm{u}) > 0$, indicating that the function on the closed interval is $[0,2]$. Thus, the function $\phi(\mathrm{u})$ gets the maximum value at $\mathrm{u} = 2$, This implies that the constant $\phi(\mathrm{u})$ is an decreasing function on the closed interval $[0,2]$. Thus, $\phi(\mathrm{u})$ obtains the maximum value at $\mathrm{u} = 0$. We accept

$$|\mathrm{a}_2\mathrm{a}_3 - \mathrm{a}_4| \le \phi(0) = \frac{2}{39\delta}(1-\beta).$$

This clearly concludes the demonstration of the aforementioned Theorem.

Theorem 3. Let $f(z) \in \mathrm{C}(\beta, \delta)$, $0 \le \beta < 1, \delta > 0$. Then, we have

$$\left|\mathrm{a}_3 - \mathrm{a}_2^2\right| \le \frac{1}{7\delta}(1-\beta), \tag{30}$$

$$|\mathrm{a}_3| \le \frac{4}{9\delta^2}(1-\beta)^2 + \frac{1}{7\delta}(1-\beta) \tag{31}$$

Proof. By using the equation (24) and applying Lemma 1, we derive the result (31).

The definition of what comes after the Fekete-Szegö functional applies to $\mu \in \mathbb{C}$ with $f \in \mathrm{C}(\beta, \delta)$,

$$a_3 - \mu a_2^2 = \frac{(1-\beta)^2 u_1^2}{9\delta^2}(1-\mu) + \frac{(1-\beta)(u_2 - v_2)}{28\delta}.$$

By Lemma 1, we receive

$$\left|a_3 - \mu a_2^2\right| \le \frac{4}{9\delta^2}(1-\beta)^2(1-\mu) + \frac{1}{7\delta}(1-\beta),$$

When the value of μ is equal to 1, the result we acquire is (30).

Theorem 4. Let $f(z) \in C(\beta, \delta), 0 \le \beta < 1, \delta > 0$. Then we hold

$$|a_4| \le \frac{(1-\beta)}{\delta}\left[\frac{32}{351\delta^2}(1-\beta)^2 + \frac{65}{273\delta}(1-\beta) + \frac{2}{39}\right], \quad (32)$$

$$|a_5| \le \frac{(1-\beta)}{\delta}\left[\frac{13720}{7371\delta^3}(1-\beta)^3 + \frac{95732}{40131\delta^2}(1-\beta)^2 + \frac{162337}{1872780\delta}(1-\beta) + \frac{1}{42}\right]. \quad (33)$$

Proof. By applying Lemma 1 to the given information in (25), we obtain the result stated in (32).

By computing the difference between the numbers (21) and (17), we obtain

$$168\delta a_5 = 504\delta a_2 a_4 + 252\delta a_3^2 + 156\delta a_2^2 a_3 + 1020\delta a_2^4 - 1764\delta a_2 a_3 + (1-\beta)(u_4 - v_4).$$

By substituting properly (22), (24) and (25), we have

$$\begin{aligned} a_5 = {} & \frac{20580}{176904\delta^4}(1-\beta)^4 u_1^4 + \frac{49920}{1100736\delta^3}(1-\beta)^3 u_1^2(u_2 - v_2) \\ & + \frac{504}{39312\delta^2}(1-\beta)^2 u_1(u_3 - v_3) + \frac{252}{131712\delta^2}(1-\beta)^2(u_2 - v_2)^2 \\ & - \frac{1764}{4536\delta^3}(1-\beta)^3 u_1^3 - \frac{1764}{14112\delta^2}(1-\beta)^2 u_1(u_2 - v_2) \\ & + \frac{1}{168\delta}(1-\beta)(u_4 - v_4). \end{aligned}$$

By using Lemma 1, we derive equation (33).

Theorem 5. Consider a function $f(z) \in C(\beta, \delta), 0 \le \beta < 1, \delta > 0$. Next, we possess

$$|H_3(1)| \le \begin{cases} \mathcal{K}\mathcal{K}_1 - \mathcal{K}_2\left(\frac{8}{39\delta}(1-\beta)\left[\frac{(1-\beta)^2}{\delta^2} + \frac{1}{4}\right]\right) + \mathcal{K}_3\mathcal{K}_4, & n \le u \le 2 \\ \mathcal{K}\mathcal{K}_1 - \frac{2}{39\delta}(1-\beta), & 0 \le u \le n, \end{cases} \quad (34)$$

where $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$ and n are obtained by equation (31), (7), (32), (33) and (30), respectively.

Proof. Since

$$|H_3(1)| = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5\left(a_3 - a_2^2\right).$$

By utilizing the triangle inequality, we receive the result (3).

Substituting $|a_3| \le \frac{4}{9\delta^2}(1-\beta)^2 + \frac{1}{7\delta}(1-\beta)$,

$$|a_2a_4 - a_3^2| \le \frac{(1-\beta)^2}{\delta^2}\left[\frac{16}{117}(1-\beta)^2 + \frac{4}{117}\right],$$

$$|a_4| \le \frac{(1-\beta)}{\delta}\left[\frac{32}{351\delta^2}(1-\beta)^2 + \frac{65}{273\delta}(1-\beta) + \frac{2}{39}\right],$$

$$|a_5| \le \frac{(1-\beta)}{\delta}\left[\frac{13720}{7371\delta^3}(1-\beta)^3 + \frac{95732}{40131\delta^2}(1-\beta)^2 + \frac{162337}{187278\delta}(1-\beta) + \frac{1}{42}\right]$$

and

$$\left|a_3 - a_2^2\right| \le \frac{1}{7\delta}(1-\beta)$$

in

$$|H_3(1)| \le |a_3|\left|a_2a_4 - a_3^2\right| - |a_4||a_4 - a_2a_3| + |a_5|\left|a_3 - a_2^2\right|,$$

we obtain (34).

This concludes the demonstration of the aforementioned Theorem.

3. Discussion

Our research enhances the comprehensive understanding of bi-univalent functions, their subclasses, and their prospective applications across several mathematical domains. The data acquired may provide a basis for subsequent research into the characteristics and uses of bi-univalent functions and their subclasses. Future research endeavors may investigate more improvements of the boundaries and analyse other subclasses of bi-univalent functions to reveal new insights into their properties and potential applications. This study facilitates a more profound investigation of the intriguing domain of bi-univalent functions and their significance in mathematics.

4.Conclusions:

This article conducted a thorough examination of the third Hankel determinant $H_3(1)$ for a specific subclass of bi-univalent functions, $C(\beta,\delta)$. This subclass holds considerable significance in multiple mathematical domains, including complex analysis and geometric function theory. We defined the bi-univalent functions $C(\beta,\delta)$ and established constraints on the coefficients $|a_n|$. Our findings established the top bounds for bi-univalent functions within this newly created subclass, specifically for $n = 2,3,4$ and 5.

References

[1] S. A. Al-Ameedee, W. G. Atshan and F. A. Al-Maamori, Second Hankel determinant for certain subclasses of bi-univalent functions, J. Phys. Conf. Ser., (2020), 1664, 012044.

[2] W. G. Atshan, R. A. Al-Sajjad, S. Altinkaya, On the Hankel determinant of m-fold symmetric bi-univalent functions using a new operator, Gazi Univ. J. Sci., (2023), 36, 349-360.

[3] W. G. Atshan, R. A. Al-Sajjad, Some Applications of quasi-subordination for bi-univalent functions using Jackson's convolution operator, Iraqi Journal of Science, 63(10) (2022), pp. 4417-4428.

[4] W. G. Atshan and A. H. Abada, On subclass of k-uniformly convex functions of complex order involving multiplier transformations, Abstract and Applied Analysis, Volume 2012, Issue 1, (2012), Article ID 150571, 10 pages.

[5] W. G. Atshan, E. H. Abd and S. Yalcin, An application of subclasses of harmonic univalent functions involving hypergeometric function, Adiyaman University Journal of Science, 10(2)(2020), pp. 595-607.

[6] W. G. Atshan and A. S. Joudah, On a class of univalent functions with negative coefficients defined by generalized Ruscheweyh derivatives II, European Journal of Scientific Research, 64(2) (2011), pp. 340-352.

[7] K. O. Babalola, On $H_3(1)$ Hankel determinant for some classes of univalent functions, Inequal. Theory Appl., (2010) ,6, 1-7.

[8] E. I. Badiwi, W. G. Atshan, A. N. Alkiffai and A. A. Lupas, Certain results on subclasses of analytic and bi-univalent functions associated with coefficient estimates and quasi-subordination, Symmetry, 15(12)(2023), 2208, 1-12.

[9] M. Buyankara, M. Caglar, Hankel and Toeplitz determinants for a subclass of analytic functions, Math. Stud., (2023), 60, 132-137.

[10] A. M. Darweesh, W. G. Atshan, A. H. Battor and M. S. Mahdi, On the third Hankel determinant of certain subclass of bi-univalent, Mathematical Modelling of Engineering Problems, 10(3)(2023), pp. 1087-1095.

[11] P. L. Duren, Univalent Functions. In Grundlehren der Mathematischen Wissenschaften, Band 259; Springer: New York, NY, USA; Berlin/Hidelberg, Germany; Tokyo, Japan, (1983).

[12] E. Deniz, M. Caglar and H. Orhan, Second Hankel determinant for bi-starlike and bi-convex functions of order α, Appl. Math. Comput., (2015), 271, 301-307.

[13] M. Fekete, G. Szegö, Eine Bemerkung über ungerade Schlichte Funktionen, J. Lond. Math. Soc., (1933), 1, 85-89.

[14] S. K. Gebur and W. G. Atshan, Second Hankel determinant and Fekete-Szegö problem for a new class of bi-univalent functions involving Euler polynomials, Symmetry, 16(5) (2024), 530, 1-16.

[15] H. O. Guney, G. Murugusundaramoorthy and K. Vijaya, Coefficient bounds for subclasses of bi-univalent functions associated with the Chebyshev polynomials, J. Complex Anal., (2017), 2017, 4150210.

[16] U. Grenander, G. Szegö, Toeplilz Forms and Their Applications, California Monographs in Mathematical Sciences; University California Press: Berkeley, CA, USA, (1958).

[17] H. Ö. Güney, G. Murugusundaramoorthy and H. M. Srivastava, The second hankel determinant for a certain class of bi-close-to convex functions, Results Math., (2019), 74, 93.

[18] A. Janteng, S. A. Halim and M. Darus, coefficient inequality for a function whose derivative has a positive real part. J. Inequal. Pure Appl. Math., (2006), 7, 1-5.

[19] A. Khan, M. Haq, L. I. Cotîrlă, G. I. Oros, Bernardi Integral Operator and Its Application to the Fourth Hankel Determinant , J. Funct. Spaces, (2022), 4227493.

[20] A. K. Mishra, P. Gochhayat, Second hankel determinant for a class of analytic functions defined by fractional derivative, Int. J. Math. Sci., (2008), 153280.

[21] S. Mahmood, H. M. Srivastava, N. Khan, Q. Z. Ahmad, B. Khan and I. Ali, Upper bound of the third hankel determinant for a subclass of q-starlike functions, Symmetry, (2019), 11, 347.

[22] W. Noonan, D. K. Thomas, On the second Hankel determinant of a really mean p-valent functions, Trans. Am. Math. Soc., (1976), 223, 337-346.

[23] H. Orhan, M. Caglar and L. I. Cotirla, Third Hankel Determinant for a Subfamily of Holomorphic Functions Related with Lemniscate of Bernoulli, Mathematics, (2023), 11, 1147.

[24] I. A. R. Rahman and W. G. Atshan, Results on coefficients bounds for new subclasses by using quasi-subordination of analytic function estimates, AIP Conference Proceedings, 2398 (2022), 060021,

[25] I. A. R. Rahman, W. G. Atshan, G. I. Oros, New concept on fourth Hankel determinant of a certain subclass of analytic functions, Afr. Mat., (2022), 33, 7.

[26] P. O. Sabir, H. M. Srivastava, W. G. Atshan, P. O. Mohammed, N. Chorfi and M. Vivas-Cortez, A family of holomorphic and m-fold symmetric bi-univalent functions endowed with coefficient estimate problems, Mathematics, 11(18) (2023), 3970, 1-13.

[27] Q. A. Shakir and W. G. Atshan, On third Hankel determinant for certain subclass of bi-univalent functions, Symmetry, 16(2) (2024), 239, 1-10.

[28] G. Singh, G. Singh, On the second hankel determinant for a new subclass of analytic functions, J. Math. Sci. Appl., (2014), 2, 1-3.

[29] H. M. Srivastava, Q. Z. Ahmad, N. Khan and B. Khan, Hanlel and toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain, Mathematics, (2019), 7, 181.

[30] H. Srivastava, N. Raza, E. S. Abujarad, G. Srivastava and M. H. Abujarad, Fekete-szegö inequality for classes of (p,q)-starlike and (p,q)-convex functions, Rev. Real Acad. Cienc. Exactas Fisicas Nat. Ser. A Matematicas, (2019),113,3563-3584.

[31] H. M. Srivastava, Q. Z. Ahmad, M. Darus, N. Khan, B. Khan, N. Zaman and H. H. Shah, Upper bound of the third hankel determinant for a subclass of close-to-convex functions associated with the lemniscate of Bernoulli, Mathematics, (2019), 7, 848.

[32] L. Shi, H. M. Srivastava, M. Arif, S. Hussain and H. Khan, An investigation of the third hankel determinant problem for certain subfamilies of univalent functions involving the exponential function, Symmetry, (2019), 11, 598.

[33] S. R. Swamy and W. G. Atshan, Coefficient bounds for regular and bi-univalent functions linked with Gegenbauer polynomials, Gulf Journal of Mathematics, 13(2) (2022), pp. 67-77.